ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2014, Article ID ama0169, 23 pages
ISSN 2307-7743
http://scienceasia.asia

A GENERALIZED POWER LINDLEY DISTRIBUTION WITH APPLICATIONS

GAYAN WARAHENA-LIYANAGE AND MAVIS PARARAI

Abstract. The Exponentiated Power Lindley (EPL) distribution which is an extension of the Power Lindley Distribution is introduced and its properties are explored. This new distribution represents a more flexible model for the lifetime data. Some statistical properties of the proposed distribution including the shapes of the density and hazard rate functions, the moments and moment generating function, skewness and kurtosis are explored. Entropy measures and the distribution of the order statistics are given. The maximum likelihood estimation technique is used to estimate the model parameters and finally an application of the model with a real data set is presented for the illustration of the usefulness of the proposed distribution.

1. Introduction

The power Lindley (PL) distribution was proposed by Ghitany et al. [4]. This distribution is an extension of the Lindley (L) distribution which was proposed by Lindley [7] in the context of fiducial and Bayesian statistics. Properties and applications of the Lindley distribution have been studied in the context of reliability analysis by Ghitany et al. [3]. Several other authors including Sankaran [13], Asgharzadeh et al. [1] and Nadarajah et al. [8] proposed and developed the mathematical properties of the generalized Lindley distribution. The probability density function (pdf) of the Lindley distribution is given by,

$$f(y;\beta)=\frac{\beta^2}{\beta+1}(1+y)e^{-\beta y},\quad y>0,\beta>0. \tag{1.1}$$

Using the transformation $X=Y^{\frac{1}{\alpha}}$, Ghittany et al. [4] derived the power Lindley (PL) distribution given by

$$f(x;\alpha,\beta)=\frac{\alpha\beta^2}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha},\quad x>0,\alpha>0,\beta>0. \tag{1.2}$$

The survival function and cumulative distribution function (cdf) of the power Lindley distribution are

$$S(x)=\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}, \tag{1.3}$$

2010 *Mathematics Subject Classification.* 62F10, 62F12.

Key words and phrases. Exponentiated power Lindley distribution, Power Lindley distribution, Maximum likelihood estimation.

and

$$F(x) = 1 - S(x) = 1 - \left(1 + \frac{\beta x^{\alpha}}{\beta + 1}\right) e^{-\beta x^{\alpha}}, \tag{1.4}$$

for $x > 0, \alpha, \beta > 0$, respectively.

The purpose of this paper is to develop a three-parameter alternative to several lifetime distributions including the gamma, Weibull, exponentiated Weibull, exponentiated Lindley, and lognormal distributions. In this context, we propose and develop the statistical properties of the exponentiated power Lindley (EPL) distribution and show that it is a far better model for reliability analysis.

Our aim in this paper is to discuss some important statistical properties of the EPL distribution. This discussion includes the shapes of the density function and hazard rate function, reversed hazard rate function, moments, moment generating function, distribution of order statistics and model parameter estimation by using the maximum likelihood method. Finally, applications of the model to real data sets in order to illustrate the applicability and usefulness of the EPL distribution are presented.

This paper is organized as follows: In section 2, the model and some of its statistical properties including shapes and behavior of the hazard function are presented. Distribution of order statistics, Moments and related measures are given in section 3. Section 4 contains entropy measures. In section 5, mean deviations, Lorenz and Bonferroni curves are presented. In section 6, we present the maximum likelihood method for estimating the parameters of the distribution. Applications are given in section 7 and thereafter the concluding remarks.

2. The Model, Sub-models and Properties

In this section, the cdf, pdf, hazard and reverse hazard functions, and their shapes are presented. We will define $G(x) = [F(x)]^{\omega}$ for $\omega > 0$ where $F(x)$ is the cdf of the power Lindley distribution given in (1.4).

$$G(x; \alpha, \beta, \omega) = \left[1 - \left(1 + \frac{\beta x^{\alpha}}{\beta + 1}\right) e^{-\beta x^{\alpha}}\right]^{\omega}, \quad x > 0, \alpha > 0, \beta > 0, \omega > 0. \tag{2.1}$$

We refer to $G(x)$ as cdf of the EPL distribution. Plots of the cdf for the EPL distribution for several values of the parameters, α, β and ω are given in Figure 2.1.

