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ON SOME RESULTS ON LINEAR ORTHOGONALITY SPACES

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ABSTRACT. The existence of an inner product provides a means of introducing the notion of orthogonality in normed linear spaces. The most natural definition of orthogonality is: "$x \perp y$, if and only if the inner product is zero." This paper introduces different notions of orthogonality in normed linear spaces and shows that they are all equivalent in real Hilbert spaces. Also, some characterizations of linear orthogonality spaces are given.

1. Introduction

The following are some definitions of orthogonality in normed linear spaces: Let E be a normed linear space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. x is orthogonal to y in E.

(a) in the sense of Birkhoff ([1, 12]) , if for every $\lambda \in \mathbb{K}$.

$$\|x + \lambda y\| \geq \|x\|;$$

(b) in the sense of Roberts ([10, 15]), if for every $\lambda \in \mathbb{K}$

$$\|x + \lambda y\| = \|x - \lambda y\|;$$

(c) in the isosceles sense [10], if and only if

$$\|x + y\| = \|x - y\|;$$

(d) in the Pythagorean sense [10], if and only if

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2;$$

(e) in the sense of Singer [20], if and only if

$$\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|;$$

(f) in the sense of Lumer-Giles (relative to semi-inner product [.,.]) [6, 14, 5], iff $[y, x] = 0$;

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(g) in the sense of Saidi [17, 18, 19], if for every $a, b \in \mathbb{K}$

$$\|ax + by\| = \||a|x + |b|y\|.$$

The concept of orthogonality has certain desired properties, the most important being that every two dimensional subspace contains nonzero orthogonal elements [2 and 7].

In [10], James showed that isosceles and Pythagorean orthogonality are equivalent in real Hilbert spaces and are also symmetric, additive and homogeneous. However, they are not additive and homogeneous in general normed linear spaces.

In [14], Lumer introduced a form which is a generalization of the inner product. If a vector spaces on which a form $[x, y]$ which is linear in one component only, strictly positive and satisfies schwartz inequality is defined, then such a form induces a norm by setting $\|x\| = [x, x]^{1/2}$ and it is called a semi-inner product and every normed linear space can be made into a semi-inner product space. Let E be a vector space over $\mathbb{C}$, the complex field. Let $[.,.]$ be a semi-inner product on E. The pair $(E, [.,.])$ is called a semi-inner product space. For any $x, y \in E$, x and y are said to be orthogonal in the sense of Lumer-Giles and written $x \perp_{LG} y$, if and only if $[y, x] = 0$. A continuous semi-inner product (s.i.p) space is a semi-inner product space where the s.i.p. has the additional property: for every $x, y \in S$ and $\lambda \in \mathbb{R}$

$$\lim_{\lambda \to 0} Re\{[y, x + \lambda y]\} \longrightarrow Re\{[y, x]\},$$

where $S := \{x \in E : ||x|| = 1\}$
In a continuous semi-inner product space, Lumer-Giles orthogonality is equivalent to Birkhoff's orthogonality. It is known that semi-inner product is not commutative, Lumer-Giles orthogonality relation $(\perp_{LG})$, is not symmetric. However, it follows from a property of semi-inner product space that the relation is additive, i.e., if $x \perp_{LG} y$ and $x \perp_{LG} z$ then $x \perp_{LG} \lambda y + \mu z$ for all complex λ, μ [3, 5, 6].
Ivan Singer gave the following characterization of best approximants in terms of Birkhoff's orthogonality: Let $(E, \| \cdot \|)$ be a normed linear space, G a linear subspace of E, $x \in E \backslash G$ and $g_0 \in G$. Then $g_0 \in P_G(x)$ if and only if $x - g_0 \perp_B G$ [21]; where

$$P_G(x) := \left\{ g_0 \in G : \|x - g_0\| = \inf_{g \in G} \|x - g\| \right\}.$$

Dragomir and Koliha [5] also gave a new characterization of Birkhoff's orthogonality which was used to present an application of Birkhoff's orthogonbality in the theory of best approximation in normed linear spaces: Let $(E, \| \cdot \|)$ be a normed linear space, and x, y are two vectors

in E. Then $x\perp_{B}y$ if and only if $x\perp_{LG}y$ relative to some semi-inner product $[.,.]$ which generates the norm $\|\cdot\|$.
Let $(E,\|\cdot\|)$ be a normed linear space, G a linear subspace of E, $x\in E\backslash\bar{G}$ and $g_0\in G$. Then, the following statements are equivalent
(i) $g_0\in P_G(x)$; (ii) For some semi-inner product [.,.] generating the norm of E, g_0 and x satisfy the inequality $[x-g_0, x-g_0+w]\le \|x-g_0+w\|^2$ for all $w\in G$.

