ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2013, Article ID ama0112, 10 pages
ISSN 2307-7743
http://scienceasia.asia

BOUNDARY VALUE PROBLEM FOR TWO-LAYER HALF-PLANE

O. YAREMKO

ABSTRACT. In the article the formulas for the modeling of conservative fields in piecewise infinite plate with a thin inclusion found. The accuracy of the found formulas is of order equal to the thickness of the outer layer. The problem for higher accuracy solved by means of asymptotic formulas.

1. INTRODUCTION

Problems about structure of the conservative field of a two-layer flat plate leads to a separate system of the Laplace's equation with a boundary condition to Dirichlet and internal coupling conditions [1], [2]. In the work the method of transformation operators is applied. Yaremko O. E. developed this method [1], [2], [3], [4], [5]. As a result the conservative field for the two-layer piecewise plate can be interpreted as a deformation of the conservative field of a homogeneous piecewise plate. Thus the deforming transformation operator is written out. The deforming transformation operator is recursive, and convenient for practical realization on the computer. In this case, use the analytic representation of a field by means of the deforming transformation operator is difficult.

We consider the case of small thickness l cross-border layer of the plate. Let's consider that the physical properties of the layers very different, that is the ratio of the thermal conductivity of the layers close to zero.

The method of the deforming transformation operator gives algorithm inconvenient for realization. Formulas which have no these defect are found based on the Euler–Maclaurin formula.

Numerical methods of research lead to ill-conditioned [8] systems of the linear algebraic equations.

2. THE PLANE CASE

2.1. **The half-plane with internal coupling condition.** Modelling of potential fields for a semi-limited infinite plate with thin inclusion leads to the Dirichlet problem for two-layer half-plane with coupling conditions on the direct. Let's consider a boundary problem about the solving of separate system of the Laplace's equation

$$a_1^2\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0,(x,y);0<x<l,-\infty<y<\infty \tag{1}$$

2010 *Mathematics Subject Classification.* 65R10.

Key words and phrases. conservative field, internal coupling conditions, problem of Robin, Dirichlet problem.

$$a_2^2\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0,(x,y);l<x,-\infty<y<\infty \tag{2}$$

by the boundary condition for direct $x=l$:

$$u_1(x,y)=\hat{u}(x,y),x=0 \tag{3}$$

by the internal coupling conditions [1], [2], [3], [4], [5] for direct $x=l$:

$$u_1(x,y)=u_2(x,y),k\frac{\partial u_1}{\partial x}(x,y)=\frac{\partial u_2}{\partial x}(x,y),k>0, \tag{4}$$

Here function $\hat{u}=\hat{u}(x,y)$- is harmonic in the right half-plane, $H=\{(x,y):0<x,y\in R\}$ and continuous in $\{(x,y):0\leq x,y\in R\}$ and the conditions are met

$$\hat{u}=0(\sqrt{x^2+y^2}),$$

$$\int_{-\infty}^{\infty}\frac{\hat{u}(0,y)}{1+y^2}dy<\infty.$$

The problem (1)-(4) is ill-posed [8] in the case of a thin inclusion, that is, small value l , and in the case of a strong heterogeneity, that is, large differences in the parameters a_1,a_2. The solution of this problem by known methods is accompanied by considerable computing difficulties. New analytical methods to solve this problem it is necessary to develop.

2.2. **Thin strip.** Function $\hat{u}=\hat{u}(x,y)$ is a harmonic function in the right half-plane $H=\{(x,y):0<x,y\in R\}$ continuous in $\{(x,y):0\leq x,y\in R\}$ and the conditions are met

$$\hat{u}=0(\sqrt{x^2+y^2}),$$

$$\int_{-\infty}^{\infty}\frac{\hat{u}(0,y)}{1+y^2}dy<\infty.$$

Let's consider the Dirichlet problem for Laplace's equation in the strip $H_l=\{(x,y):0<x<l,y\in R\}$

$$\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0,(x,y);0<x<l,-\infty<y<\infty \tag{5}$$

with boundary conditions

$$u(0,y)=\hat{u}(0,y),u(l,y)=0. \tag{6}$$

The problem (5)-(6) is ill-posed [8] at small value l.

