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FIXED POINT THEOREMS FOR GENERAL CONTRACTIVE MULTIVALUED MAPPINGS

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Abstract. We prove the existence of common fixed point for Multivalued maps satisfying general contractive type conditions.

1. Introduction

Banach fixed point theorem plays an important role in several branches of mathematics. For instance, it has been used to show the existence of solutions of nonlinear Volterra integral equations, nonlinear integro-differential equations in Banach spaces and to show the convergence of algorithms in computational mathematics. The generalizations to multivalued case are enormous. One of these generalizations is the Nadler's theorem [4] as follows.

Theorem 1.1. Let (X, d) be a complete metric space and T be a multivalued map on X with Tx is nonempty closed bounded subset of X for each $x \in X$. If there exists $c \in [0, 1)$ such that

$$H(Tx, Ty) \leq c\ d(x, y)$$

then T has a fixed point in X.

In [1], the authors proved the next Theorem 1.2 which is a generalization of Theorem 1.1.

Theorem 1.2. Let (X, d) be a complete metric space and T be a multivalued map on X with Tx is nonempty closed bounded subset of X for each $x \in X$. If there exists $c \in [0, 1)$ such that for each $x, y \in X$

$$H(Tx, Ty) \leq c\ \max\left\{d(x, y), d(x, Tx,), d(y, Ty), \frac{1}{2}\left\{d(x, Ty) + d(y, Tx)\right\}\right\},$$

then T has a fixed point in X provided that $x \to d(x, Tx)$ is lower semicontinuous.

In particular, in [2] the authors gave a generalization of Theorem 1.1.

In this paper, we give a generalization of Theorem 1.2 without the condition of which $x \to d(x, Tx)$ is lower semicontinuous. This will be done in Theorem 2.1. In [8], the authors proved the next theorem 1.3 which is a positive response of the conjecture

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proposed by Reich[5]. In [3], the authors replaced the condition $\lim_{r\to t^+} k(r) < 1$ for all $0 < t < \infty$ by $\lim_{r\to t^+} k(r) < 1$ for all $0 \le t < \infty$, where $k : [0,\infty) \to [0,1)$. In [1], the authors gave an alternative proof of theorem 5 of [3].

Theorem 1.3. Let (X, d) be a complete metric space and T be a multivalued map on X with Tx is nonempty closed bounded subset of X for each $x \in X$ and if for each $x, y \in X$

$$H(Tx,\ Ty) \le k\ \left(d(x,\ y)\right) d(x,\ y),$$

where $k : (0,\infty) \to [0,1)$ is a function such that $\lim \sup_{r\to t^+} k(r) < 1$ for every $t \in [0,\infty)$, then T has a fixed point in X.

In this paper, we give a generalization of theorem 5 of [3] and theorem 2.1 of [1].

Let (X, d) be a metric space. We denote by $CB(X)$ the family of nonempty closed bounded subsets of X. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $CB(X)$. That is, for $A,B \in CB(X)$,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,\ b)\right\}$$

where $d(a,B) = \inf \left\{d(a,\ b) : b \in B\right\}$ is the distance from the point a to the subset B.

Lemma 1.4[4]. Let (X, d) be a metric space and let $A,B \in CB(X)$ with $H(A,B) < \varepsilon$. Then for each $a \in A$, there exists a point $b \in B$ such that $d(a,b) < \varepsilon$.

From now on, let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing function such that

(φ_1) $\phi(0) = 0$,

(φ_2) $0<\phi(t) < t$ for each $t > 0$,

(φ_1) $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t \in (0,\infty)$.

2. Main Result

Theorem 2.1. Let (X, d) be a complete metric space and $T, S : X \to CB(X)$ be multivalued maps satisfying for each $x, y \in X$,

$$H(Sx,\ Ty) \le \phi\left(\max\left\{\frac{d(x,\ y)+d(x,\ Sx)}{2},\ d(y,\ Ty),\frac{d(x,\ \mathrm{T}y)+d(y,\ Sx)}{2}\right\}\right) \tag{2.1.1}$$

Then T and S have a common fixed point in X. That is, there exists a point $p \in X$ such that $p \in Tp \cap Sp$.

Proof. Let $x_0 \in X$ and $x_1 \in Sx_0$. Let $c \in X$ be such that $\phi(\mathrm{d}(\mathrm{x}_0,\ \mathrm{x}_1)) < \phi(\mathrm{c})$.

