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FURTHER RESULTS ON COCOMPLETE BIPARTITE GRAPHS

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Abstract. Let $G = (V_1, V_2, E)$ be a bipartite graph. Then G is called cocomplete bipartite graph, if for any two vertices $u, v \in V_i$, $i = 1, 2$ there exists P_3 containing them. In this paper, we introduce the concepts of Weak and Strong cocomplete bipartite graphs. We study some properties of these graphs. We also develop furthers results on cocomplete bipartite graph.

1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. For graph theoretical terminology we refer to [6].

A bipartite graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2. If G contains every edge joining V_1 and V_2, then G is a complete bipartite graph. If V_1 and V_2 have n and m vertices in complete bipartite graph, we write $G = K_{n,m} = K(n, m)$. A bipartite graph $G = (V_1, V_2, E)$ is called cocomplete bipartite graph, if for any two vertices $u, v \in V_i$, $i = 1, 2$ there exists P_3 containing them [1].

A graph is connected if every pair of its vertices are joined by a path. A subgraph of G is a graph having all of its vertices and edges in G. A spanning subgraph is a subgraph containing all the vertices of G.

We define $\lfloor x \rfloor$ to be greatest integer not exceeding x and $\lceil x \rceil$ to be smallest integer not smaller than x [6].

Define $N(v) = \{u \in V(G)|u \text{ is adjacent to } v\}$, $N[v] = N(v) \cup \{v\}$, A set S of vertices of G is neighborhood set of G if $G = \cup_{v \in S} \langle N[v] \rangle$ [7]. The degree of a vertex v of G, denoted by $deg(v)$, is the number of vertices adjacent to v. We denote the maximum degree of G by $\Delta(G)$, the minimum degree of G by $\delta(G)$ [6]. A path P_n is a graph with vertices $u_1, \ldots, u_n$ and edges $u_1u_2, u_2u_3, \ldots, u_{n-1}u_n$. It can also be called a path from u_1 to u_n [4].

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A balanced bipartite graph is a bipartite graph in which both partite sets are of the same cardinality [5].

A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers. [3].

A connected graph G is said to be geodetic, if a unique shortest path joins any two of its vertices. The shortest path between two vertices u and v is called geodetic path between u and v.

The line graph of G, denoted by $L(G)$, is the graph whose vertex set is $E(G)$ with two vertices adjacent in $L(G)$ whenever the corresponding edges of G are adjacent [6].

The complement of a graph G, denoted by $\overline{G}$ has the same vertex set as G and two vertices u and v are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A graph G is said to be self-complementary if it is isomorphic to its complement $\overline{G}$. Let $G = (V_1, V_2, E)$ be cocomplete bipartite graph, we define complement of G denoted by $\overline{G} = (V_1, V_2, \overline{E})$ as follows:

(1) Every two vertices in V_1 are adjacent,
(2) Every two vertices in V_2 are adjacent,
(3) $u \in V_1$, $v \in V_2$ are adjacent in $\overline{G}$ if and only if $u \in V_1$, $v \in V_2$ and u, v are not adjacent in G.

We use the following results to prove our main results.

Observation 1.1[1]. Any complete bipartite graph is cocomplete bipartite graph, but the converse is not true.

Observation 1.2[1]. C_n is cocomplete bipartite graph if and only if $n = 4$ or 6.

Theorem 1.3[1]. Let $G = (V_1, V_2, E)$ be a cocomplete bipartite graph with $|V_1| = n$, $|V_2| = m$ and $|E| \geqslant 1$. Then G has minimum number of edges $q = n + m - 1$ if and only if $G \cong S_{n,m}$.

Theorem 1.4[1]. Let $G = (V_1, V_2, E)$ be a balanced bipartite graph of order p, $p \geqslant 4$ with minimum degree $\delta(G) > \lfloor \frac{p}{4} \rfloor$. Then G is cocomplete bipartite graph.