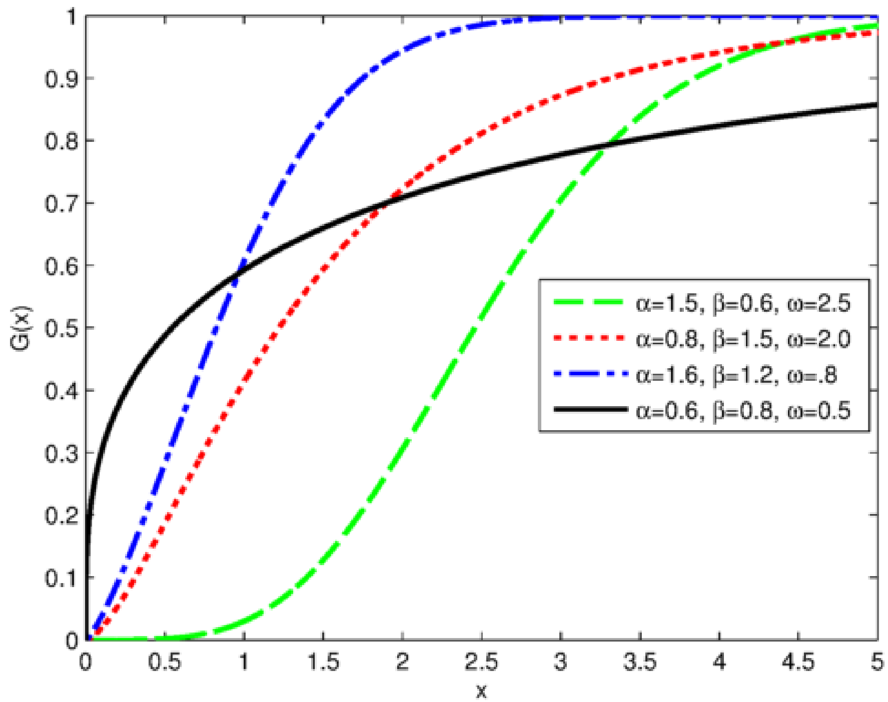

FIGURE 2.1. Plot of the CDF for different values of α, β and ω

The pdf of the EPL distribution is given by

$$g(x;\alpha,\beta,\omega) \;=\; \frac{\alpha\beta^2\omega}{\beta+1}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega-1},$$

for $x>0, \alpha>0, \beta>0, \omega>0$. Plots for the pdf of EPL distribution are given below and each plot has been generated by fixing one parameter at a time.

- α is fixed: see figure 2.1.
- β is fixed: see figure 2.2.
- ω is fixed: see figure 2.3.

