

CURVATURE PROPERTIES OF HOMOGENEOUS MATSUMOTO METRIC

NARASIMHAMURTHY S.K, KAVYASHREE A.R AND MALLIKARJUN Y
KUMBAR

ABSTRACT. In this paper, we find the formula for the S-curvature of homogeneous Matsumoto metric. Further, we obtain the formula of the mean Berwald curvature using the equation of S-curvature.

1. INTRODUCTION

The concept of (α, β) -metrics were introduced by M. Matsumoto in 1972. The study of Finsler space with (α, β) -metric was studied by many authors ([1],[2],[7] etc.) and it is quite old concept, but it is a very important aspects of Finsler geometry and its applications to physics. An (α, β) -metric is scalar function on TM defined by, $F = \alpha\phi(s)$, $s = \beta/\alpha$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form in the manifold M.

Some of very important examples of (α, β) -metrics are Randers metric, Matsumoto metric and Berwald metric etc. The Matsumoto metric is one of the interesting (α, β) -metric with $\phi = 1/1 - s$, introduced by M. Matsumoto ([10]) by using gradient of slope, speed and gravity.

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This metric formulates the model of a Finsler space and many authors ([3],[12],[15]) have studied in different perspectives.

Also, curvature properties of (α, β) -metrics have been studied by various authors

([10],[13],[15]etc.). There are several interesting curvatures in Finsler geometry, among them the flag curvature is the most important one, which is the natural generalization of sectional curvature in Riemannian geometry. On the other hand, Z.Shen introduced the notion of any Riemann manifold has vanishing S-curvature. It is important to note that S-curvature and flag curvature are subtly related with each other. It is an interesting problem to compute the geometric quantities, particularly the curvatures of homogeneous spaces. In 1976, J.Milnor used the formula of the sectional curvature of left invariant Riemannian metric on a Lie group to study the curvature properties of such spaces and obtained some interesting results [6].

In 2007, S.Deng and Z.Hu proved that a homogeneous Finsler spaces with non-positive flag curvature and strictly negative Ricci scalar is a simply connected[11]. In 2009, S.Deng obtained the explicit formula of S-curvature of homogeneous Randers spaces and proved that a homogeneous Randers space with almost isotropic S-curvature must have vanishing S-curvature [12].

The main purpose of this paper is to give formula for S-curvature of homogeneous Matsumoto metric and also, we find the formula of the mean Berwald curvature E_{ij} of homogeneous Matsumoto metric.

2. PRELIMINARIES

Let M be an n -dimensional C^∞ manifold and $TM = \bigcup_{x \in M} T_x M$ denote the tangent bundle of M . A Finsler metric on M is a functions $F : TM \rightarrow [0, \infty)$ with the following properties:

- a) F is C^∞ on $TM \setminus \{0\}$;
- b) At each point $x \in M$, $F_x(y) = F(x, y)$ is a Minkowskian norm on $T_x M$.

The pair (M, F) is called Finsler manifold;

Let (M, F) be a Finsler manifold and

$$(2.1) \quad g_{ij}(x, y) = \frac{1}{2}[F^2(x, y)]_{y^i y^j}.$$

For a vector $y = y^i \frac{\partial}{\partial x^i} |_x \neq 0$, F induces an inner product g_y on $T_x M$ as follows

$$g_{ij}(u, v) = g_{ij} u^i v^j,$$

where $u = u^i \frac{\partial}{\partial x^i} |_x$ and $v = v^i \frac{\partial}{\partial x^i} |_x$.

Let V be an n -dimensional real vector space and F be a Minkowski norm on V . For a basis $\{e_i\}$ of V , let

$$\sigma_F = \frac{Vol(B^n)}{Vol\{y^i\} \in R^n | F(y^i e_i) < 1},$$

where Vol means the volume of a subset in the standard Euclidean space R^n and B^n is the open ball of radius 1. This quantity is generally depends on the choice of basis e_i . But it is easy to see that

$$\tau(y) = \frac{\ln \sqrt{\det(g_{ij}(y))}}{\sigma_F}, \quad y \in V \setminus \{0\}$$

is independent of the choice of the basis. $\tau = \tau(y)$ is called the distortion of (V, F) . Now let (M, F) be a Finsler space. Let $\tau(x, y)$ be the

distortion of the Minkowski norm F_x on T_xM . For $y \in T_xM - \{0\}$, let $\tau(t)$ be the geodesic with $\tau(0) = x$ and $\dot{\tau}(0) = y$. Then the quantity

$$(2.2) \quad S(x, y) = d/dt[\tau(\sigma(y), \dot{\sigma}(t))] |_{t=0}$$

is called the S-curvature of the Finsler space (M, F) .

Definition 2.1. *A Finsler space (M, F) is said to have almost isotropic S-curvature if there exists a smooth function $c(x)$ on M and a closed 1-form η such that:*

$$S(x, y) = (n + 1)(c(x)F(y) + \eta(y)), \quad x \in M, \quad y \in T_xM$$

If in the above equation $\eta = 0$, then (M, F) is said to have isotropic S-curvature. If $\eta = 0$ and $c(x)$ is a constant, then (M, F) is said to have constant S-curvature.