3. The axial case

3.1. **The circle with the internal coupling conditions.** Let the function $\hat{u}=\hat{u}(x,y)$- is a harmonic function in the unit circle, $B=\{(x,y):x^2+y^2<1\}$ and continuous in its closure $\bar{B}$. The boundary problem about the solving a separate system of the Laplace's equation consider in the circle B

$$\Delta u_1=0,(x,y)\in K_R,$$

$$\Delta u_2=0,(x,y)\in B_R \tag{7}$$

Here K_R - annulus with the smaller radius R and larger radius 1, B_R - the circle of radius R. Let the boundary condition on the circle satisfied S:

$$u_1(x,y) = \hat{u}(x,y), (x,y) \in S \tag{8}$$

internal coupling condition on a circle of radius R:

$$u_1(x,y) = u_2(x,y),$$

$$kL_0u_1(x,y) = L_0u_2(x,y), k > 0, (x,y) \in S_R, \tag{9}$$

$$L_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Let's notice that the problem (7)-(9) is ill-posed [8] at values $R \approx 1$.

3.2. **Thin annulus.** Let the function $\hat{u} = \hat{u}(x,y)$ - is a harmonic function in the unit circle, $B = \{(x,y) : x^2 + y^2 < 1\}$ and continuous in its closure $\bar{B}$. In annulus K_R- with the smaller radius R and larger radius 1, consider the Dirichlet problem about the solution of the Laplace's equation

$$\Delta u_1 = 0, (x,y) \in K_R, \tag{10}$$

by the boundary conditions on the circumferences S, S_R:

$$u = \hat{u}, (x,y) \in S; u = 0, (x,y) \in S_R. \tag{11}$$

The problem (10) is ill-posed [8] by the values $R \approx 1$.

4. Transformation operators

In the works of the author [1], [2], [3], [4] and [5] the concept of operator transformation J, $J : \hat{u} \to u$, allowing a known solution $\hat{u}$ of the model problem to solve u any of the four above-listed problems determined.

The half-plane with the internal coupling conditions

Transformation operator J has the form

$$J : \hat{u} \to u, u = \chi(H_1)u_1 + \chi(H_2)u_2,$$

where:

$$u_1(x,y) = \sum_{j=0}^{\infty} \left(\frac{1-k}{1+k}\right)^j \left(\hat{u}(x+2lj,y) - \frac{1-k}{1+k}\hat{u}(2l-x+2lj,y)\right), \quad 0 < x < l,$$

$$u_2(x,y) = \frac{2k}{k+1}\sum_{j=0}^{\infty} \left(\frac{1-k}{1+k}\right)^j \hat{u}\left(\frac{a_1}{a_2}(x-l) + l + 2lj, y\right), \quad l < x, \tag{12}$$

$$k = \frac{\lambda_1}{\lambda_2}\frac{a_2}{a_1},$$

$\chi-$ is the characteristic function of the set [6].

Thin strip

Transformation operator $J : \hat{u} \to u$, can be determined by the formula

$$u(x,y) = \sum_{j=0}^{\infty} (\hat{u}(x+2lj,y) - \hat{u}(2l-x+2lj,y)); \quad 0 < x < l. \tag{13}$$

Circle with the internal coupling conditions

Transformation operator is expressed by the formula:

$$J:\hat{u}\to u, u=\chi(K_R)u_1+\chi(B_R)u_2,$$

where:

$$u_1(x,y)=\sum_{j=0}^{\infty}\left(\frac{1-k}{1+k}\right)^j\left(\hat{u}(xR^{2lj},yR^{2lj})-\frac{1-k}{1+k}\hat{u}\left(\frac{R^{2j+2}}{x},\frac{R^{2j+2}}{y}\right)\right), \tag{14}$$

$$u_2(x,y)=\frac{2k}{k+1}\sum_{j=0}^{\infty}\left(\frac{1-k}{1+k}\right)^j\hat{u}(xR^{2j},yR^{2j}),$$

χ – is the characteristic function of the set [6].

Thin annulus

Transformation operator $J:\hat{u}\to u$, can be determined by the formula

$$u(x,y)=\sum_{j=0}^{\infty}\left(u\left(xR^{2j},yR^{2j}\right)-u\left(\frac{R^{2j+2}}{x},\frac{R^{2j+2}}{y}\right)\right), \tag{15}$$

Formulas (12) and (14) are useful only if the values $k\approx 1$. In the case of the two-layer body, which components are sharply different properties, that is at values k close to zero, and for large values k, transformation operators cannot be used, in view of the slow convergence of the series at $\frac{1-k}{1+k}\approx 1$.

Formulas(13) and (15) cannot be used in view of the slow convergence of the series at $l\approx 1$ and $R\approx 1$, respectively.