In [18], Saidi also gave another extension of the notion of orthogonality in Hilbert spaces to general Banach spaces. Let H be a Hilbert space. A vector x is orthogonal to y in H if and only if

$$\|x+\lambda_1 y\| = \|x+\lambda_2 y\|, \tag{I}$$

for all $\lambda_1,\lambda_2\in \mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $|\lambda_1|=|\lambda_2|$.
Equation (I) is equivalent to

$$\|\lambda x+\mu y\| = \||\lambda|x+|\mu|y\|, \tag{II}$$

for all $\lambda,\mu\in\mathbb{K}$ in any Banach space.
Using equation (II), Saidi showed that given a finite or infinite sequence $(x_n)_{n\in L}$ in a Banach space E, then $(x_n)_{n\in L}$ is orthogonal if and only if

$$\left\|\sum_{n\in L} a_n x_n\right\| = \left\|\sum_{n\in L} |a_n| x_n\right\|, \quad \text{for each} \quad \sum_{n\in L} a_n x_n \in E,$$

where L:= {1, 2, . . . , N}, for some natural number $\mathbb{N}$ or L. As a result of Saidi's orthogonality, there is an establishment of a characterization of orthogonality in Banach spaces $\ell^p_S(\mathbb{C}), 1\le p<\infty$, where S is a set of positive integers and $\mathbb{C}$ is the field of complex numbers, and also generalizations of the usual characterization of orthogonality in the Hilbert spaces $\ell^2_S(\mathbb{C})$, via inner products. For example, two elements x_1 and x_2 in $\ell^p_2(\mathbb{C}), 2<p<\infty$, are orthogonal if and only if they have disjoint supports. If $p\in[1,\infty)$ is not an even integer, then $(a_j)_{j\in S}\perp(b_j)_{j\in S}$ in $\ell^p_S(\mathbb{C})$ if and only if, for every real number $r>0$, and for all integers $m\ge 1$

$$\sum_{j\in J_r} |b_j|^p \left(\frac{a_j}{b_j}\right)^m = 0,$$

Or

$$\sum_{j\in J_r} |a_j|^p \left(\frac{b_j}{a_j}\right)^m = 0,$$

Also, if p is a positive even integer, then $(a_j)_{j\in J}\perp(b_j)_{j\in S}$ in $\ell^p_S(\mathbb{C})$ if and only if

$$\sum_{j\in J}|b_j|^p\left|\frac{a_j}{b_j}\right|^{2k}\left(\frac{a_j}{b_j}\right)^m=0,$$

for all integers $m,k,1\leq m\leq p/2,0\leq k\leq(p/2)-m$. Or

$$\sum_{j\in J}|a_j|^p\left|\frac{b_j}{a_j}\right|^{2k}\left(\frac{b_j}{a_j}\right)^m=0,$$

for all integers $m,k,1\leq m\leq p/2,0\leq k\leq(p/2)-m$, where

$$\begin{aligned}J: &= supp(a_j)\cap supp(b_j):=\{j\in S:a_jb_j\neq 0\};\\ J_r: &= \left\{j\in J:\left|\frac{b_j}{a_j}\right|=r\right\}.\end{aligned}$$

The characterization of compact operators in Hilbert spaces was extended to any Banach space that admits an orthonormal schauder basis: suppose that $\{e_n)_{n=1}^{\infty}$ is an orthonormal schauder basis of the Banach space E and that F is a normed space. For each positive integer k, let P_k be the projection on $[e_n:1\leq n\leq k]$ defined by

$$P_k\left(\sum_{n=1}^{\infty}\alpha_ne_n\right)=\sum_{n=1}^{k}\alpha_ne_n,\quad\sum_{n=1}^{\infty}\alpha_ne_n\in E.$$

Then an operator $T\in L(F,E)$ is compact if and only if $P_k\circ T$ converges to T in L(F,E) [18, Theorem 5].

If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in a Hilbert space H and if T is the operator on E defiend by

$$T(x):=\sum_{n=1}^{\infty}\lambda_n(e_n^*,x)e_n\quad\text{for all}\quad x\in E$$

where e_n^* is the coefficient functional in $[e_k:k\geq 1]^*$ associated with e_n, then T is compact if and only if $\lim_{n\to\infty}\lambda_n=0$. Its extension to general Banach space is as follows: If $\{e_n\}_{n=1}^{\infty}$ is an orthonormal sequence in a Banach space E, then the operator T defined as above is compact if and only if $\lim_{n\to\infty}\lambda_n=0$ [18, Corollary 3.]