2. Main Results

Definition 2.1. A cocomplete bipartite graph G is said to be weak cocomplete bipartite graph if and only if removal of any edge from G results in a non cocomplete bipartite graph.

Definition 2.2. A cocomplete bipartite graph G is said to be strong cocomplete bipartite graph if and only if there is at least one edge of G whose removal from G results in a cocomplete bipartite graph.

Observations:

(1) Every complete bipartite graph is strong cocomplete bipartite graph. The converse is not true in view of Observation 1.1.

(2) C_n is weak cocomplete bipartite graph if $n = 6$ and strong cocomplete bipartite graph if $n = 4$.
(3) $S_{n,m}$ is strong cocomplete bipartite graph if and only if $n = m = 1$.
(4) A connected cocomplete bipartite graph G is weak cocomplete bipartite graph if and only if G is not strong cocomplete bipartite graph.

Remark 2.1. Every weak cocomplete bipartite graph is cocomplete bipartite graph, but the converse is not true by Observation 4.

Remark 2.2. Every strong cocomplete bipartite graph is cocomplete bipartite graph, but the converse is not true by Observation 4.

Remark 2.3. A cocomplete bipartite graph G is neither weak cocomplete bipartite graph nor strong cocomplete bipartite graph if and only if G is disconnected.

Theorem 2.1. *Let G be strong cocomplete bipartite graph. Then G has minimum number of edges $q_s = n + m$ if and only if $G - e \cong S_{n,m}$, where e is an edge of G not incident with the center point of G.*

Proof. Let $G = (V_1, V_2, E)$ be strong cocomplete bipartite graph. By Remark 2.4, G is cocomplete bipartite graph. Suppose that $G - e \cong S_{n,m}$, where e is an edge of G not incident with the center point of G. By Theorem 1.3 the number of edges in $S_{n,m}$ is q, where $q = n + m - 1$.
Hence the minimum number of edges in G is $q_s = n + m$. Since by Theorem 1.3 the cocomplete bipartite graph with minimum number of edges is $S_{n,m}$, it follows that G has minimum number of edges.
Conversely, Suppose that G has minimum number of edges $q_s = n + m$. Since G is strong cocomplete bipartite graph, there exists an edge $e \in E(G)$ such that $G - e$ is cocomplete bipartite graph with $n+m-1$ edges. By Theorem 1.3, $G - e \cong S_{n,m}$. □

Theorem 2.2. *(Characterisation for weak cocomplete bipartite graph) Let $G = (V_1, V_2, E)$ be cocomplete bipartite graph. Then G is weak cocomplete bipartite graph if and only if for each edge uv of G there is a unique path of length two from u to a vertex of $N(v)$ or v to a vertex of $N(u)$.*

Proof. Let G be weak cocomplete bipartite graph. Let $e = uv \in E(G)$. By definition of weak cocomplete bipartite graph, $G - e$ is not cocomplete bipartite graph. So, there exist $x, y \in V_i(G - e) = V_i(G)$, for some $i = 1, 2$ such that there is no path of length 2 from x to y in $G - e$. But G is cocomplete bipartite graph, and hence there is a path (x, w, y) from x to y in G where $w \in V_j(G)$, $j \neq i$. Since $e = uv$ is the only edge deleted from G, it follows that either $e = xw$ or $e = wy$. If $e = xw$ then $u = x$, $v = w$, $y \in N(v)$ and (u, v, y) is the unique path of length 2 from u to y. If $e = wy$ then $u = y$, $v = w$, $x \in N(v)$ and (u, v, x) is the unique path of length 2 from u to x. Hence there is a unique path of length two from u to a vertex of $N(v)$ and v to a vertex of $N(u)$.
Conversely, suppose that for $u, v \in V_i$, $i = 1, 2$ we have $w \in V_j$, $j \neq i$ such that (u, w, v) is a unique path of length 2 between u and v. Then $G - uw$ is not cocomplete bipartite graph. Hence G is weak cocomplete bipartite graph.