The purpose of our research is to consists in creation of a design of transformation operators comfortable in these cases.

5. Auxiliary result

Lemma 1. If the function y = f(x) is defined on the line segment [0,1] and has limited variation [10] on the line segment [0,1], then for each value R the following assessment for the difference of integral and its integral sum is carried

$$\left|\int_0^1 x^{-1}f(x)dx-\ln\frac{1}{R^2}\sum_{j=0}^{\infty}f(R^{2j})\right|\le\ln\frac{1}{R^2}\mathop{V}_0^1(f)$$

Proof. Let's spread integral into the sum (n +1) of the summand: $\int_0^1 x^{-1}f(x)dx=\sum_{j=0}^{n-1}\int_{R^2}^1 x^{-1}f(R^{2j}x)dx+\int_0^1 x^{-1}f(R^{2n}x)dx$Variable substitution in the last integral results in equality:

$$\int_0^1 x^{-1}f(x)dx=\sum_{j=0}^{n-1}\int_{R^2}^1 f(R^{2j})\frac{dx}{x}+\int_0^{R^{2n}}x^{-1}f(x)dx$$

Considering that $\left|f(R^{2j}x)-f(R^{2j})\right|\le\mathop{V}_0^1(f)$, We obtain the required assessment:

$$\left|\int_0^1 x^{-1}f(x)dx-\sum_{j=0}^{\infty}\int_{R^2}^1 f(R^{2j})\frac{dx}{x}\right|\le$$

$$\leq \sum_{j=0}^{\infty} \int_{R^2}^{1} \left| f(R^{2j}x) - f(R^{2j}) \right| \frac{dx}{x} \leq \overset{1}{\underset{0}{V}}(f) \ln \frac{1}{R^2}$$

$\overset{1}{\underset{0}{V}}(f)$ – variation of function [10] f on the line segment [0,1]. The flat analog of the Lemma is well known [8].

Lemma 2. If the function y = f(x) is defined on the ray [0;∞) and has limited variation [10] on the ray [0; ∞); then for each value l the following assessment for the difference of integral and its integral sum is carried

$$\left| \int_{0}^{\infty} f(x)dx - 2l \sum_{j=0}^{\infty} f(2lj) \right| \leq 2l \overset{\infty}{\underset{0}{V}}(f)$$

The Bernoulli numbers [11] are defined using the generating function

$$\frac{z}{e^z - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} z^n.$$

The Euler–Maclaurin formula applied to the function $f(2lx)$ on the ray [0;∞) results in equality

$$\sum_{j=0}^{\infty} f(2lj) \cong \frac{1}{2l} \int_{0}^{\infty} f(x)dx + \frac{f(0)}{2} - \sum_{k=1}^{\infty} \frac{(2l)^{2k-1} B_{2k}}{k!} f^{2k-1}(0),$$

$$\sum_{j=0}^{\infty} f(R^{2j}) \cong \frac{1}{\ln \frac{1}{R^2}} \int_{0}^{\infty} f(x)dx + \frac{f(1)}{2} - \sum_{k=1}^{\infty} \frac{(2l)^{2k-1} B_{2k}}{k!} L_0^{2k-1} f(1)$$

Corollary 1. When $k \in (0,1)$ completed asymptotically

$$\sum_{j=0}^{\infty} \left(\frac{1-k}{1+k} \right)^j f(x + 2lj) \cong$$

$$\cong \frac{1}{2l} \int_{0}^{\infty} e^{h\varepsilon} f(x+\varepsilon) d\varepsilon + \frac{f(x)}{2} - \sum_{k=1}^{\infty} \frac{(2l)^{2k-1} B_{2k}}{k!} L_h^{2k-1} f(x).$$

When $k \in (1, \infty)$completed asymptotically

$$\sum_{j=0}^{\infty} \left(\frac{1-k}{1+k} \right)^j f(x + 2lj) \cong \frac{f(x)}{2} - \sum_{k=1}^{\infty} \frac{(2l)^{2k-1} B_{2k}}{k!} (2^{2k} - 1) L_h^{2k-1} f(x).$$

The operator L_{2h} has the form

$$L_{2h} = 2h + \frac{d}{dx}.$$

Corollary 2. When $k \in (0,1)$ completed asymptotically

$$\sum_{j=0}^{\infty} \left(\frac{1-k}{1+k} \right)^j f(rR^{2j}) \cong$$

$$\cong \frac{1}{\ln \frac{1}{R^2}} \int\limits_0^1 \varepsilon^{h-1} f(r\varepsilon) d\varepsilon + \frac{f(r)}{2} + \sum_{k=1}^{\infty} \frac{\left(\ln \frac{1}{R^2}\right)^{2k-1} B_{2k}}{k!} L_{2h}^{2k-1} f(r).$$

