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# ON THE DERIVATIVE OF JACOBI'S POLYNOMIAL

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**Abstract:** In this paper, we consider Hermite interpolation on the nodes , which are obtained by vertically projected zeros of the $(1-x^2)P_n^{(\alpha,\beta)'}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial. We obtain the explicit forms and establish a convergence theorem for the interpolatory polynomial.

## §1. Introduction

Surányi and Turán [8] and Balázs and Turán [4, 5] initiated the study of Lacunary interpolation in the special case when the function values and its second derivatives are prescribed on the zeros of $\Pi_n(x) = (1-x^2)P'_{n-1}(x)$, where $P_{n-1}(x)$ is the (n-1)th Legendre polynomial. Saxena and Sharma [9], considered the case of (0, 1, 3) and Saxena [10] considered (0, 1, 2, 4) – Interpolation on the zeros of $\Pi_n(x)$ and obtained results analogous to the above results of P.Turán and his associates. Saxena modified the problem of (0,2) -interpolation on the zeros of $\Pi_n(x)$ by prescribing two additional conditions, namely the third derivative at +1 and -1. He [11] also studied the problem of mixed type Lacunary (0,2;0,1) -interpolation with the second derivatives prescribed at the zeros of $\Pi_n(x)$ and the first derivatives at the zeros of $P_{n-1}(x)$. S. Xie [15] studied the problem of (0, 2) -interpolation taking the nodes as the zeros of $xW_n(x)$ , for n even or $W_n(x)/x$ ,for n odd, where $W_n(x) = (1-x^2)P'_{n-1}(x)$.

Many authors considered a Hermite interpolation problem for different set of nodes and obtained the explicit forms, estimates and convergence. Also, in a paper, S. Bahadur [1] proved the convergence of Hermite interpolation polynomial based on the nodes obtained by projecting vertically the zeros of $\Pi_n(x)$ on the unit circle and also, authors [2] have considered the Hermite interpolation on the unit circle and established a convergence theorem for that. These have motivated us to consider the problem on the zeros of derivatives of the Jacobi polynomial instead of the zeros of the Jacobi polynomial. The answer of this question leads us to the results of this paper.

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In this paper, we have considered the zeros of $(1-x^2)P_n^{(\alpha,\beta)'}(x)$, which are vertically projected onto the unit circle ($P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial). We obtain the explicit forms and establish a convergence theorem for the interpolatory polynomial. In section 2, we give some preliminaries and in section 3, we describe the problem and obtained the regularity of the same. In section 4, we give the explicit formulae of the interpolatory polynomials. In sections 5 and 6, estimation and convergence of interpolatory polynomials are considered respectively.

## §2. Preliminaries

In this section, we shall give some well-known results, which we shall use.

(2.1) $$(1-x^2)P_n^{(\alpha,\beta)''}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0$$

(2.2) $$H(z)=\prod_{k=1}^{2n-2}(z-t_k)=K_n^{*}P_n^{(\alpha,\beta)'}\left(\frac{1+z^2}{2z}\right)z^{n-1}$$

We shall require the fundamental polynomials of Lagrange interpolation based on the nodes as zeros of H(z) is given by:

(2.3) $$l_k(z)=\frac{H(z)}{H'(t_k)(z-t_k)},\ k=1(1)2n-2$$

We will also use the following well-known inequalities (see [12])

(2.4) $$(1-x^2)^{\frac{1}{2}}\left|P_n^{(\alpha,\beta)}(x)\right|=o\left(n^{\alpha-1}\right)\ \text{for}\quad \alpha>0,\ x\in[-1,1]$$

(2.5) $$\left(1-x_k^2\right)^{-1}\sim\left(\frac{k}{n}\right)^{-2}$$

(2.6) $$\left|P_n^{(\alpha,\beta)'}(x_k)\right|\sim k^{-\alpha-\frac{3}{2}}n^{\alpha+2}$$