Also, there exists comparison between isosceles and Birkhoff orthogonality: If x and y in E are orthogonal (in isosceles sense) i.e. $\|x+y\|=\|x-y\|$, then $\|x+\lambda y\|\geq\|x\|$ for $|\lambda|\geq 1$. Some standard text containing relevant material can be sourced from [4,8,9,11,13 and 16].

2. Linear Orthogonality Space

2.1 Definition (Linear orthogonality space). A pair $(E, \perp)$ is a linear orthogonality space provided E is a real linear space with $\dim E \geq 2$ and $\perp \subset E^2$ is a relation such that

($L01$) $x \perp y$ if and only if $y \perp x$ and $x \perp y$ for every $x \in E$ implies $y = 0$, or $y \perp x$ for all $y \in E$ implies $x = 0$

($L02$) if $x, y \in E \backslash \{0\}$ and $x \perp y$, then x and y are linearly independent,

($L03$) if $x, y, z \in E$, $x \perp y$ and $x \perp z$ imply $x \perp y + z$

($L04$) if $x, y \in E$ and $x \perp y$, then $ax \perp by$ for every $a, b \in \mathbb{R}$,

($L05$) if P is a 2-dimensional subspace of E, $x \in P$ and $a \in \mathbb{R}^+$, then there exists $y \in P$ with $x \perp y$ and $x + y \perp ax - y$.

Any linear space can be made into a linear orthogonality space if we define $x \perp 0$, $0 \perp x$ for all x, and for non-zero vectors x, y define $x \perp y$ iff x, y are linearly independent. An inner product space is a linear orthogonality space.

2.2 EXAMPLES.

2.2.1 Euclidean Space R^n. The space $\mathbb{R}^n$ is a linear orthogonality space with inner product defined by

$$< x, y >= \sum_{i=1}^{n} x_i y_i$$

where

$$x = (x_i) = \{x_1, x_2, x_3 ..., x_n\}$$

and

$$y = (y_i) = \{y_1, y_2, y_3 ..., y_n\}$$

(i) Let $x, y \in \mathbb{R}^n$, then
$x \perp y \Rightarrow < x, y >= \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i =< y, x >= 0$
$\Rightarrow y \perp x$

(ii) Let $x, y \in \mathbb{R}^n - \{0\}$ and $x \perp y$, then for $\alpha_i \in \mathbb{R}^+$
$< \alpha x, \alpha y >= \sum_{i=1}^{n} |\alpha_i|^2 x_i y_i = 0 \Rightarrow |\alpha_i|^2 = 0$
since $x, y \in \mathbb{R}^n - \{0\}$. This implies $\alpha_i = 0 \Rightarrow x$ and y are linearly independent.

2.2.2 Unitary Space C^n. The space $\mathbb{C}^n$ is a linear orthogonality space with inner product defined by

$$< x, y >= \sum_{i=1}^{n} x_i \bar{y}_i$$

where

$$x = \{x_1, x_2, x_3 ..., x_n\}$$

and

$$y = \{y_1, y_2, y_3..., y_n\}$$

(i) Let $x, y \in \mathbb{C}^n$, then
$x \perp y \Rightarrow < x, y >= \sum_{i=1}^{n} x_i \bar{y}_i = \sum_{i=1}^{n} y_i \bar{x}_i =< y, x >= 0$
$\Rightarrow y \perp x$

(ii) Let $x, y \in \mathbb{C}^n - \{0\}$ and $x \perp y$, then for $\alpha_i \in \mathbb{R} - \{0\}$
$< \alpha x, \alpha y >= \sum_{i=1}^{n} \alpha_i x_i \overline{\alpha_i y_i} \Rightarrow \sum_i^n \alpha_i \bar{\alpha}_i x_i \bar{y}_i = \sum_i^n |\alpha_i|^2 x_i \bar{y}_i = 0$
$\Rightarrow |\alpha_i|^2 = 0$ which implies $\alpha_i = 0$.
Therefore x and y are linearly independent.

(iii) $x \perp y \Rightarrow ax \perp by$ for $x, y \in \mathbb{C}^n - \{0\}$ and $a, b \in \mathbb{R} - \{0\}$,
Then
$< ax, by >= \sum_i^n a_i \bar{b}_i x_i \bar{y}_i = a\bar{b} < x, y >= 0$
Therefore, $ax \perp by$.

2.3 Definition (Orthogonally Additive Functions). A real-valued function $f : E \longrightarrow \mathbb{R}$ on an inner product space E is orthogonally additive if and only if $f(x + y) = f(x) + f(y)$ whenever $x \perp y \; \forall \; x, y \in E$.