□

Theorem 2.3. *(Characterisation for strong cocomplete bipartite graph) Let $G = (V_1, V_2, E)$ be cocomplete bipartite graph. Then G is strong cocomplete bipartite graph if and only if for some $u \in V(G)$, there exists $v \in N(u)$ such that there are at least two paths of length two from u to every vertex of $N(v)$ and v to every vertex of $N(u)$.*

Proof. Let $G = (V_1, V_2, E)$ be cocomplete bipartite graph. Suppose that for some $u \in V(G)$, there exist $v \in N(u)$ such that there are at least two paths of length two from u to every vertex of $N(v)$ and v to every vertex of $N(u)$. Then $G - uv$, is still cocomplete bipartite graph. Hence G is strong cocomplete bipartite graph. Conversely, suppose that for every $u \in V(G)$ and for every $v \in N(u)$, there is $w \in N(v)$ such that (u, v, w) is the only path of length 2 from u to w. Hence $G-uv$ is not cocomplete bipartite graph, and hence G is weak cocomplete bipartite graph, a contradiction. □

Definition 2.3. A bipartite graph G which is not cocomplete bipartite graph is said to be almost cocomplete bipartite graph if and only if there exist two non adjacent vertices u and v in G such that $G + uv$ is cocomplete bipartite graph.

Example : P_5 is almost cocomplete bipartite graph. But C_8 is not almost cocomplete bipartite graph.

Definition 2.4. Let G be cocomplete bipartite graph. If v is a vertex of G such that $G - v$ is cocomplete bipartite graph. Then v is called cordial vertex of G.

Example : Each vertex of C_4 is cordial vertex, because C_4 is cocomplete bipartite graph and for any $v \in V(G)$, $C_4 - v = P_3$ is also cocomplete bipartite graph.
Example : No vertex of C_6 is cordial, because C_6 is cocomplete bipartite graph but $C_6 - v$ is not cocomplete bipartite graph for any $v \in V(G)$.

Observations.

(1) A path P_n is almost cocomplete bipartite graph if and only if $n = 5$ or 6.
(2) If G is weak cocomplete bipartite graph and e is any edge of G, then $G - e$ is almost cocomplete bipartite graph.
(3) Each vertex of complete bipartite graph $K_{n,m}$, $n, m \geqslant 2$ is cordial vertex.
(4) Each pendant vertex of $S_{n,m}$, $n, m \geqslant 2$ is cordial vertex.

Theorem 2.4. *Let $G = (V_1, V_2, E)$ be a balanced bipartite graph of order p, $p \geqslant 4$ with minimum degree $\delta(G) > \lceil \frac{p}{4} \rceil$. Then G has cordial vertex.*

Proof. Let $G = (V_1, V_2, E)$ be a balanced bipartite graph of order p, $p \geqslant 4$ then $|V_1| = |V_2|$. Let $u, w \in V_1$. Since $\delta(G) > \lceil \frac{p}{4} \rceil$, we have $\delta(G) > \lfloor \frac{p}{4} \rfloor$. By Theorem 1.4, G is cocomplete bipartite graph. So, there exists $v \in V_2$ such that (u, v, w) is a path. Since $p \geqslant 4$ and G is balanced there exists $w' \in V_2$. Since $\delta(G) > \lfloor \frac{p}{4} \rfloor$, u and w are adjacent to at least $\lfloor \frac{|V_2|}{2} \rfloor + 1$ vertices and hence are adjacent to v and w', so that (u, w', w) is also a path. This true for all $u, w \in V_i$, $i = 1, 2$ and $v \in V_j$, $j \neq i$.
Hence $G - v$ is cocomplete bipartite graph. Thus v is cordial vertex. □

Theorem 2.5. *Let G be a weak cocomplete bipartite graph. If G has pendant vertex, then G has cordial vertex.*

Proof. Let $G = (V_1, V_2, E)$ be a weak cocomplete bipartite graph. Suppose $\delta(G) = 1$. Let $v \in V(G)$ such that $deg(v) = 1$. Then $G - v$ is cocomplete bipartite graph, as removal of v does not alter the common neighborhood v of any other pair of vertices. So, v is cordial vertex of G. □

Remark 2.4. The converse of above Theorem is not true in view of Figure 1.