(2.7) $$\left|P_n^{(\alpha,\beta)}(x)\right|=o\left(n^{\alpha}\right),$$

(2.8) $$\left|P_n^{(\alpha,\beta)'}(x)\right|=o\left(n^{\alpha+2}\right)$$

(2.9) $$\left|P_n^{(\alpha,\beta)}(x_k)\right|\sim k^{-\alpha-\frac{1}{2}}n^{\alpha}$$

(2.10) $$(1-x^2)\left|P_n^{(\alpha,\beta)'}(x)\right|=o\left(n^{\alpha+1}\right)$$

**§3. The Problem and Regularity:** Let $T_n = \{t_k : k = 0(1)2n-1\}$ satisfying:

$$(3.1)\qquad T_n = \{t_0 = 1, t_{2n-1} = -1, \quad t_k = \cos\phi_k + i\sin\phi_k, t_{n+k} = -t_k, k = 1(1)n-1\},$$

where $\{u_k = \cos\phi_k : k = 1(1)n-1\}$ are the zeros of the derivative of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ such that $1 > u_1 > u_2 .......... > u_{n-1} > -1$. Here we are interested to determine the interpolatory polynomial $R_n(z)$ satisfying the conditions:

$$(3.2)\qquad \begin{cases} R_n(t_k) = \alpha_k, k = 0(1)2n-1 \\ R_n'(t_k) = \beta_k, \quad k = 1(1)2n-2 \end{cases}$$

where $\alpha_k$ and $\beta_k$ are arbitrary given complex numbers. We shall also establish the convergence theorem for $R_n(z)$.

**Theorem 1:** Hermite interpolation is regular on $T_n$.

**Proof:** It is sufficient, if we show, the unique solution of (3.2) is $R_n(z) \equiv 0$, when all data $\alpha_k = \beta_k = 0$. In this case, we have $R_n(z) = H(z)q(z)$,

where q (z) is a polynomial of degree $\leq$ 2n -1. Obviously $R_n(t_k) = 0$ for k = 1(1)2n-1. Then from $R_n'(t_k) = 0$, we have $q(t_k) = 0$. As $H'(t_k) \neq 0$, we get $q(z) = (az+b)H(z)$. In addition q $(\pm 1) = 0$, we get $R_n(z) \equiv 0$, provided $H(\pm 1) \neq 0$.

Hence, the theorem follows.

## §4. Explicit Representation of Interpolatory Polynomials

We shall write $R_n(z)$ satisfying (3.2) as:

$$(4.1)\qquad R_n(z) = \sum_{k=0}^{2n-1} \alpha_k A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z),$$

where $A_k(z)$ and $B_k(z)$ are unique polynomials, each of degree at most 4n-3 satisfying the conditions:

$$(4.2)\qquad \begin{cases} A_k(t_j) = \delta_{jk}, \quad j,k = 0(1)2n-1 \\ A_k'(t_j) = 0, \quad j = 1(1)2n-2, k = 0(1)2n-1 \end{cases}$$

$$(4.3)\qquad \begin{cases} B_k(t_j) = 0, \quad j = 0(1)2n-1, k = 1(1)2n-2 \\ B_k'(t_j) = \delta_{jk}, \quad j,k = 1(1)2n-2 \end{cases}$$

**Theorem 2:** For $k = 1(1)2n\text{-}2$

$$(4.4)\qquad B_k(z) = \frac{(z^2-1)H(z)l_k(z)}{(t_k^2-1)H'(t_k)}$$

**Theorem 2:** For $k = 1(1)2n\text{-}2$

(4.5) $$A_k(z) = l_k^2(z) - 2l_k'(t_k)B_k(z)$$

For $k = 0, 2n\text{-}1$

(4.6) $$A_k(z) = \frac{(1 + t_k z)H^2(z)}{2H^2(t_k)}$$

## §5. Estimations of fundamental polynomials

**Lemma 1[3]:** Let $l_k(z)$ be given by (2.3). Then

(5.1) $$\max_{|z|=1} \sum_{k=1}^{2n-2} |l_k(z)| \leq \frac{c}{k^{-\alpha-\frac{1}{2}}}$$

where $c$ is a constant independent of $n$ and $z$.