We have

$$d(x_1,Tx_1) \le H\left(Sx_0,\ Tx_1\right)$$
$$\le \phi\left(\max\left\{\frac{d\left(x_0,x_1\right)+d\left(x_0,\ Sx_0\right)}{2},\ d\left(x_1,\ Tx_1\right),\frac{d\left(x_0,\ \mathrm{T}x_1\right)+d\left(x_1,\ Sx_0\right)}{2}\right\}\right)$$
$$\le \phi\left(\max\left\{\frac{d\left(x_0,x_1\right)+d\left(x_0,\ x_1\right)}{2},\ d\left(x_1,\ Tx_1\right),\frac{d\left(x_0,\ \mathrm{T}x_1\right)+d\left(x_1,\ x_1\right)}{2}\right\}\right) \tag{2.1.2}$$

If $d\left(x_0,\ x_1\right) \le d\left(x_1,\ Tx_1\right)$, then $d\left(x_1,\ \mathrm{T}x_1\right) \le \phi\left(d\left(x_1,\ Tx_1\right)\right)$ which implies $d\left(x_1,\ \mathrm{T}x_1\right) = 0$ and so $d\left(x_0,\ x_1\right) = 0$. Thus we have $x_1 \in Tx_1 \cap Sx_1$, That is, T and S have a common fixed point in X.

Suppose that $d\left(x_1,\ \mathrm{T}x_1\right) \le d\left(x_0,\ x_1\right)$. From (2.1.2) we have

$$d\left(x_1,\ \mathrm{T}x_1\right) \le \phi\left(d\left(x_0,\ x_1\right)\right) < \phi(c)$$

We can take $x_1 \in Tx_1$ such that $d\left(x_1,\ x_2\right) < \phi(c)$.

Similarly, we have

$$d(x_2,Sx_2) \le H\left(Sx_2,\ Tx_1\right)$$
$$\le \phi\left(\max\left\{\frac{d\left(x_1,x_2\right)+d\left(x_2,\ Sx_2\right)}{2},\ d\left(x_1,\ Tx_1\right),\frac{d\left(x_2,\ \mathrm{T}x_1\right)+d\left(x_1,\ Sx_2\right)}{2}\right\}\right)$$
$$\le \phi\left(\max\left\{\frac{d\left(x_1,x_2\right)+d\left(x_2,\ Sx_2\right)}{2},\ d\left(x_1,\ x_2\right),\frac{d\left(x_2,\ x_2\right)+d\left(x_1,\ Sx_2\right)}{2}\right\}\right)$$
$$\le \phi\left(d(x_1,x_2)\right)$$
$$< \phi^2\left(c\right)$$

We can take $x_3 \in Sx_2$ such that $d(x_2,x_3) < \phi^2(c)$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1} \in Sx_{2n} \in Tx_{2n+1}, d(x_{n,}x_{n+1}) < \phi^n(c)$$

Thus we have $\sum_{n=0}^{\infty} d(x_{n,}x_{n+1}) < \sum_{n=0}^{\infty}\phi^n(c) < \infty.$ Therefore , $\{x_n\}$ is a Cauchy sequence in X. and let $\lim_{n\to\infty} x_n = p.$ From (2.1.1) we have

$$d(x_{2n+1},Tp) \le H\left(Sx_{2n},\ Tp\right)$$
$$\le \phi\left(\max\left\{\frac{d\left(x_{2n},p\right)+d\left(x_{2n},\ Sx_{2n}\right)}{2},\ d\left(p,\ Tp\right),\frac{d\left(x_{2n},\ Tp\right)+d\left(p,\ Sx_{2n}\right)}{2}\right\}\right)$$

$$\leq \phi\left(\max\left\{\frac{d\left(x_{2n}, p\right)+d\left(x_{2n}, x_{2n+1}\right)}{2}, d\left(p, Tp\right), \frac{d\left(x_{2n}, Tp\right)+d\left(p, x_{2n+1}\right)}{2}\right\}\right)$$

(2.1.3)

Letting $n\to\infty$ in (2.1.3) we have $d(p,Tp)\leq\phi\left(d(p,Tp)\right)$. Thus $d(p,Tp)=0$, or $p\in Tp$.

Similarly, we can show $p\in Sp$. Therefore, we have $p\in Tp\cap Sp$.

Corollary 2.1. Let (X, d) be a complete metric space and $T: X\to CB(X)$ be multivalued maps satisfying for each $x, y\in X$,

$$H\left(Tx, Ty\right)\leq\phi\left(\max\left\{\frac{d\left(x, y\right)+d\left(x, Tx\right)}{2}, d\left(y, Ty\right), \frac{d\left(x, Ty\right)+d\left(y, Tx\right)}{2}\right\}\right)$$

Then T has a fixed point in X.

If we have $\emptyset(t) = kt$, $k\in[0, 1)$ in Theorem 2.1 (Corollary 2.1), then the conclusion of Theorem 2.1 (Corollary 2.1) is satisfied.

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