By [2], in a local coordinate system, the S-curvature of (α, β) -metrics $F = \alpha\phi(s)$ with the underlying Riemannian metric α can be expressed as

$$S = (2\Psi - \frac{f'(b)}{bf(b)})(r_0 + s_0) - \alpha^{-1} \frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0),$$

where $Q = \frac{\phi'}{\phi - s\phi'}$, $\Delta = 1 + sQ + (b^2 + s^2)Q'$, $\Psi = \frac{Q'}{2\Delta}$,

$$\Phi = -(Q - sQ')n\Delta + 1 + sQ - ((b^2 - s^2)(1 + sQ)Q''),$$

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \quad b^i = b_j a^{ji},$$

$$r_j = b^i r_{ij}, \quad s_j = b^i s_{ij}, \quad r_{00} = r_{ij} y^i y^j, \quad s_0 = s_i y^i, \quad r_0 = r_i y^i$$

and the function $f(b)$ in the formula is defined as follows. The Busemann-Hausdorff volume form $dV_{BH} = \sigma_{BH}(x)dx$ is defined by

$$\sigma_{BH}(x) = \frac{\omega_n}{Vol\{(y^i) \in \mathbb{R}^n : F(x, y^i \frac{\partial}{\partial x^i}) < 1\}},$$

and the Holmes-Thomson volume form, $dV_{HT} = \sigma_{HT}(x)dx$ is defined by

$$\sigma_{HT}(x) = \frac{1}{\omega_n} \int_{\{(y^i) \in \mathbb{R}^n : F(x, y^i \frac{\partial}{\partial x^i}) < 1\}} \det(g_{ij}) dy,$$

where Vol denotes the Euclidean volume, $g_{ij} = \frac{\partial^2}{\partial y^i \partial y^j} [F^2]$, and

$$\omega_n = Vol(B^n(1)) = \frac{1}{n} Vol(S^{n-1}) = \frac{1}{n} Vol(S^{n-2}) \int_0^\pi \sin^{n-2}(t) dt.$$

When $F = \sqrt{g_{ij}(x)y^i y^j}$ is a Riemannian metric, both volume forms reduce to the same Riemannian volume form $dV_{HT} = dV_{BH} = \sqrt{\det(g_{ij}(x))} dx$. Now for the (α, β) -metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, $b = \|\beta_x\|_\alpha$, let $dV = dV_{BH}$ or dV_{HT} . Then

$$(2.3) \quad f(b) = \begin{cases} \left\{ \begin{array}{l} \frac{\int_0^\pi \sin^{n-2} t dt}{\int_0^\pi \frac{\sin^{n-2} t dt}{\phi(b \cos t)}}, & \text{if } dV = dV_{BH}; \\ \frac{\int_0^\pi (\sin^{n-2} t) T(b \cos t) dt}{\int_0^\pi \sin^{n-2} t dt}, & \text{if } dV = dV_{HT}. \end{array} \right. \end{cases}$$

where $T(s) = \phi(\phi - s\phi')^{n-2} \{(\phi - s\phi') + (b^2 - s^2)\phi''\}$. Then the volume form dV is given by $dV = f(b)dV_\alpha$, where $dV_\alpha = \sqrt{\det(\alpha_{ij})} dx$, denote the Riemannian volume form of α .

Recall that the group $I(M, F)$ of isometries of a Finsler space (M, F) is a Lie transformation group of M [13]. If $I(M, F)$ acts transitively on M, then (M, F) is called homogeneous. By using Levi-civita connection of $(G/H, \alpha)$, Shaoqing and Wang proved the following:

Theorem 2.1. [13] *Let $F = \alpha\phi(s)$ be a G -invariant (α, β) -metric on the reduction homogeneous manifold G/H with a decomposition of the Lie algebra*

$$g = h + m.$$

Then the S -curvature of F has the form

$$(2.4) \quad S(o, y) = -\frac{1}{\alpha(y)} \frac{\Phi}{2\Delta^2} (-c\langle [u, y]_m, y \rangle - \alpha(y)Q\langle [u, y]_m, u \rangle), \quad y \in m,$$

where u is the vector in m corresponding to the 1-form β , and we have identified m with the tangent space of G/H at origin $o=H$.

Definition 2.2. Let $(G/H, F)$ be a homogeneous (α, β) metric of the form $F = \alpha\phi(s)$, where $s = \beta/\alpha$ with α a G -invariant Riemannian metric on G/H and β a G -invariant vector field on G/H . As pointed out in [13], β corresponds to a unique vector u in $T_o(G/H)$ which is fixed under linear isotropy representation of H on $T_o(G/H)$ and $o = H$ is the origin of G/H .

It is clear that $b = \|\beta_x\|_\alpha$ is a constant. Also, G/H is a reductive homogeneous manifold in the sense of Nomizu[9], i.e, the Lie algebra of G has a decomposition:

$$g = h + m(\text{direct sum of subspaces})$$

such that $Ad(m) \subset m, \forall h \in H$.

3. S-CURVATURE OF HOMOGENEOUS MATSUMOTO METRIC

In this section we find the formula for the homogeneous Matsumoto metric by computing the S -curvature.