When $k \in (1, \infty)$ completed asymptotically

$$\sum_{j=0}^{\infty} \left(\frac{1-k}{1+k}\right)^j f(rR^{2j}) \cong \frac{f(r)}{2} + \sum_{k=1}^{\infty} \frac{(\ln \frac{1}{R^2})^{2k-1} B_{2k}}{k!} (2^{2k}-1) L_{2h}^{2k-1} f(r).$$

The operator L_{2h} has the form

$$L_{2h} = 2h + \frac{d}{dx}.$$

6. Main result

6.1. **The plane case.** Study the case of a thin shell, that is the case of a small thickness of the final layer l. Physically this means that the coefficients of heat capacity layers are very different.

Theorem 1. Let the condition is fulfilled

$$\frac{1-k}{1+k} = e^{2hl},$$

and the value k is small. For component u_1 and u_2 of the problem solution (1)-(4) approximate formulas are valid

$$u_2 \approx \frac{1-e^{2hl}}{2l} \hat{u}_3,$$

$$u_1 \approx \frac{1}{2l} \hat{u}_3 (x, y) - \frac{e^{2h}}{2l} \hat{u}_3 (2l - x, y), (x, y) \in H_2,$$

Where $\hat{u}_3$ - solution of the problem of Robin [12] with the boundary condition

$$2h\hat{u}_3 + \frac{\partial}{\partial n} \hat{u}_3 = \hat{u}, (x, y), x = 0.$$

The assessment is fair

$$\left| \frac{1-e^{2hl}}{2l} \int\limits_0^{\infty} e^{\varepsilon h} \hat{u}(x+\varepsilon, y) d\varepsilon - u_2(x, y) \right| \leq (1-e^{2hl}) \mathop{V}_0^{\infty} (e^{\varepsilon h} \hat{u}(x+\varepsilon, y)).$$

Proof. Let's apply the lemma 2. As a result we obtain the formula:

$$\left| \frac{2k}{k+1} \frac{1}{2l} \int\limits_0^{\infty} e^{\varepsilon h} \hat{u}(x+\varepsilon, y) d\varepsilon - u_2(x, y) \right| \leq \frac{2k}{k+1} \mathop{V}_0^{\infty} (e^{\varepsilon h} \hat{u}(x+\varepsilon, y)).$$

We will express k through h. Get the assessment, which proved for u_2. The formula links solutions the Dirichlet problem and the Robin problem from [12]

$$u_2 = -\int\limits_0^{\infty} e^{\varepsilon h} \hat{u}(x+\varepsilon, y) d\varepsilon.$$

This theorem allows the physical interpretation: component of the solution u_2 with the accuracy to a numerical multiplier it is approximately equal to the solution of the third homogeneous boundary problem with boundary condition

$$2h\hat{u}_3 + \frac{\partial}{\partial n}\hat{u}_3 = \hat{u}, (x, y), x = 0.$$

Similarly

Theorem 2. Let the condition is fulfilled

$$\frac{1-k}{1+k} = e^{2hl},$$

and the value k>>1. For the problem solution (1)-(4) approximate formulas are valid

$$u_1 \approx \frac{1-e^{2hl}}{1+e^{2hl}}\frac{1}{2l}\left(\left(\hat{u}_3(x,y) - e^{2hl}\hat{u}_3(x+2l,y)\right) + ...\right),$$

$$\left(... + e^{2hl}(\hat{u}_3(2l-x,y) - e^{2hl}\hat{u}_3(4l-x,y))\right),$$

$$u_2 \approx \frac{1-e^{2hl}}{2l}\frac{1}{2l}\left(\hat{u}_3(x,y) - e^{2hl}\hat{u}_3(x+2l,y)\right).$$

Theorem 2 is proved on the basis of the Lemma 2.