3. Main Results

We show by using the following theorem that the above notions of Orthogonality are all equivalent in real Hilbert space.

3.1 Theorem. Let $(E, (.,.))$ be a real Hilbert space and $x, y \in E$.
$< x, y >= 0$ if and only if
(i) $\|x - y\| = \|x + y\|$,
(ii) $\|x - y\|^2 = \|x\|^2 + \|y\|^2$,
(iii) $\|x + \lambda y\| \geq \|x\|$, for all $\lambda \in \mathbb{R}$
(iv) $\|x + \lambda y\| = \|x - \lambda y\|$, for all $\lambda \in \mathbb{R}$.
Whenever $< x, y >=< y, x >= 0$

Proof: Consider (i). Suppose $< x, y >=< y, x >= 0$ and since $x, y \in E$, then

$$\begin{aligned} \|x - y\|^2 &= |x\|^2 + \|y\|^2 \text{ and} \\ \|x + y\|^2 &= \|x\|^2 + \|y\|^2 \\ \Rightarrow \|x - y\|^2 &= \|x + y\|^2 \\ \Rightarrow \|x - y\| &= \|x + y\| \end{aligned}$$

Conversely, suppose $\|x - y\| = \|x + y\|$, then

$$\begin{aligned} \|x + y\|^2 - \|x - y\|^2 &= 0. \\ \Rightarrow \|x + y\|^2 - \|x - y\|^2 &= 2 < x, y > +2 < y, x >= 0 \\ &= 4 < x, y >= 0 \end{aligned}$$

Therefore, $< x, y >= 0$.

Consider (ii) and suppose $< x, y >=< y, x >= 0$, it follows from (i) that

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

Conversely, suppose $\|x - y\|^2 = \|x\|^2 + \|y\|^2$, then

$$\begin{aligned} \|x - y\|^2 - \|x\|^2 - \|y\|^2 &= -2 < x, y >= 0. \\ \Rightarrow \quad < x, y > &= 0. \end{aligned}$$

Consider (iii) and suppose $< x, y >=< y, x >= 0$, then

$$\|x + \lambda y\|^2 = \|x\|^2 + \lambda^2 \|y\|^2$$

Therefore,

$$\|x + \lambda y\| \geq \|x\| \qquad \text{for all} \quad \lambda \in \mathbb{R}.$$

Conversely, suppose $\|x + \lambda y\| \geq \|x\|$, for all $\lambda \in \mathbb{R}$ then

$$\begin{aligned} \|x + \lambda y\|^2 - \|x\|^2 &\geq 0. \\ \Rightarrow \lambda^2 \|y\|^2 + 2\lambda < x, y > &\geq 0. \end{aligned}$$

Let $\quad \lambda = \dfrac{\alpha < x, y >}{< y, y >}, \quad$ then

$$\left(\alpha^2 + 2\alpha\right) \frac{< x, y >}{< y, y >} \geq 0 \text{ and } y \neq 0.$$

Setting $\beta > 1$ and $\alpha = (-\beta)^{-1}$, then, $(\alpha^2 + 2\alpha)$ will always be negative. But, $\dfrac{< x, y >}{< y, y >}$ cannot be negative.

Thus, $(\alpha^2 + 2\alpha) \frac{<x,y>}{<y,y>} \geq 0$ is therefore a contradiction.

Therefore, $\quad < x, y >= 0$.

Consider (iv). The result follows from (i).

Note:- The notion of equality would fail if $x, y \in E$ and $(E, < ., . >)$ is a complex Hilbert space.

3.2 Lemma. Let $(E, \perp)$ be a linear orthogonality normed space and let $f : E \longrightarrow \mathbb{R}$ such that $f(x) =< x, x >$ is orthogonally additive, if

(a) f is odd, then f is linear

(b) f is even, then $f(\alpha x) = \alpha^2 f(x)$ for all $\alpha \in \mathbb{R}$ and if $x, y \in E$, then $f(x) = f(y)$.