For a Matsumoto metric $\phi = \frac{1}{1-s}$, we have the following

$$\begin{aligned} \Theta &= \frac{1-4s}{2(1-3s+2b^2)}, \\ Q &= \frac{1}{1-2s}, \\ \Psi &= \frac{1}{1-3s+2b^2}. \end{aligned}$$

Since $(G/H, F)$ is homogeneous, we only need to compute at the origin $0 = H$. Let $(U, (x^1, x^2, \dots, x^n))$ be the local coordinate system. According to the formula of the S-curvature in local coordinate system ([13]), we need to compute the following quantities at the origin:

i) $r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i})$ and the b_i 's are defined by $\beta = b_i dx^i$. Further, $s_i = b_j s_i^j$ and s_j^i is defined by $s_j^i = a^{ih} s_{hj}$, where $s_{ij} = \frac{1}{2}(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i})$ and (a^{kl}) is the inverse matrix of (a_{ij}) ;

ii) $s_0 = s_i y^i$;

iii) $\rho_0 = \rho_{x^i} y^i$, where $\rho = \ln \sqrt{1 - \|\beta\|^2}$ and $\|\beta\|^2$ is the length of the form β with respect to α .

The quantity of type (iii) is easy. In fact $\rho_{x^i} = 0$ for any i , since β , as an invariant form on G/H , has constant length. Therefore $\rho_0 = 0$.

Next we compute r_{00} and s_0 .

First, since

$$b_i = \beta\left(\frac{\partial}{\partial x^i}\right) = \langle \tilde{u}, \frac{\partial}{\partial x^i} \rangle = c \left\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i} \right\rangle,$$

we have

$$\begin{aligned} \frac{\partial b_i}{\partial x^j} &= c \frac{\partial}{\partial x^j} \left\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i} \right\rangle \\ (3.1) \quad &= c \left(\left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^n} \right\rangle + \left\langle \frac{\partial}{\partial x^n}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \right\rangle \right). \end{aligned}$$

Hence at the origin we have (here we use the symmetry of the connection: $\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = [\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}] = 0$)

$$s_{ij}(0) = \frac{1}{2} c \left(\left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^n} \right\rangle - \left\langle \nabla_{\frac{\partial}{\partial x^n}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle \right) \Big|_0.$$

By ([12]) we have

$$(3.2) \quad s_{ij}(0) = \frac{1}{2} c \langle [u_i, u_j]_m, u_n \rangle.$$

Since at the origin we have $(a_{ij}) = I_n$, we get

$$s_j^i(0) = a^{ik}(0)s_{kj}(0) = \sum_{k=1}^n \delta_{ik}\delta_{kj}(0) = s_{ij}(0).$$

Therefore

$$s_i(0) = b_i(0)s_i^l(0) = cs_i^n(0) = cs_{ni}(0).$$

Thus for $y = y^l u_l \in m$, we have

$$\begin{aligned} s_0(y) &= y^l s_l(0) = cy^l s_{nl}(0) = \frac{1}{2}c^2 y^l \langle [u_n, u_l]_m, u_n \rangle \\ &= \frac{1}{2} \langle [cu_n, y^l u_l]_m, cu_n \rangle \\ (3.3) \quad &= \frac{1}{2} \langle [u, y]_m, u \rangle. \end{aligned}$$

Next we compute r_{ij} . Suppose $i \geq j$. Then we have

$$\begin{aligned} r_{ij}(0) &= \frac{1}{2}(b_{i;j} + b_{i;j})|_0 \\ &= \frac{1}{2} \left(\frac{\partial b_i}{\partial x^j} - b_l \Gamma_{ji}^l + \frac{\partial b_j}{\partial x^i} - b_l \Gamma_{ij}^l \right) |_0 \\ &= \frac{1}{2} \left(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i} \right) |_0 - c \Gamma_{ij}^n(0). \end{aligned}$$

By (3.1) and ([12]) we have

$$(3.4) \quad \frac{1}{2} \left(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i} \right) |_0 = -\frac{1}{2}c \langle [u_i, u_j]_m, u_n \rangle, \quad i \geq j.$$

By combining equation in ([12]) with (3.4) we get

$$(3.5) \quad r_{ij}(0) = -\frac{1}{2}c \left(\langle [u_n, u_j]_m, u_j \rangle + \langle [u_n, u_j]_m \rangle \right), \quad i \geq j.$$

Moreover,

$$(3.6) \quad r_{00}|_0 = r_{ij}(0)y^i y^j = -c \langle [u_n, y]_m, y \rangle.$$

Note that r_{ij} is symmetric with respect to the indices i, j and the right hand of 3.5 is also symmetric with respect to i, j . We conclude

that 3.5 is also valid for $i \geq j$.

At the origin, for a Matsumoto metric direct computation shows that

$$(3.7) \quad S(0, y) = \frac{(n+1)}{\alpha^2(y)} [\Theta - Q\alpha^2(y)\langle [u, y]_m, u \rangle + \alpha(y)c\langle [u_n, y]_m, y \rangle].$$

This is the S-curvature of homogeneous Matsumoto metric. Then we state the following:

Theorem 3.2. *Let $F = \alpha\phi(s)$, where $\phi(s) = \frac{1}{1-s}$ be a G -invariant Matsumoto metric on the reductive homogeneous manifold G/H with a decomposition of the Lie algebra*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

Then the S-curvature of F has the form as in 3.7, where $y \in \mathfrak{m}$, u is the vector in \mathfrak{m} corresponding to the 1-form β , and we have identified \mathfrak{m} with the tangent space of G/H at origin $0 = H$.