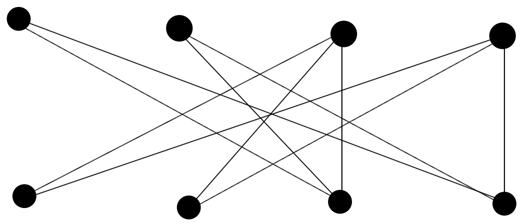

Figure 1

Theorem 2.6. *(Characterisation of cordial vertex) Let $G = (V_1, V_2, E)$ be a cocomplete bipartite graph with $\delta(G) \geqslant 2$. Then $v \in V(G)$ is cordial vertex if and only if there are at least two paths of length two between any two neighbors of v in G.*

Proof. Let $G = (V_1, V_2, E)$ be cocomplete bipartite graph with $\delta(G) \geqslant 2$. Let $v \in V(G)$ be a cordial vertex of G. Then $G - v$ is also cocomplete bipartite graph. Since $\delta(G) \geqslant 2$, let $u, w \in V(G)$, such that $u, w \in N(v)$, i.e. (u, v, w) is a path in G. Since $G - v$ is also cocomplete bipartite graph, there exists $w' \in V(G - v)$, such that (u, w', w) is a path in $G - v$ and hence in G. Hence there are at least two paths of length two between any two neighbors of v in G.
Conversely, suppose that G is cocomplete bipartite graph with $\delta(G) \geqslant 2$. Let $v \in V(G)$ be such that there are two paths of length two between any two neighbors of G. Suppose that $v \in V_1$. Let $u, w \in V(G)$. Since deletion of v from G dose not alter the neighborhood of any vertex in the set $V_1(G)$ and $\delta(G) \geqslant 2$, we con assume that $u, w \in V_2(G)$.
Case 1 : $u, w \in N(v)$. By hypothesis, there exists $w' \neq v \in V_1$, such that (u, w', w) is a path.
Case 2 : One of u, w is not neighbors of v. Assume that $u \in N(v)$. Then v, w are not adjacent in G. Since G is cocomplete bipartite graph, there exists $w' \neq v \in V_1$, such that (u, w', w) is a path in G hence in $G - v$. Hence $G - v$ is cocomplete bipartite graph.

Case 3 : $u, w \notin N(v)$. Since G is cocomplete bipartite graph, there exists $w' \neq v \in V_1$, such that (u, w', w) is a path in G. So, this path is not affected by deleting of v from G. Thus v is a cordial vertex. □

Corollary 2.1. *Let G be a cocomplete bipartite graph of order p, $p \geqslant 3$. Then $v \in V(G)$ is cordial vertex if and only if either v is pendant vertex of G or any two neighbors of v are joined by at least two paths of length two.*

Proof. Since removal of a pendant vertex from cocomplete bipartite graph G results in cocomplete bipartite graph the results follows from Theorem 2.14.

□

Theorem 2.7. *Let $K_{n,n}$ be a balanced complete bipartite graph. If $S \subset E(K_{n,n})$ with $|S| < n$, then every connected spanning subgraph of $K_{n,n} - S$ is cocomplete bipartite graph.*

Proof. Let $H = (V_1, V_2, E')$ be a connected spanning subgraph of $K_{n,n} = (V_1, V_2, E)$. Since $K_{n,n}$ is balanced bipartite graph, $|V_1| = |V_2| = n$. Let $E = \{e_1, e_2, \ldots, e_{n^2}\}$ and $E' = E - S$ where $|S| < n$. Let $u, v \in V_i$, for some i. Since $|S| < n$, there exists a vertex $w \in V_j$, $j \neq i$ such that (u, w, v) is a path in $K_{n,n} - S$. Since u and v are arbitrary, $K_{n,n} - S$ is cocomplete bipartite graph. □