Proof

(a) $f(x) = ||x||^2$

Since $f : E \longrightarrow \mathbb{R}$ is orthogonally additive,

if f is odd, then $\forall x, y \in E$,

$f(-(x + y)) = -f(x + y) = -||x + y||^2$

$-||x+y||^2 = -(f(x)+f(y))$
$= -(||x||^2 + ||y||^2)$
$\Rightarrow \quad ||x+y||^2 = ||x||^2 + ||y||^2$
Therefore,
$f(x+y) = f(x) + f(y)$
(b) If f is even, then for all $\alpha \in \mathbb{R}$, $x \in E$,

$$f(\alpha x) = f(-\alpha x) = ||-\alpha x||^2 = |\alpha|^2 ||x|| = \alpha^2 f(x)$$

whenever P is 2-dimensional subspace of E and $x, y \in P$, then by (LO5), $x+y \perp x-y$, we have,

$$f((x+y)+(x-y)) = f(x+y)+f(x-y)$$

$$= f(x) + f(y) + f(x) + f(-y)$$

$$= f(x) + f(y) + f(x) + f(y). \quad (since \quad f \quad is \quad even)$$

Therefore,

$$f(2x) = 2f(x) + 2f(y)$$

$$4f(x) = 2f(x) + 2f(y)$$

$$2f(x) = 2f(y)$$

$$\Rightarrow f(x) = f(y)$$

3.3 Theorem. Let $(E, \perp)$ be a linear orthogonality normed space and E a Hilbert space. Let $f : E \longrightarrow \mathbb{R}$ be orthogonally additive and odd. Then,

(i) There exist a unique vector $y_0 \in E$ such that

$$f(x) = \langle x, y_0 \rangle \quad \text{for each } x \in E$$

(ii) Moreover, $\|f\| = \|y_0\|$

Proof.

(i) Let $u, z, x \in E$ and let $u = x - \frac{f(x)}{f(z)}$.
Since f is odd, by Lemma 3.2

$$f(u+z) = f\left(x - \frac{f(x)}{f(z)} + z\right) = f\left(x - \frac{f(x)}{f(z)}\right) + f(z)$$

This implies $x - \dfrac{f(x)}{f(z)} \perp z$ since f is orthogonally additive.
Therefore, $u \perp z$, implying that $\langle u, z \rangle = 0$.

Taking the inner product of u with z, we have

$$\begin{aligned}
\langle u, z\rangle &= \left\langle x - \frac{f(x)}{f(z)}z, z\right\rangle = \langle x, z\rangle - \langle\frac{f(x)}{f(z)}z, z\rangle = 0 \\
\Rightarrow &\qquad \langle x, z\rangle - \frac{f(x)}{f(z)}\langle z, z\rangle = 0 \\
\Rightarrow &\qquad \frac{f(x)}{f(z)}\langle z, z\rangle = \langle x, z\rangle \\
\Rightarrow &\qquad f(x) = \frac{f(z)}{\langle z, z\rangle}\ \langle x, z\rangle \\
&\qquad = \left\langle x, \frac{\overline{f(z)}}{\overline{\langle z, z\rangle}}z\right\rangle
\end{aligned}$$

Let $\dfrac{\overline{f(z)}}{\overline{\langle z, z\rangle}}z = y_0$ and for arbitrary $z \in E$.
Then

$$f(x) = \langle x, y_0\rangle \text{ for each } x \in E$$

This prove (i).

(ii) From (i), we have

$$f(x) = \langle x, y_0\rangle$$

$$\Rightarrow |f(x)| \leq \|x\| \cdot \|y_0\|$$

so that

$$\|f\| \leq \|y_0\|$$

Also,

$$\begin{aligned}
f(y_0) &= \langle y_0, y_0\rangle \\
f(y_0) &= \|y_0\|^2 \\
\|f\| &= \|y_0\| \text{ as required.}
\end{aligned}$$

Uniqueness

Let $y_1, y_2 \in E$ such that

$$f(x) = \langle x, y_1\rangle = \langle x, y_2\rangle\ \ \forall\, x \in E.$$

Then

$$\langle x, y_1\rangle = \langle x, y_2\rangle$$

$$\Rightarrow \langle x, y_1\rangle - \langle x, y_2\rangle = 0$$

$$\Rightarrow \langle x, y_1 - y_2\rangle = 0 \text{ for all } x \in E$$

This implies that $x \perp y_1 - y_2$ and

$$\begin{aligned} f(x+(y_1-y_2) &= f(x)+f(y_1-y_2) \\ &= f(x)+f(y_1)+f(-y_2) \\ &= f(x)+f(y_1)-f(y_2) \text{ since } f \text{ is odd} \\ \text{Then,} & \\ \|x+(y_1-y_2)\|^2 &= \|x\|^2+\|y_1\|^2-\|y_2\|^2 \end{aligned}$$

From Theorem 3.1 and (ii), we have,

$$\|x\|^2+\|y_1-y_2\|^2 = \|x\|^2+\|f\|-\|f\|$$

Therefore, $|y_1-y_2\|^2 = 0$

$\Rightarrow \quad y_1 = y_2$ □

Since y_1 and y_2 are arbitrarily chosen, then the theorem is proved.

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