4. MEAN BERWALD CURVATURE OF HOMOGENEOUS MATSUMOTO METRIC

In this section we apply the theorem 2.1 and corollary 2.1 to give a formula of mean Berwald curvature of homogeneous Matsumoto metric.

The mean Berwald curvature (E-curvature) is an important non-Riemannian quantity defined by (see [3])

$$E_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{\partial G^m}{\partial y^m} \right),$$

where $G^m = G^m(x, y)$ are the spray coefficients. We know that

$$S = \frac{\partial G^m}{\partial y^m} - (\ln \sigma(x))_{,k} y^k,$$

here $(\ln\sigma(x))_{x^k}$ is the function of x because $\ln\sigma(x)$ is the function of x . hence

$$0 = \frac{\partial^2}{\partial y^i \partial y^j} [(\ln\sigma(x))_{x^k} y^k].$$

This means that

$$\frac{\partial^2 S}{\partial y^i \partial y^j} = \frac{\partial^2}{\partial y^i \partial y^j} \left[\frac{\partial G^m}{\partial y^m} - (\ln\sigma(x))_{x^k} y^k \right] = \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{\partial G^m}{\partial y^m} \right) = 2E_{ij}.$$

Now we compute

$$\frac{\partial^2 S}{\partial y^i \partial y^j}(o, y) = \frac{\partial^2 S(o, y)}{\partial y^i \partial y^j} = 2E_{ij}(o, y)$$

By Theorem 2.1 we've

$$\frac{\partial^2 S(o, y)}{\partial y^i \partial y^j} = \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{c\Phi}{2\Delta^2 \alpha(y)} \langle [u, y]_m, y \rangle \right) + \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{Q\Phi}{2\Delta^2} \langle [u, y]_m, u \rangle \right).$$

Before the computation, we recall that

$$\frac{\partial s}{\partial y^m} = \frac{1}{\alpha} (b_m - s \frac{y_m}{\alpha}), \quad \frac{\partial \alpha}{\partial y_m} = \frac{y_m}{\alpha},$$

where $y_m = a_{mj} y^j$. Since u_1, u_2, \dots, u_m is an orthonormal basis, we have $a_{mj}|_o = \delta_j^m$. Therefore at the origin we have $y_m = y^m$.

Theorem 4.3. *Let $(G/H, F)$ be a homogeneous Matsumoto metric of the form $F = \alpha\phi(s)$ where $s = \beta/\alpha$ with α a G -invariant Riemannian metric G/H and β a G -invariant vector field on G/H . Then the mean Berwald curvature of homogeneous Matsumoto metric is given in 4.2.*

Proof: Now we consider the special case of Matsumoto metric $\phi =$

$$\frac{1}{1-s}.$$

Let $\psi = \phi - s\phi'$. Then we have $Q = \frac{\phi'}{\psi} = \frac{1}{1-2s}$, $Q' = 2/(1-2s^2)$,

$$\psi = \frac{Q'}{2\Delta} = \frac{1}{1-3s+2b^2}, \quad \Delta = 1 + sQ + (b^2 - s^2)Q' = \frac{1-3s+2b^2}{(1-2s)^2},$$

$$\Theta = \frac{Q-sQ'}{2\Delta} = \frac{1-4s}{2(1-3s+2b^2)},$$

$$\begin{aligned}
 \Phi &= -(Q - sQ')\{n\Delta + 1 + sQ\} - (b^2 - s^2)(1 + sQ)Q'' \\
 &= -\frac{1}{(1-2s)^4}\{16s^4 - 32s^3 + 2s^3(6n - 8b^2 + 11) + s[8b^2(6 - n) - 7n - 11] + \\
 &\quad (n + 2) + 2b^2(n - 4)\}.
 \end{aligned}$$

Letting $c = 1$ we get

$$\begin{aligned}
 S(o, y) &= \frac{1}{\alpha(y)} \frac{\Phi}{2\Delta^2} (-c\langle [u, y]_m, y \rangle - \alpha(y)Q\langle [u, y]_m, u \rangle) = -\frac{\Phi}{2\Delta^2\alpha(y)} \langle [u, y]_m, y \rangle - \\
 &\quad \frac{\Phi Q}{2\Delta^2} \langle [u, y]_m, u \rangle,
 \end{aligned}$$

$$\Phi = -\frac{A_1}{(1-2s)^4},$$

$$\begin{aligned}
 \text{where } A_1 &= 16s^4 - 32s^3 + 2s^2(6n - 8b^2 + 11) + s[8b^2(6 - n) - 7n - 11] + \\
 &\quad (n + 2) + 2b^2(n - 4),
 \end{aligned}$$