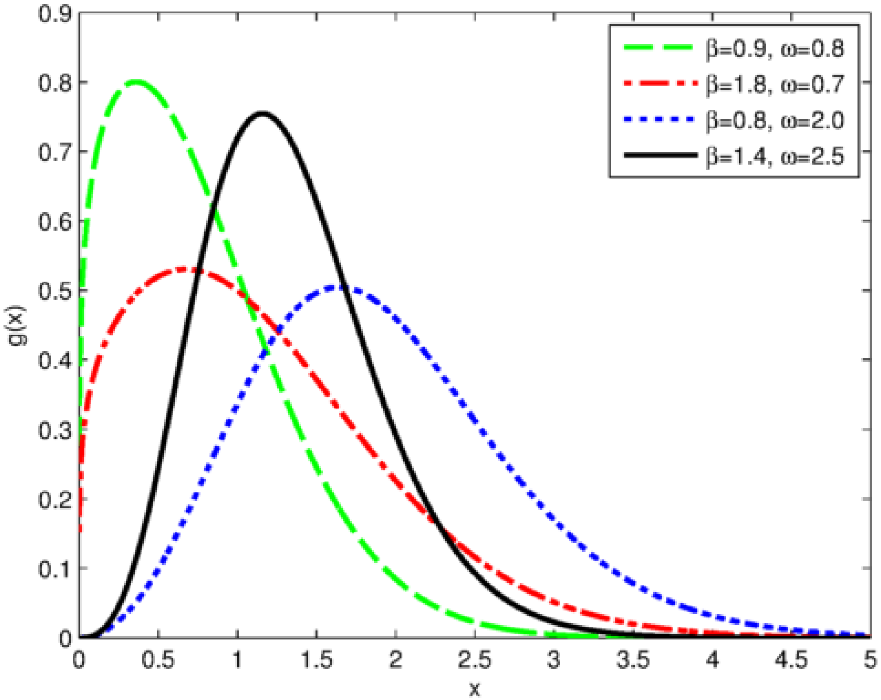

FIGURE 2.2. Plot of the PDF for different values of β, ω and $\alpha = 1.5$

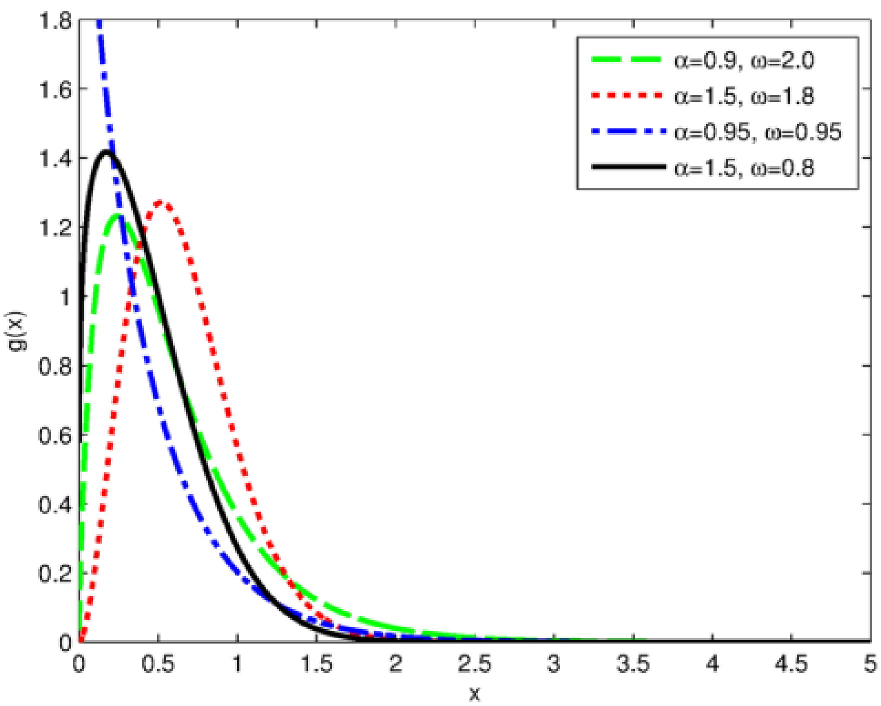

FIGURE 2.3. Plot of the PDF for different values of α, ω and $\beta = 2.0$

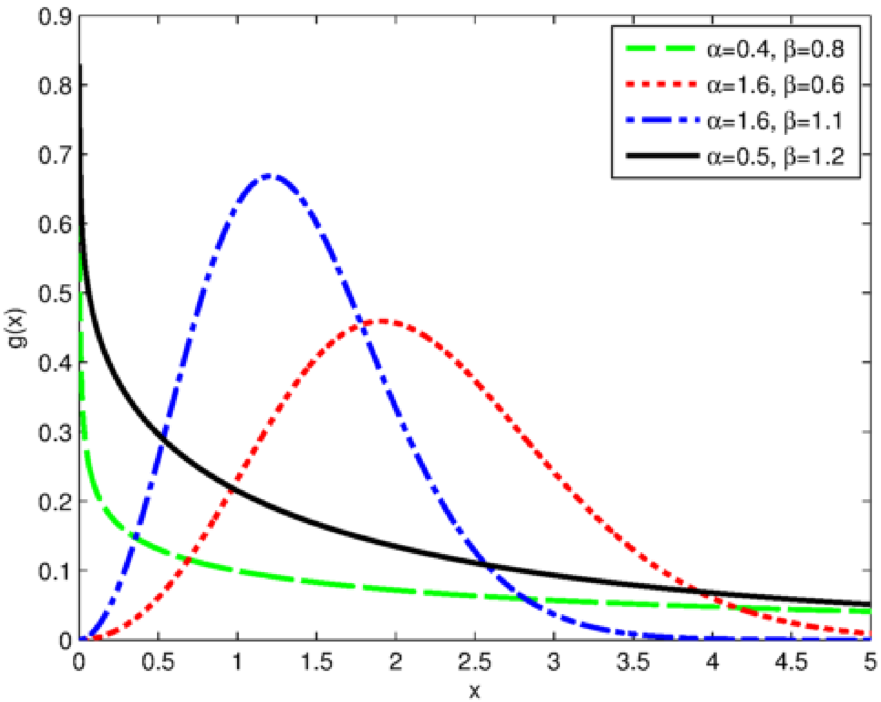

FIGURE 2.4. Plot of the PDF for different values of α, β and $\omega = 1.8$

The pdf of the EPL distribution is unimodal. It increases and decreases for various values of the parameters giving the shapes obtained in the above plots. The plots in figs 2.3 and 2.4 show that for $\alpha < 1$, the pdf is decreasing and for values of $\alpha > 1$, the pdf is unimodal. For values of $\omega > 1, \beta > 1$, the graph seems almost symmmetric.

2.1. **Sub-models.** Some sub-models of the EPL distribution are presented in this section.