Thin strip

We apply Lemma 2. For the solution of the Dirichlet problem in the strip (5)-(6) assessment is fair

$$\left|\int_0^\infty \frac{\hat{u}(x+\varepsilon,y) - \hat{u}(2l-x+\varepsilon,y)}{2l}d\varepsilon - u(x,y)\right| \leq \overset{\infty}{\underset{0}{V}},$$

here

$$\overset{\infty}{\underset{0}{V}} = \overset{\infty}{\underset{0}{V}}\left(\hat{u}(x+\varepsilon,y) - \hat{u}(2l-x+\varepsilon,y)\right)$$

- variation of function [10] taken from a variable ε on the interval [0;∞). Thus, the following theorem is obtained

Theorem 3. The solution of the Dirichlet problem in the strip (5)-(6) in the strip can be found on the approximate formula

$$u(x,y) \approx \frac{\hat{u}_2(x,y) - \hat{u}_2(2l-x,y)}{2l},$$

Where $\hat{u}_2$ - the solution of the Neumann problem for the equation Laplace's in a circle with the boundary condition

$$\frac{\partial \hat{u}_2}{\partial x} = \hat{u}, x = 0.$$

6.2. **The axial case.** The circle with with the conditions internal coupling. Study the case of a thin shell, that is the case of a small thickness of the outer layer 1-R. Let's consider also magnitude k of the small, that is consider the case when the coefficients of heat capacity layers differ greatly. Assuming

$$\frac{1-k}{1+k} = R^{2h}.$$

Formula follow from Lemma 1

$$\left|\frac{2k}{k+1}\frac{1}{\ln\frac{1}{R^2}}\int\limits_0^1 \varepsilon^{h-1}\hat{u}(x\varepsilon, y\varepsilon)d\varepsilon - u_2(x,y)\right| \le \frac{2k}{k+1}\mathop{V}\limits_0^1\left(\varepsilon^h\hat{u}(x\varepsilon, y\varepsilon)\right),$$

$$\left|\frac{1-R^{2h}}{\ln\frac{1}{R^2}}\int\limits_0^1 \varepsilon^{h-1}\hat{u}(x\varepsilon, y\varepsilon)d\varepsilon - u_2(x,y)\right| \le \mathop{V}\limits_0^1\left(\varepsilon^h\hat{u}(x\varepsilon, y\varepsilon)\right)\left(1-R^{2h}\right).$$

Therefore, we come to the next result.

Theorem 4. For the solution of the problem (7)-(9) u_1, u_2 approximate formulas are valid

$$u_1 \approx \frac{1}{\ln\frac{1}{R^2}}\hat{u}_3(x,y) - \frac{R^{2h}}{\ln\frac{1}{R^2}}\hat{u}_3\left(\frac{R^2}{x}, \frac{R^2}{y}\right), \quad u_2 \approx \frac{1-R^{2h}}{\ln\frac{1}{R^2}}\hat{u}_3.$$

This theorem allows the physical interpretation: component of the solution u_2 with the accuracy to a numerical multiplier it is approximately equal to the solution of the third homogeneous boundary problem with boundary condition

$$2h\hat{u}_3 + \frac{\partial}{\partial n}\hat{u}_3 = \hat{u}, (x,y) \in S.$$

It is necessary to consider the formula from [3], The formula links solutions of the first and third homogeneous boundary problems:

$$\hat{u}_3 = \int\limits_0^1 \varepsilon^{h-1}\hat{u}(x\varepsilon, y\varepsilon)d\varepsilon.$$

Study the case of small values of thickness of the external layer $1 - R \approx 0$, thus we consider coefficient k>>1. If the expression component u_2 rewritten in the form:

$$u_2(x,y) = \frac{2k}{k+1}\sum_{j=0}^{\infty}\left(\frac{k-1}{1+k}\right)^{2j}\hat{u}(xR^{4j}, yR^{4j}) -$$

$$-\frac{2k}{k+1}\frac{k-1}{1+k}\sum_{j=0}^{\infty}\left(\frac{k-1}{1+k}\right)^{2j}\hat{u}(xR^2R^{4j}, yR^2R^{4j}),$$

define the number h using the formula

$$\frac{k-1}{1+k} = R^{2h}.$$

and lemma 1. We get the approximate formulas for component of the problem solution (7)-(9):

$$u_1 \approx \frac{1-R^{2h}}{1+R^{2h}}\frac{1}{2\ln\frac{1}{R^2}}\left(\left(\hat{u}_3(x,y) - R^{2h}\hat{u}_3\left(R^2x, R^2y\right)\right)\right) +$$

$$+R^{2h}\left(\hat{u}_3\left(\frac{R^2}{x}, \frac{R^2}{y}\right) - R^{2h}\hat{u}_3\left(\frac{R^4}{x}, \frac{R^4}{y}\right)\right)$$

$$u_2 \approx \frac{1-R^{2h}}{2\ln\frac{1}{R^2}}\left(\hat{u}_3(x,y) - R^{2h}\hat{u}_3\left(R^2x, R^2y\right)\right).$$

Thin ring

Let _ the function $\hat{u} = \hat{u}(x,y)$ - is harmonic in the unit circle, $B = \{(x,y) : x^2 + y^2 < 1\}$ continuous in its closure $\bar{B}$. In the ring K_R- with the smaller radius R and larger radius 1, consider the Dirichlet problem (10)-(11).