$$\begin{aligned}
 (4.1) \\
 S(o, y) &= \frac{A_1}{2(1 - 3s + 2b^2)^2} \langle [u, y]_m, y \rangle - \frac{A_1}{2(1 - 3s + 2b^2)^2(1 - 2s)} \langle [u, y]_m, u \rangle.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 E_{ij}(o, y) &= \frac{\partial^2 S(o, y)}{\partial y^i \partial y^j} = \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{A_1}{2(1 - 3s + 2b^2)^2 \alpha(y)} \langle [u, y]_m, y \rangle \right. \\
 &\quad \left. - \frac{A_1}{2(1 - 3s + 2b^2)^2 (1 - 2s)} \langle [u, y]_m, u \rangle \right) \\
 &= \frac{\partial^2}{\partial y^i \partial y^j} \left[\frac{A_1}{2(1 - 3s + 2b^2)^2 \alpha(y)} \langle [u, y]_m, y \rangle \right] \\
 &\quad - \frac{\partial^2}{\partial y^i \partial y^j} \left[\frac{A_1}{2(1 - 3s + 2b^2)^2 (1 - 2s)} \langle [u, y]_m, u \rangle \right].
 \end{aligned}$$

Now,

$$\frac{\partial^2}{\partial y^i \partial y^j} \left[\frac{A_1}{2(1-3s+2b^2)^2\alpha(y)} \langle [u, y]_m, y \rangle \right]$$

$$\begin{aligned}
&= \frac{1}{2} \{ \langle [u, y]_m, y \rangle \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{A_1}{(1-3s+2b^2)^2 \alpha(y)} \right) \right. \\
&+ \frac{A_1}{(1-3s+2b^2)^2 \alpha(y)} \frac{\partial^2 \langle [u, y]_m, y \rangle}{\partial y^i \partial y^j} \\
&+ \frac{\partial \langle [u, y]_m, y \rangle}{\partial y^j} \frac{\partial}{\partial y^i} \left(\frac{A_1}{(1-3s+2b^2)^2 \alpha(y)} \right) \\
&\left. + \frac{\partial \langle [u, y]_m, y \rangle}{\partial y^i} \frac{\partial}{\partial y^j} \left(\frac{A_1}{(1-3s+2b^2)^2 \alpha(y)} \right) \right\},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial \langle [u, y]_m, y \rangle}{\partial y^i} &= \langle [u, u_i]_m, y \rangle + \langle [u, y]_m, u_i \rangle, \\
\frac{\partial \langle [u, y]_m, y \rangle}{\partial y^i \partial y^j} &= \langle [u, u_i], u_j \rangle + \langle [u, u_j]_m, u_i \rangle.
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial}{\partial y^j} \left(\frac{A_1}{(1-3s+2b^2)^2 \alpha(y)} \right) &= \frac{A_1' (b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)^2 \alpha^2(y)} + \frac{6A_1 (b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)^3 \alpha^2(y)} \\
&+ \frac{A_1 y^j}{(1-3s+2b^2)^2 \alpha^3(y)},
\end{aligned}$$

where

$$A_1' = 48s^3 - 96s^2 + 6s(6n - 8b^2 + 11) + 8b^2(6 - n) - 7n - 11.$$

Hence

$$\begin{aligned}
 & \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{A_1}{(1-3s+2b^2)^2 \alpha(y)} \right) \\
 = & \frac{\partial}{\partial y^i} \left(\frac{A_1'(b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)^2 \alpha^2(y)} \right) + 6 \frac{\partial}{\partial y^i} \left(\frac{A_1(b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)^3 \alpha^2(y)} \right) \\
 + & \frac{\partial}{\partial y^i} \left(\frac{A_1 y^j}{(1-3s+2b^2)^2 \alpha^3(y)} \right) \\
 = & \frac{1}{(1-3s+2b^2)^2 \alpha^2(y)} \left[A_1 \delta_i^j - \frac{A_1' s \delta_i^j}{\alpha(y)} + \frac{A_1' s y^i y^j}{\alpha^3(y)} - \frac{3A_1 y^i y^j}{\alpha^2(y)} \right. \\
 + & \frac{(6A_1' + 54A_1)}{(1-3s+2b^2) \alpha(y)} (b_j - s \frac{y^j}{\alpha(y)}) (b_i - s \frac{y^i}{\alpha(y)}) + \frac{6A_1 s y^i y^j}{\alpha^3(y)} - \frac{6A_1 s \delta_j^i}{\alpha(y)} \\
 + & \left(\frac{A_1'' b_j}{\alpha(y)} - \frac{A_1'' s y^j}{\alpha^2(y)} + \frac{6A_1' b_j}{\alpha(y)} - \frac{A_1' y^j}{\alpha^2(y)} \frac{6A_1' s y^j}{\alpha^2(y)} \right) \\
 - & \left. \frac{6A_1 y^j}{\alpha^2(y)} (b_i - s \frac{y^i}{\alpha(y)}) - \left(\frac{2A_1'}{\alpha^2(y)} + \frac{12A_1 y^i}{\alpha^2(y)} \right) (b_j - s \frac{y^j}{\alpha(y)}) \right],
 \end{aligned}$$