- When $\omega = 1$, we obtain the PL distribution.
- When $\alpha = 1$, we obtain the exponentiated Lindley (EL) distribution.
- When $\omega = \alpha = 1$, we obtain the Lindley (L) distribution.
- When $\beta = 1$, we obtain exponentiated two-component mixture of Weibull distribution (with shape parameter α and scale 1) and a gamma distribution (with shape parameter 2α and scale 1).
- When $\alpha = 2$, we obtain an exponentiated two-component mixture of Rayleigh distribution (with scale β and scale 2) and a gamma distribution (with shape 4 and scale β).

2.2. **Shapes and Stochastic Orders.** In this section, we present the mode and discuss the shape, as well as stochastic orders of the EPL distribution. To obtain the mode, we solve the equation $\frac{d\ln(g(x))}{dx} = 0$, for x. Note that

$$\begin{aligned} \log(g(x)) &= \log(\alpha) + 2\log(\beta) + \log(\omega) - \log(1+\beta) + \log(1+x^{\alpha}) - \beta x^{\alpha} \\ &+ (\alpha-1)\log(x) + (\omega-1)\log\left[1 - \left(1 + \frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]. \end{aligned} \tag{2.2}$$

so that $\frac{d\ln(g(x))}{dx} = 0$, results in

$$\frac{\alpha x^{\alpha-1}}{1+x^\alpha} - \alpha\beta x^{\alpha-1} + \frac{\alpha-1}{x} + (\omega-1)\left[\frac{\alpha\beta^2(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}}{(1+\beta)-[1+\beta(1+x^\alpha)]\,e^{-\beta x^\alpha}}\right] = 0.$$

Note that $\lim_{x\to 0} g(x) = \infty$, and $\lim_{x\to\infty} g(x) = 0$.

Let X_i be distributed according to $EPL(\alpha,\beta,\omega)$, with cdf and pdf G_i and g_i, respectively for $i=1,2$. We say X_2 is stochastically greater than X_1 in likelihood ratio if $g_2(x)/g_1(x)$ is an increasing function of x. It is well known that likelihood ratio order implies failure rate order which in turn implies stochastic order, see Shaked and Shanthikumar [12] for additional details.

- If $\beta_1=\beta_2$ and $\alpha_1=\alpha_2$, then X_2 is stochastically greater than X_1 with respect to likelihood ratio order if and only if $\omega_2>\omega_1$.
- If $\alpha_1=\alpha_2$ and $\omega_1=\omega_2$ then X_2 is stochastically larger than X_1 with respect to likelihood ratio order if and only if $\delta_1>\delta_2$.

Note that

$$\begin{aligned}\frac{g_2(x)}{g_1(x)} &= \frac{(1+\beta_1){\alpha_2}^2\omega_2\beta_2(1+x^{\alpha_2})x^{\alpha_2-\alpha_1}e^{\beta_1x^{\alpha_1}-\beta_2x^{\alpha_2}}}{(1+\beta_2){\alpha_1}^2\omega_1\beta_1(1+x^{\alpha_1})}\\ &\times \frac{\left[1-\left(1+\frac{\beta_2x^{\alpha_2}}{\beta_2+1}\right)e^{-\beta_2x^{\alpha_2}}\right]^{\omega_2-1}}{\left[1-\left(1+\frac{\beta_1x^{\alpha_1}}{\beta_1+1}\right)e^{-\beta_1x^{\alpha_1}}\right]^{\omega_1-1}}. \end{aligned} \tag{2.3}$$

If $\beta_1=\beta_2$, and $\alpha_1=\alpha_2$, then

$$K(x) = \frac{\omega_2}{\omega_1}\left[1-\left(1+\frac{\beta x^\alpha}{1+\beta}\right)\exp(-\beta x^\alpha)\right]^{\omega_2-\omega_1}, \tag{2.4}$$

and is such that

$$\begin{aligned}K'(x) &= \frac{\omega_2(\omega_2-\omega_1)}{\omega_1}\left[\frac{\alpha\beta^2}{\beta+1}(1+x^\alpha)x^{\alpha-1}\exp(-\beta x^\alpha)\right]\\ &\times \left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega_2-\omega_1-1} \geq 0,\end{aligned} \tag{2.5}$$

if and only if $\omega_2-\omega_1\geq 0$.