Theorem 2.8. *A spanning subgraph H of $K_{n,m}$, $n \geqslant m$ with minimum degree $\delta(H) > \lceil \frac{n}{2} \rceil$ is cocomplete bipartite graph.*

Proof. Let $H = (V_1, V_2, E')$ be a connected spanning subgraph of $K_{n,m} = (V_1, V_2, E)$. Since $\delta(H) > \lceil \frac{n}{2} \rceil$, then degree of each vertex of H is at least $\lceil \frac{n}{2} \rceil + 1$. Thus H is connected. Let $u, v \in V_i$, $i = 1, 2$ such that $deg(v) = \delta(H)$, $deg(u) \geqslant deg(v)$. Since $\delta(H) > \lceil \frac{n}{2} \rceil$, then both u and v are adjacent to at least $\lfloor \frac{|V_j|}{2} \rfloor + 1$ vertices of V_j, $j \neq i$. Hence we can find $w \in V_j$, $j \neq i$ such that (u, w, v) is a path. Hence H is cocomplete bipartite graph. □

Remark 2.5. Every strong cocomplete bipartite graph contains a spanning subgraph which is cocomplete bipartite graph.

Theorem 2.9. *Let $G = (V_1, V_2, E)$, $|V_1| = n \geqslant 3$, $|V_2| = m \geqslant 3$, be a bipartite graph. If the subgraph $H = (V_1 - u, V_2 - v)$ of G is cocomplete bipartite graph for each $u \in V_1$ and $v \in V_2$, then G is cocomplete bipartite graph.*

Proof. Let $G = (V_1, V_2, E)$, $|V_1| = n \geqslant 3$, $|V_2| = m \geqslant 3$, be a bipartite graph and a subgraph $H_{i,j} = (V_1 - u_i, V_2 - v_j, E_{i,j})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ of G be cocomplete bipartite graph. Then for all $u_k, u_l \in H_{i,j}$, $(k \neq l \neq i)$ there exists v_t, $(t \neq j)$ such that $(u_k, , v_t, u_l)$ is a path.
Similarly, for all $v_k, v_l \in H_{i,j}$, $(k \neq l \neq j)$ there exists u_t, $(t \neq i)$ such that (v_k, u_t, v_l) is a path. This holds for every $H_{i,j}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, since $H_{i,j}$ is cocomplete bipartite graph.
Hence any two vertices in V_i, $i = 1, 2$ are joined by a path of length two. Thus G is cocomplete bipartite graph. □

Remark 2.6. The converse of above Theorem is not true. For example $C_6 = (V_1, V_2, E)$ is cocomplete bipartite graph. But there exist $u, v \in V(C_6)$ such that $(V_1 - u, V_2 - v)$ is not cocomplete bipartite graph.

Theorem 2.10. *The line graph of an isolate-free graph G is cocomplete bipartite graph if and only if G is one of the following graphs:*

(1) $G \cong 2K_2$,
(2) $G \cong C_n$, $n = 4, 6$,
(3) $G \cong P_n$, $3 \leqslant n \leqslant 5$.

Proof. Suppose that $H = L(G)$ is cocomplete bipartite graph. If $\Delta(H) \geqslant 3$ in H, then $K_{1,3}$ would be an induced subgraph of H and $H \neq L(G)$ for any graph, a contradiction to the hypothesis. So, $\Delta(H) \leqslant 2$. We consider the following cases.
Case 1 : $\Delta(H) = 0$. Then $H = \overline{K}_2$, and hence $G = 2K_2$.
Case 2 : $\Delta(H) = 1$. then $H = K_2$ and hence $G = P_3$.
Case 3 : $\Delta(H) = 2$. then $H = C_n$ or P_n. We consider the following Subcases.
Subcase 3.1 : $H = C_n$. Since C_4 and C_6 are the only cycles which are cocomplete bipartite graph, it follows that $H = C_4$ or C_6.
Hence $G = C_4$ or C_6.
Subcase 3.2 : $H = P_n$. Since P_3 and P_4 are the only paths with $\Delta(H) = 2$ which are cocomplete bipartite graph, it follows that $H = P_3$ or P_4. So corresponding $G = P_4$ or P_5.
Hence from all the above cases. G is one of the following graphs:

(1) $G \cong 2K_2$,
(2) $G \cong C_n$, $n = 4, 6$,
(3) $G \cong P_n$, $3 \leqslant n \leqslant 5$.

The converse is obvious. □

Theorem 2.11. *Let $G = (V_1, V_2, E)$ be cocomplete bipartite graph. Then the complement of G is cocomplete bipartite graph if and only if G is one of the following graphs:*

(1) $G \cong K_2$,
(2) $G \cong \overline{K}_2$,
(3) $G \cong P_4$.

Proof. Let $G = (V_1, V_2, E)$ be cocomplete bipartite graph such that $\overline{G} = (V_1, V_2, E')$ is cocomplete bipartite graph. If $|V_1| \geqslant 3$ or $|V_2| \geqslant 3$, then K_3 is an induced subgraph of $\overline{G}$ and $\overline{G}$ is not cocomplete bipartite graph a contradiction to the hypothesis. So, $|V_1| \leqslant 2$ and $|V_2| \leqslant 2$. We consider the following cases.
Case 1 : $|V_1| = |V_2| = 1$. Let $u \in V_1$, $v \in V_2$. We consider the following subcases.
Subcase 1.1 : u and v are adjacent. then $G \cong K_2$ and hence $\overline{G} \cong \overline{K}_2$.
Subcase 1.2 : u and v are not adjacent. then $G \cong \overline{K}_2$ and hence $\overline{G} \cong K_2$.
Case 2 : Suppose $|V_1| = 1$ and $V_2 = 2$. $G \cong P_3$, and hence $\overline{G} \cong K_2 \cup K_1$.
Case 3 : $|V_1| = 2$ and $|V_2| = 2$. We consider the following subcases.
Subcase 3.1 : $G \cong P_4$, and hence $\overline{G} \cong P_4$
Subcase 3.2 : $G \cong C_4$, and hence $\overline{G} \cong 2K_2$.

Hence from all the above cases. Since G is cocomplete bipartite graph, then G is one of the following graphs:

(1) $G \cong K_2$,
(2) $G \cong \overline{K}_2$,
(3) $G \cong P_4$.

The converse is obvious. □

Corollary 2.2. *Let $G = (V_1, V_2, E)$ be cocomplete bipartite graph. Then G is Self-Complementary cocomplete bipartite graph if and only if $G \cong P_4$.*

Proof. Proof follows from above Theorem. □

Theorem 2.12. *Let $G = (V_1, V_2, E)$ be a connected cyclic cocomplete bipartite graph. Then the girth of G is 4 or 6.*

Proof. Result follows by Observation 1.2.

□

Theorem 2.13. *Let $G = (V_1, V_2, E)$ be a connected cocomplete bipartite graph of order p. Then G is geodetic graph if and only if G is one of the following graphs:*

(1) $G \cong K_{1,n}$,
(2) $G \cong S_{n,m}$.

Proof. Let $G = (V_1, V_2, E)$ be cocomplete bipartite graph. Suppose G is geodetic graph, then G is a cyclic and hence it is a tree.
Hence G is one of the following graphs:

(1) $G \cong K_{1,n}$,
(2) $G \cong S_{n,m}$.

The converse is obvious. □

Corollary 2.3. *Let $G = (V_1, V_2, E)$ be a connected cocomplete bipartite graph. Then any geodetic graph $G \ncong K_2$ is weak cocomplete bipartite graph.*

Proof. Proof follows from above Theorem. □

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