where

$$A_1'' = 144s^2 - 192s + 6(6n - 8b^2 + 11).$$

Therefore we have

$$\begin{aligned}
 & \frac{\partial^2}{\partial y^i \partial y^j} \left[\frac{A_1}{2(1-3s+2b^2)} \langle [u, y]_m, y \rangle \right] \\
 = & \frac{A_1}{(1-3s+2b^2)^2 \alpha(y)} (\langle [u, u_i]_m, u_j \rangle + \langle [u, u_j]_m, u_i \rangle) + (\langle [u, u_j]_m, y \rangle + \langle [u, y]_m, u_j \rangle) \\
 + & \left[\frac{A_1'(b_i - s \frac{y^i}{\alpha(y)})}{(1-3s+2b^2)^2 \alpha^2(y)} + \frac{6A_1(b_i - s \frac{y^i}{\alpha(y)})}{(1-3s+2b^2)^2 \alpha^2(y)} - \frac{A_1 y^i}{(1-3s+2b^2)^3 \alpha^3(y)} \right] \\
 + & \left[\frac{A_1'(b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)^2 \alpha^2(y)} + \frac{6A_1(b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)^2 \alpha^2(y)} - \frac{A_1 y^j}{(1-3s+2b^2)^3 \alpha^3(y)} \right]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\langle [u, y]_m, y \rangle}{(1-3s+2b^2)^2 \alpha^2(y)} \left[A_1 \delta_i^j - \frac{A_1' s \delta_i^j}{\alpha(y)} + \frac{A_1' s y^i y^j}{\alpha^3(y)} - \frac{3A_1 y^i y^j}{\alpha^2(y)} \right. \\
& + \frac{(6A_1' + 54A_1)}{(1-3s+2b^2) \alpha(y)} \left(b_j - s \frac{y^j}{\alpha(y)} \right) \left(b_i - s \frac{y^i}{\alpha(y)} \right) + \frac{6A_1 s y^i y^j}{\alpha^3(y)} - \frac{6A_1 s \delta_j^i}{\alpha(y)} \\
& + \left(\frac{A_1'' b_j}{\alpha(y)} - \frac{A_1'' s y^j}{\alpha^2(y)} + \frac{6A_1' b_j}{\alpha(y)} - \frac{A_1' y^j}{\alpha^2(y)} - \frac{6A_1' s y^j}{\alpha^2(y)} \right) \\
& \left. - \frac{6A_1 y^j}{\alpha^2(y)} \left(b_i - s \frac{y^i}{\alpha(y)} \right) - \left(\frac{2A_1'}{\alpha^2(y)} + \frac{12A_1 y^i}{\alpha^2(y)} \right) \left(b_j - s \frac{y^j}{\alpha(y)} \right) \right].
\end{aligned}$$

Note that

$$\begin{aligned}
& \frac{\partial^2}{\partial y^i \partial y^j} \left[-\frac{A_1}{2(1-3s+2b^2)^2(1-2s)} \langle [u, y]_m, u \rangle \right] \\
& = \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{A_1}{(1-3s+2b^2)(1-2s)} \right) \langle [u, y]_m, u \rangle \\
& - \frac{1}{2} \left\{ \frac{A_1}{(1-3s+2b^2)(1-2s)} \frac{\partial^2 \langle [u, y]_m, u \rangle}{\partial y^i \partial y^j} \right. \\
& + \frac{\partial}{\partial y^i} \left(\frac{A_1}{(1-3s+2b^2)(1-2s)} \right) \frac{\partial \langle [u, y]_m, u \rangle}{\partial y^j} + \frac{\partial}{\partial y^j} \left(\frac{A_1}{(1-3s+2b^2)(1-2s)} \right) \\
& \left. \frac{\partial \langle [u, y]_m, u \rangle}{\partial y^i} \right\}.
\end{aligned}$$

where

$$\begin{aligned}
& \frac{\partial \langle [u, y]_m, u \rangle}{\partial y^j} = \langle [u, u_j]_m, u \rangle, \quad \frac{\partial^2 \langle [u, y]_m, u \rangle}{\partial y^i \partial y^j} = \frac{\partial}{\partial y^i} \langle [u, u_j]_m, u \rangle = 0, \\
& \frac{\partial}{\partial y^j} \left(\frac{A_1}{(1-3s+2b^2)(1-2s)} \right) = \\
& \frac{A_1' (b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)(1-2s)\alpha(y)} + \frac{3A_1 (b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)^2(1-2s)\alpha(y)} + \frac{2A_1 (b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)(1-2s)^2\alpha(y)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& \frac{\partial}{\partial y^i} \left(\frac{A_1}{(1-3s+2b^2)(1-2s)} \right) = \\
& \frac{A_1' (b_i - s \frac{y^i}{\alpha(y)})}{(1-3s+2b^2)(1-2s)\alpha(y)} + \frac{3A_1 (b_i - s \frac{y^i}{\alpha(y)})}{(1-3s+2b^2)^2(1-2s)\alpha(y)} + \frac{2A_1 (b_i - s \frac{y^i}{\alpha(y)})}{(1-3s+2b^2)(1-2s)^2\alpha(y)},
\end{aligned}$$