2.3. **Quantile Function.** The quantile function is the solution of the equation

$$\left[1-\left(\frac{1+\beta+\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^\omega = p \quad \text{where } 0<p<1.$$

Thus, the quantile function, say $Q(p)$, defined by $G(Q(p))=p$ is the root of the equation,

$$\left[1-\left(\frac{1+\beta+\beta Q(p)^\alpha}{\beta+1}\right)\exp(-\beta Q(p)^\alpha)\right]^\omega = p \quad \text{where } 0<p<1.$$

Let $Z(p) = -1 - \beta - \beta Q(p)^{\alpha}$.
We have

$$\begin{aligned}\left[1 + \left(\frac{Z(p)}{\beta+1}\right)\exp\{Z(p)+1+\beta\}\right]^{\omega} &= p\\ 1 + \left(\frac{Z(p)}{\beta+1}\right)\exp\{Z(p)+1+\beta\} &= p^{1/\omega}\\ Z(p)\exp\{Z(p)\} &= \frac{-(\beta+1)(1-p^{1/\omega}}{\exp(1+\beta)}.\end{aligned}$$

So the solution for $Z(p)$ is

$$Z(p) = W\left(-(\beta+1)(1-p^{1/\omega})\exp(-1-\beta)\right).$$

for $0 < p < 1$, where $W(.)$ is the Lambert W function, [2].
Now,

$$\begin{aligned}-1-\beta-\beta Q(p)^{\alpha} &= W\left(-(\beta+1)(1-p^{1/\omega})\exp(-1-\beta)\right)\\ \beta Q(p)^{\alpha} &= -1-\beta-W\left(-(\beta+1)(1-p^{1/\omega})\exp(-1-\beta)\right).\end{aligned}$$

So we have

$$Q(p) = \left[-1-\frac{1}{\beta}-\frac{1}{\beta}W\left(-(\beta+1)(1-p^{1/\omega})\exp(-1-\beta)\right)\right]^{1/\alpha}. \tag{2.6}$$

2.4. **Hazard and Reverse Hazard Functions.** The survival function for the EPL distribution is given by,

$$\begin{aligned}\overline{G}(x;\alpha,\beta,\omega) &= 1 - G(x;\alpha,\beta,\omega)\\ &= 1 - \left[1 - \left(1 + \frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{\omega}.\end{aligned} \tag{2.7}$$

The hazard and reverse hazard functions are given by

$$\begin{aligned}\lambda_G(x;\alpha,\beta,\omega) &= \frac{g(x;\alpha,\beta,\omega)}{\overline{G}(x;\alpha,\beta,\omega)}\\ &= \frac{\frac{\alpha\beta^2\omega}{\beta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{\omega-1}}{1-\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{\omega}},\end{aligned}$$

and

$$\begin{aligned}\tau_G(x;\alpha,\beta,\omega) &= \frac{g(x;\alpha,\beta,\omega)}{G(x;\alpha,\beta,\omega)}\\ &= \frac{\frac{\alpha\beta^2\omega}{\beta+1}(1+x^{\alpha})x^{\alpha-1}e^{-\beta x^{\alpha}}\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{\omega-1}}{\left[1-\left(1+\frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right]^{\omega}},\end{aligned}$$

respectively. Plots of the hazard function are given below:

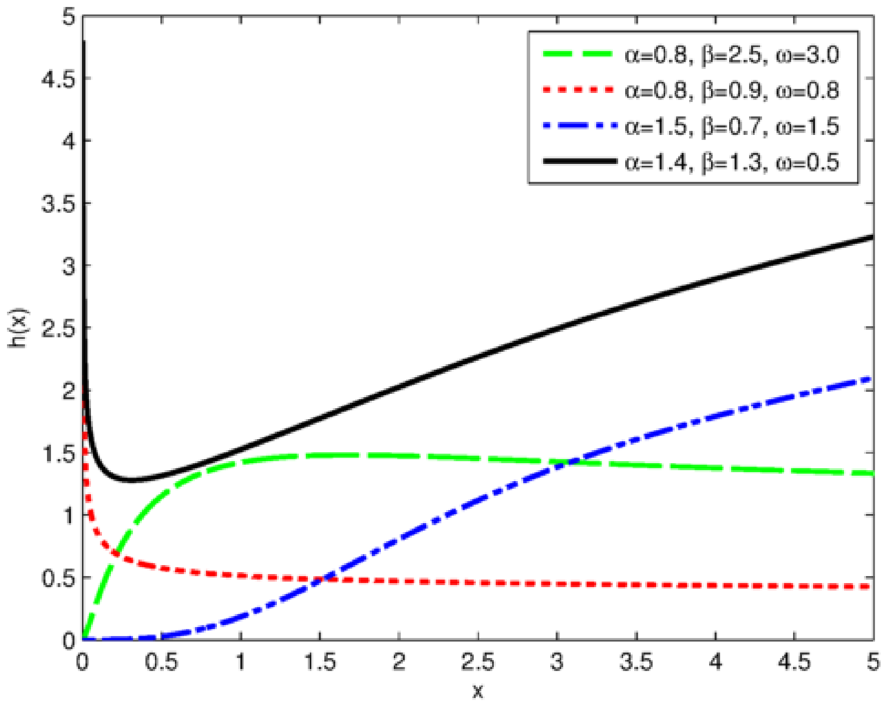

FIGURE 2.5. Plot of the hazard function for different values of α, β and ω

The graphs of the hazard function for four combinations of the values of the model parameters show various shapes including monotonically increasing, monotonically decreasing, uni-modal, bathtub, and upside down bathtub shapes with four combinations of the values of the parameters. This attractive flexibility makes the EPL hazard rate function useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