We apply Lemma 1, we obtain the following assessment of solutions of the Dirichlet problem in the annulus (10)-(11)

$$\left| \int_0^1 \frac{\hat{u}(x\varepsilon, y\varepsilon) - \hat{u}\left(\frac{R^2}{x}\varepsilon, \frac{R^2}{y}\varepsilon\right)}{\varepsilon \ln \frac{1}{R^2}} d\varepsilon - u(x,y) \right| \leq \overset{\infty}{\underset{0}{V}},$$

here

$$\overset{1}{\underset{0}{V}} = \overset{1}{\underset{0}{V}} \left(\hat{u}(x\varepsilon, y\varepsilon) - \hat{u}\left(\frac{R^2}{x}\varepsilon, \frac{R^2}{y}\varepsilon\right)\right) -$$

variation of function [10] taken from a variable " on the line segment [0; 1]. Thus, the following theorem is valid.

Theorem 5. For the solutions of the Dirichlet problem in the annulus (10)-(11) approximate formula is valid

$$u(x,y) \approx \frac{\hat{u}_2(x,y) - \hat{u}_2\left(\frac{R^2}{x}, \frac{R^2}{y}\right)}{\ln \frac{1}{R^2}},$$

Where $\hat{u}_2$ - the solution of the Neumann problem for the equation Laplace's in a circle with the boundary condition

$$\frac{\partial \hat{u}}{\partial n} = \hat{u}, (x,y) \in S.$$

7. Conclusion

From the results given in article follows, that the accuracy of the found formula is order equal to the thickness of the outer layer. Therefore, obtaining the formulas of higher accuracy class represents an important interest. This problem is solved by using asymptotic formulas presented in corollary 1, 2. The problem of distribution of the results on the multilayered plates and also on the boundary problems with the more general boundary conditions is actual.

References

[1] Yaremko O.E. Transformation operator and boundary value problems Differential Equation. Vol.40, No. 8, 2004, pp.1149-1160.

[2] Yaremko, O.E. Matrix integral Fourier transforms for problems with discontinuous coefficients and transformation operators (2007) Doklady Mathematics, 76 (3), pp. 876-878.

[3] Bavrin, I.I., Yaremko, O.E. Transformation Operators and Boundary Value Problems in the Theory of Harmonic and Biharmonic Functions (2003) Doklady Mathematics, 68 (3), pp. 371-375.

[4] Bavrin, I.I., Yaremko, O.E. The operator method in the theory of integral transforms for piecewise homogeneous media (2001) Doklady Mathematics, 64 (1), pp. 44-47.

[5] Yaremko, O.E. The method of transformation operators as applied to boundary value problems in spherically symmetric domains (2006) Doklady Mathematics, 74 (1), pp. 507-511.

[6] A.N.Kolmogorov, S.V.Fomin, Elements of the Theory of Functions and Functional Analysis, The main edition of physico-mathematical literature, Nayka, M., 1976.

[7] Polya G., Szego G., Problems and Theorems in Analysis, The main edition of physico-mathematical literature, M.: Nayka, 1978., p. 392.

[8] Tikhonov, A. N.; Arsenin, V. Y. (1977). Solutions of Ill-Posed Problems. New York: Winston. ISBN 0-470-99124-0.

[9] Folland, G.B. (1999). Real Analysis: Modern Techniques and Their Applications (Second ed.). John Wiley & Sons, Inc.

[10] Rudin, Walter (1966), Real and complex analysis, McGraw-Hill, ISBN 0-07-054234-1.

[11] Abramowitz, M.; Stegun, C. A. (1972), "§23.1: Bernoulli and Euler Polynomials and the Euler-Maclaurin Formula", Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables(9th printing ed.), New York: Dover, pp. 804–806.

[12] Mei, Zhen (2000). Numerical bifurcation analysis for reaction-diffusion equations. Berlin; New York: Springer. ISBN 3-540-67296-6.

Physics and Mathematics faculty, Penza state university, Penza, Russia