where

$$A_1' = 48s^3 - 96s^2 + 6s(6n - 8b^2 + 11) + 8b^2(6 - n) - 7n - 11.$$

Hence

$$\begin{aligned}
 & \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{A_1}{(1-3s+2b^2)(1-2s)} \right) = \\
 & \frac{\partial}{\partial y^i} \left[\frac{A_1'(b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)(1-2s)\alpha(y)} + \frac{3A_1(b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)^2(1-2s)\alpha(y)} + \frac{2A_1(b_j - s \frac{y^j}{\alpha(y)})}{(1-3s+2b^2)(1-2s)^2\alpha(y)} \right] \\
 & = \frac{1}{(1-3s+2b^2)(1-2s)\alpha^2(y)} \left\{ [A_1''b_j - \frac{(A_1''s + A_1')y^i}{\alpha(y)} + \frac{3A_1'b_j}{(1-3s+2b^2)} \right. \\
 & - \frac{3A_1's}{(1-3s+2b^2)\alpha(y)} - \frac{3A_1y^j}{(1-3s+2b^2)\alpha(y)} + 2A_1'b_j - \frac{2[A_1sy^j + A_1y^j]}{\alpha(y)}] \\
 & (b_i - s \frac{y^i}{\alpha(y)}) - [\frac{A_1'y^i}{\alpha(y)} + \frac{3A_1y^i}{(1-3s+2b^2)\alpha(y)} + \frac{2A_1y^i}{(1-2s)\alpha(y)}] (b_j - s \frac{y^j}{\alpha(y)}) \\
 & + [3A_1' + \frac{2A_1'}{(1-2s)} + \frac{18A_1}{(1-3s+2b^2)^2} \\
 & + \frac{12A_1}{(1-3s+2b^2)(1-2s)} 8A_1] (b_i - s \frac{y^i}{\alpha(y)}) (b_j - s \frac{y^j}{\alpha(y)}) \\
 & \left. + \frac{A_1'sy^i y^j}{\alpha^3(y)} - A_1's\delta_i^j + \frac{3A_1sy^i y^j}{(1-3s+2b^2)\alpha^2(y)} - \frac{2A_1s\delta_i^j}{(1-2s)} + \frac{2A_1sy^i y^j}{(1-2s)\alpha^2(y)} \right\}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \frac{\partial^2}{\partial y^i \partial y^j} \left[-\frac{A_1}{2(1-3s+2b^2)^2(1-2s)} \langle [u, y]_m, u \rangle \right] \\
 & = \frac{\langle [u, u_j]_m, u \rangle}{(1-3s+2b^2)(1-2s)\alpha(y)} \left[A_1' + \frac{3A_1}{(1-3s+2b^2)} + \frac{2A_1}{(1-2s)} \right] (b_i - \frac{sy^i}{\alpha(y)}) \\
 & + \frac{\langle [u, u_i]_m, u \rangle}{(1-3s+2b^2)(1-2s)\alpha(y)} \left[A_1' + \frac{3A_1}{(1-3s+2b^2)} + \frac{2A_1}{(1-2s)} \right] (b_j - \frac{sy^j}{\alpha(y)}) \\
 & + \frac{\langle [u, y]_m, u \rangle}{(1-3s+2b^2)(1-2s)} \left[A_1''b_j - \frac{(A_1''s + A_1')y^i}{\alpha(y)} + \frac{3A_1'b_j}{(1-3s+2b^2)} \right. \\
 & - \frac{3A_1's}{(1-3s+2b^2)\alpha(y)} \\
 & \left. - \frac{3A_1y^j}{(1-3s+2b^2)\alpha(y)} + 2A_1'b_j - \frac{2[A_1sy^j + A_1y^j]}{\alpha(y)} \right] (b_i - s \frac{y^i}{\alpha(y)})
 \end{aligned}$$

$$\begin{aligned}
& - \left[\frac{A'_1 y^i}{\alpha(y)} + \frac{3A_1 y^i}{(1-3s+2b^2)\alpha(y)} \right. \\
& + \left. \frac{2A_1 y^i}{(1-2s)\alpha(y)} \right] (b_j - s \frac{y^j}{\alpha(y)}) + [3A'_1 + \frac{2A'_1}{(1-2s)} + \frac{18A_1}{(1-3s+2b^2)^2} \\
& + \frac{12A_1}{(1-3s+2b^2)(1-2s)} + 8A_1] (b_i - s \frac{y^i}{\alpha(y)}) (b_j - s \frac{y^j}{\alpha(y)}) \\
& + \left. \frac{A'_1 s y^i y^j}{\alpha^3(y)} - A'_1 s \delta_i^j + \frac{3A_1 s y^i y^j}{(1-3s+2b^2)\alpha^2(y)} - \frac{2A_1 s \delta_i^j}{(1-2s)} + \frac{2A_1 s y^i y^j}{(1-2s)\alpha^2(y)} \right].
\end{aligned}$$