3. Moments, Moment Generating Function and Related Measures

In this section, moments and related measures including coefficients of variation, skewness and kurtosis are presented. A table of values for mean,standard deviation, coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) is also presented.

3.1. **Moments.** The r^{th} moment about the origin of a continuous random variable X, denoted by μ'_r, is ,

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r g(x) dx \quad \text{for } r = 0, 1, 2, \ldots \quad .$$

In order to find the moments, consider the following lemma.
Lemma 1
Let,

$$\begin{aligned} L_1(\alpha, \beta, \omega, r) &= \int_0^\infty x^r (1 + x^\alpha) x^{\alpha-1} \left[1 - \left(1 + \frac{\beta x^\alpha}{\beta + 1}\right) e^{-\beta x^\alpha}\right]^{\omega-1} e^{-\beta x^\alpha} dx \\ &= \int_0^\infty (1 + x^\alpha) x^{\alpha+r-1} \left[1 - \left(1 + \frac{\beta x^\alpha}{\beta + 1}\right) e^{-\beta x^\alpha}\right]^{\omega-1} e^{-\beta x^\alpha} dx, \end{aligned}$$

then,

$$L_1(\alpha,\beta,\omega,r)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j+1}\binom{\omega-1}{i}\binom{i}{j}\binom{j+1}{k}\frac{(-1)^i\beta^j\Gamma(k+r\alpha^{-1}+1)}{\alpha(\beta+1)^i\left[\beta(i+1)\right]^{(k+r\alpha^{-1}+1)}}.$$

Proof. Using the series expansion,

$$(1-z)^{a-1}=\sum_{i=0}^{\infty}\binom{a-1}{i}(-1)^i z^i, \tag{3.1}$$

we have

$$\begin{aligned}
L_1(\alpha,\beta,\omega,r) &= \sum_{i=0}^{\infty}\binom{\omega-1}{i}(-1)^i\left[\frac{1+\beta(1+x^\alpha)}{\beta+1}\right]^i e^{-i\beta x^\alpha}\int_0^\infty (1+x^\alpha)x^{\alpha+r-1}e^{-\beta x^\alpha}dx\\
&= \sum_{i=0}^{\infty}\binom{\omega-1}{i}\frac{(-1)^i}{(\beta+1)^i}\int_0^\infty (1+x^\alpha)x^{\alpha+r-1}\left[1+\beta(1+x^\alpha)\right]^i e^{(-i\beta x^\alpha-\beta x^\alpha)}dx\\
&= \sum_{i=0}^{\infty}\binom{\omega-1}{i}\frac{(-1)^i}{(\beta+1)^i}\sum_{j=0}^{i}\binom{i}{j}\beta^j\int_0^\infty (1+x^\alpha)^{j+1}x^{\alpha+r-1}e^{(-i\beta x^\alpha-\beta x^\alpha)}dx\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j+1}\binom{\omega-1}{i}\binom{i}{j}\binom{j+1}{k}\frac{(-1)^i\beta^j}{(\beta+1)^i}\int_0^\infty x^{\alpha+\alpha k+r-1}e^{-(i+1)\beta x^\alpha}dx.
\end{aligned}$$

Now consider,

$$\int_0^\infty x^{\alpha+\alpha k+r-1}e^{-(i+1)\beta x^\alpha}dx. \tag{3.2}$$

Let $u=\beta(i+1)x^\alpha$, then $\frac{du}{dx}=\alpha\beta(i+1)x^{\alpha-1}$ and $x=\left[\frac{u}{(\beta(i+1)}\right]^{1/\alpha}$.
The above integral can rewritten by using the complete gamma function $\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}dx$ as,

$$\begin{aligned}
\int_0^\infty\left[\frac{u}{(\beta(i+1)}\right]^{\left(\frac{\alpha k+r}{\alpha}\right)}e^{-u}\frac{du}{\alpha\beta(i+1)} &= \int_0^\infty \frac{u^{(k+r\alpha^{-1})}e^{-u}}{\alpha\left[\beta(i+1)\right]^{(k+r\alpha^{-1}+1)}}du\\
&= \frac{\Gamma(k+r\alpha^{-1}+1)}{\alpha\left[\beta(i+1)\right]^{(k+r\alpha^{-1}+1)}}.
\end{aligned}$$