**Lemma 2**: For $|z| \leq 1$, we have

(5.2) $$\sum_{k=1}^{2n-2} |B_k(z)| \leq c \log n. \quad , \alpha \leq -\frac{1}{2}$$

where $B_k(z)$ is given in Theorem 2 and $c$ is a constant independent of $n$ and $z$.

**Proof:** We have,

$$|B_k(z)| = \left| \frac{(z^2 - 1)H(z)l_k(z)}{(t_k^2 - 1)H'(t_k)} \right|$$

$$= \frac{\sqrt{1 - x^2}\,|H(z)||l_k(z)|}{\sqrt{1 - u_k^2}\,|H'(t_k)|},$$

Using Lemma 1, (2.5), (2.9) and (2.10), we get the result.

**Lemma 3:** For z=$e^{i\theta}\ (0 \leq \theta < 2\pi)$, we have

(5.3) $$\sum_{k=0}^{2n-1} |A_k(z)| \leq cn \log n \quad , \alpha \leq -\frac{1}{2}$$

where $A_k(z)$ is given in Theorem 3 and $c$ is a constant independent of $n$ and $z$.

**Proof:** Using (2.5)–(2.9) and Lemmas 1 & 2, we get the required result.

## §6. Convergence

In this section, we shall prove the following main theorem:

**Theorem 4**: *Let f (z) be continuous in* $|z| \le 1$ *and analytic in* $|z| < 1$ *. Let the arbitrary number* $\beta_k$*'s be such that:*

(6.1) $\{|\beta_k| = o(n\ \omega_2(f, n^{-1})), k = 1(1)2n-2$

Then $\{R_n\}$ be defined by :

(6.2) $R_n(z) = \sum_{k=0}^{2n-1} f(t_k) A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z)$

*satisfies the relation:*

(6.3) $|R_n(z) - f(z)| = o(n\omega_2(f, n^{-1}) \log n)$ , $\alpha \le -\frac{1}{2}$,

*where* $\omega_2(f, n^{-1})$ *is the modulus of continuity of f (z).*

**Remark 1:** Let *f (z)* be continuous in $|z| \le 1$ and $f' \in \mathrm{Lip}\ \upsilon,\ \upsilon > 0$ , then the sequence $\{R_n\}$ converges uniformly to *f (z)* in $|z| \le 1$ follows from (6.3) provided

$\omega_2(f, n^{-1}) = o(n^{-1-\upsilon})$

To prove theorem 4, we shall need the following:

**Remark 2:** Let *f (z)* be continuous in $|z| \le 1$ and analytic in $|z| < 1$ .Then there exists a polynomial $F_n(z)$ of degree at most 4n-3 satisfying Jackson's inequality

(6.4) $|f(z) - F_n(z)| \le c\omega_2(f, n^{-1})$ , $z = e^{i\theta}(0 \le \theta < 2\pi)$

And also an inequality due to O. Kiŝ [6]

(6.5) $|F_n^{(m)}(z)| \le cn^m \omega_2(f, n^{-1})$ , for $m \in I^+$

**Proof:** Since $R_n(z)$ be given by (6.2) is a uniquely determined polynomial of degree $\le 4n-3$, the polynomial $F_n(z)$ satisfying (6.4) and (6.5) can be expressed as:

$$F_n(z) = \sum_{k=0}^{2n-1} F_n(t_k) A_k(z) + \sum_{k=1}^{2n-2} F'(t_k) B_k(z)$$

Then,
$$|R_n(z) - f(z)| \le |R_n(z) - F_n(z)| + |F_n(z) - f(z)|$$
$$\le \sum_{k=0}^{2n-1} |f(t_k) - F_n(t_k)||A_k(z)| + \sum_{k=1}^{2n-2} \{|\beta_k| + |F_n'(t_k)|\}|B_k(z)|$$
$$+ |F_n(z) - f(z)|$$

Using $z = e^{i\theta}(0 \le \theta < 2\pi)$ , (6.1), (6.4), (6.5) and Lemma 2 and 3, we get (6.3).

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