Finally, by direct computation we have

$$\begin{aligned}
2E_{ij}(o, y) &= \frac{\partial^2 S(o, y)}{\partial y^i \partial y^j} = \frac{\partial^2}{\partial y^i \partial y^j} \left[\frac{A_1}{2(1-3s+2b^2)^2 \alpha(y)} \langle [u, y]_m, y \rangle \right] \\
&\quad - \frac{\partial^2}{\partial y^i \partial y^j} \left[- \frac{A_1}{2(1-3s+2b^2)^2 (1-2s)} \langle [u, y]_m, u \rangle \right] \\
&= \frac{1}{(1-3s+2b^2)^2 \alpha^2(y)} \left\{ [C_1 (b_i - \frac{s y^i}{\alpha(y)}) - C_2 (b_j - \frac{s y^j}{\alpha(y)}) \right. \\
&+ C_3 (b_i - \frac{s y^i}{\alpha(y)}) (b_j - \frac{s y^j}{\alpha(y)}) \\
&+ C_4 \langle [u, y]_m, y \rangle + C_5 (\langle [u, u_j]_m, y \rangle + \langle [u, y]_m, u_j \rangle) \\
&+ C_6 (\langle [u, u_i]_m, u_j \rangle + \langle [u, u_j]_m, u_i \rangle)] \left. \right\} \\
&+ \frac{1}{(1-3s+2b^2)(1-2s)\alpha^2(y)} \left\{ [C_1^* (b_i - \frac{s y^i}{\alpha(y)}) - C_2^* (b_j - \frac{s y^j}{\alpha(y)}) \right. \\
&+ C_3^* (b_i - \frac{s y^i}{\alpha(y)}) (b_j - \frac{s y^j}{\alpha(y)}) \\
&+ C_4^* \langle [u, y]_m, u \rangle + C_5^* (\langle [u, u_j]_m, y \rangle (b_i - \frac{s y^i}{\alpha(y)}) + \langle [u, u_i]_m, y \rangle (b_j - \frac{s y^j}{\alpha(y)}))] \left. \right\},
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \frac{1}{\alpha(y)} [(A''_1 b_j + 6A'_1 b_j + A'_1 y^j) - \frac{(A'_1 s + A'_1 + 6A'_1 s + 6A') y^j}{\alpha(y)}], \\
C_2 &= \frac{(2A'_1 + 12A_1 y^i)}{\alpha^2(y)},
\end{aligned}$$

$$\begin{aligned}
 C_3 &= \frac{(6A'_1 + 54A_1)}{(1-3s+2b^2)\alpha(y)}, \\
 C_4 &= A_1\delta_i^j - [(A'_1 - 6A_1)s\delta_i^j - \frac{A'_1sy^iy^j}{\alpha(y)} + \frac{3A_1y^iy^j}{\alpha^2(y)} - \frac{6A_1y^isy^j}{\alpha^3(y)}] \frac{1}{\alpha(y)}, \\
 C_5 &= (A'_1 + 6A_1)(b_i - \frac{sy^i}{\alpha(y)}) - \frac{A_1y^i}{\alpha(y)}, \\
 C_6 &= (A'_1 + 6A_1)(b_j - \frac{sy^j}{\alpha(y)}) - \frac{A_1y^j}{\alpha(y)}, \\
 C_1^* &= [(A''_1 + 2A'_1)b_j - \frac{(A'_1s + A'_1 + 2A_1s + A_1)y^j}{\alpha(y)} + \frac{1}{(1-3s+2b^2)\alpha(y)}(3A'_1b_j\alpha(y) - \\
 & 3A'_1s - 3A_1y^j)], \\
 C_2^* &= \frac{y^i}{\alpha(y)}[A'_1 + \frac{3A_1}{(1-3s+2b^2)} + \frac{2A_1}{(1-2s)}], \\
 C_3^* &= 5A'_1 + 8A_1 + \frac{18A_1}{(1-3s+2b^2)^2} + \frac{12A_1}{(1-3s+2b^2)(1-2s)}, \\
 C_4^* &= \frac{A'_1sy^iy^j}{\alpha^3(y)} - A'_1s\delta_i^j + \frac{3A_1sy^iy^j}{(1-3s+2b^2)\alpha^2(y)} - \frac{2A_1s}{(1-2s)}(\delta_i^j + \frac{y^iy^j}{\alpha^2(y)}), \\
 C_5^* &= [A'_1 + \frac{3A_1}{(1-3s+2b^2)} + \frac{2A_1}{(1-2s)}]\alpha(y).
 \end{aligned}$$

5. CONCLUSION

It is an important problem to compute the geometric quantities such as curvature properties of homogeneous spaces. In particular, J.Milnor used the formula of the sectional curvature of left invariant Riemannian metric on a Lie group to study the curvature properties of such spaces. The formula of sectional curvature of a homogeneous Riemannian manifolds with negative or positive curvatures.

In this paper, we first present the formula for the S-curvature of a homogeneous Matsumoto metric. We know, the notion of S-curvature of a Finsler space was introduced by Z.Shen in [15]. It is a quantity to measure the rate of change of the volume form of Finsler space along the geodesics and it is a non-Riemannian quantity. Further, using the formula for the S-curvature, we find the formula for mean Berwald curvature of homogeneous Matsumoto metric.

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DEPARTMENT OF P.G. STUDIES AND RESEARCH IN MATHEMATICS, KUVEMPU
UNIVERSITY, SHANKARAGHATTA - 577451, SHIMOGA, KARNATAKA, INDIA