Consequently,

$$L_1(\alpha,\beta,\omega,n)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j+1}\binom{\omega-1}{i}\binom{i}{j}\binom{j+1}{k}\frac{(-1)^i\beta^j\Gamma(k+r\alpha^{-1}+1)}{\alpha(\beta+1)^i\left[\beta(i+1)\right]^{(k+r\alpha^{-1}+1)}}.$$

□

Now using Lemma 1, the rth moment of the EPL distribution is given by

$$\mu_r'=E(X^r)=\frac{\alpha\beta^2\omega}{\beta+1}L_1(\alpha,\beta,\omega,r). \tag{3.3}$$

The first four moments of X can be written as,

$$\begin{aligned}
\mu_1' &= E(X) = \frac{\alpha\beta^2\omega}{\beta+1}L_1(\alpha,\beta,\omega,1).\\
\mu_2' &= E(X^2) = \frac{\alpha\beta^2\omega}{\beta+1}L_1(\alpha,\beta,\omega,2).\\
\mu_3' &= E(X^3) = \frac{\alpha\beta^2\omega}{\beta+1}L_1(\alpha,\beta,\omega,3).\\
\mu_4' &= E(X^4) = \frac{\alpha\beta^2\omega}{\beta+1}L_1(\alpha,\beta,\omega,4).
\end{aligned}$$

The mean, variance, coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\mu = \mu_1' = E(X) = \frac{\alpha\beta^2\omega}{\beta+1}L_1(\alpha,\beta,\omega,1), \tag{3.4}$$

$$\sigma^2 = \mu_2' - \mu^2, \tag{3.5}$$

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1}, \tag{3.6}$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}, \tag{3.7}$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2}, \tag{3.8}$$

respectively. Table (3.1) presents the mean, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for some values of the parameters α, β and ω. Note that CV, CS and CK do not depend on the parameter α.

α	β	ω	Mean	SD	CV	CS	CK
1	2	3	2.0000	0.8165	1.1547	1.5651	2.2913
		5	9.8765	NaN	1.5411	NaN	NaN
		7	80.6584	NaN	1.8439	NaN	NaN
	4	3	0.2880	0.4910	0.3536	1.3567	2.5053
		5	0.4800	0.7494	0.8062	1.8670	2.0369
		7	1.3548	1.0488	1.0801	0.3672	1.5013
	6	3	0.0875	0.2377	NaN	1.7858	3.7775
		5	0.0714	0.2688	0.2887	2.0456	4.6601
		7	0.1000	0.3644	0.6547	1.9691	4.2695
2	2	3	1.4361	NaN	0.4668	NaN	NaN
		5	5.7117	NaN	0.6478	NaN	NaN
		7	40.2338	NaN	0.7569	NaN	NaN
	4	3	0.2991	0.4456	NaN	0.9267	1.3973
		5	0.3978	0.5672	0.3409	0.8657	1.3021
		7	0.9641	0.6522	0.4907	NaN	1.3411
	6	3	0.1127	0.2734	NaN	1.4336	2.3570
		5	0.0731	0.2570	NaN	1.8063	3.4579
		7	0.0877	0.3038	0.2816	1.7887	3.3849
5	2	3	1.2192	NaN	0.3345	NaN	NaN
		5	4.2016	NaN	0.4742	NaN	NaN
		7	26.9248	NaN	0.5525	NaN	NaN
	4	3	0.3174	0.4509	NaN	0.8706	1.2877
		5	0.3634	0.5032	0.2450	0.8288	1.2339
		7	0.7988	0.5159	0.3633	NaN	1.3525
	6	3	0.1363	0.3180	NaN	1.3956	2.2156
		5	0.0759	0.2610	NaN	1.7837	3.3524
		7	0.0824	0.2810	0.2026	1.7724	3.3090

NaN: Not Defined

Table 3.1. Table of Mean, SD, Coefficient of Variation, Skewness and Kurtosis

3.2. **Conditional Moments.** For lifetime models, it may be useful to know about the conditional moments which can be defined as $E(X^r \mid X > x)$. In order to calculate these, we consider the following lemma:

Lemma 2
Let,

$$\begin{aligned} L_2(\alpha,\beta,\omega,r,t) &= \int_t^\infty x^r(1+x^\alpha)x^{\alpha-1}\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega-1} e^{-\beta x^\alpha}dx \\ &= \int_t^\infty (1+x^\alpha)x^{\alpha+r-1}\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega-1} e^{-\beta x^\alpha}dx. \end{aligned}$$

then,

$$L_2(\alpha,\beta,\omega,r,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j+1}\binom{\omega-1}{i}\binom{i}{j}\binom{j+1}{k}\frac{(-1)^i\beta^j\Gamma(k+r\alpha^{-1}+1,\beta(i+1)t^\alpha)}{\alpha(\beta+1)^i\left[\beta(i+1)\right]^{(k+r\alpha^{-1}+1)}}.$$

Proof. Using the same procedure that was used in Lemma 1, this can be simplified into the following form.

$$L_2(\alpha,\beta,\omega,r,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j+1}\binom{\omega-1}{i}\binom{i}{j}\binom{j+1}{k}\frac{(-1)^i\beta^j}{(\beta+1)^i}\int_t^\infty x^{\alpha+\alpha k+r-1}e^{-(i+1)\beta x^\alpha}dx.$$

Now consider,

$$\int_t^\infty x^{\alpha+\alpha k+r-1}e^{-(i+1)\beta x^\alpha}dx.$$

Let $u=\beta(i+1)x^\alpha$, then $\frac{du}{dx}=\alpha\beta(i+1)x^{\alpha-1}$ and $x=\left[\frac{u}{\beta(i+1)}\right]^{1/\alpha}$.
The above integral can be rewritten by using the complementary incomplete gamma function $\Gamma(a,t)=\int_t^\infty x^{a-1}e^{-x}dx$ as,

$$\begin{aligned} \int_{\beta(i+1)t^\alpha}^\infty \left[\frac{u}{\beta(i+1)}\right]^{\left(\frac{\alpha k+r}{\alpha}\right)} e^{-u}\frac{du}{\alpha\beta(i+1)} &= \int_{\beta(i+1)t^\alpha}^\infty \frac{u^{(k+r\alpha^{-1})}e^{-u}}{\alpha\left[\beta(i+1)\right]^{(k+r\alpha^{-1}+1)}}du \\ &= \frac{\Gamma(k+r\alpha^{-1}+1,\beta(i+1)t^\alpha)}{\alpha\left[\beta(i+1)\right]^{(k+r\alpha^{-1}+1)}}. \end{aligned}$$

Consequently,

$$L_2(\alpha,\beta,\omega,r,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j+1}\binom{\omega-1}{i}\binom{i}{j}\binom{j+1}{k}\frac{(-1)^i\beta^j\Gamma(k+r\alpha^{-1}+1,\beta(i+1)t^\alpha)}{\alpha(\beta+1)^i\left[\beta(i+1)\right]^{(k+r\alpha^{-1}+1)}}.$$

□

Now using Lemma 2, the rthconditional moment of the EPL distribution is given by

$$\begin{aligned} E(X^r|X>x) &= \frac{\alpha\beta^2\omega}{\beta+1}\frac{L_2(\alpha,\beta,\omega,r,x)}{1-G(x)} \\ &= \frac{\alpha\beta^2\omega L_2(\alpha,\beta,\omega,r,x)}{\beta+1}\times\frac{1}{\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}} \\ &= \frac{\alpha\beta^2\omega}{\beta+1}\frac{L_2(\alpha,\beta,\omega,r,x)}{\left(1+\beta+\beta x^\alpha\right)e^{-\beta x^\alpha}}. \end{aligned}$$

The first four conditional moments are given by,

$$\begin{aligned}
E(X|X>x) &= \frac{\alpha\beta^2\omega}{\beta+1}\frac{L_2(\alpha,\beta,\omega,1,x)}{(1+\beta+\beta x^\alpha)\,e^{-\beta x^\alpha}}.\\
E(X^2|X>x) &= \frac{\alpha\beta^2\omega}{\beta+1}\frac{L_2(\alpha,\beta,\omega,2,x)}{(1+\beta+\beta x^\alpha)\,e^{-\beta x^\alpha}}.\\
E(X^3|X>x) &= \frac{\alpha\beta^2\omega}{\beta+1}\frac{L_2(\alpha,\beta,\omega,3,x)}{(1+\beta+\beta x^\alpha)\,e^{-\beta x^\alpha}}.\\
E(X^4|X>x) &= \frac{\alpha\beta^2\omega}{\beta+1}\frac{L_2(\alpha,\beta,\omega,4,x)}{(1+\beta+\beta x^\alpha)\,e^{-\beta x^\alpha}}.
\end{aligned}$$

The mean residual lifetime function is given by $E(X|X>x)-x$.

3.3. **Moment Generating Function.** The moment generating function (MGF) of a continuous random variable X, where it exists, is given by,

$$M_X(t)=E(e^{tX})=\int_{-\infty}^{\infty}e^{tx}g(x)dx. \tag{3.9}$$

The MGF of the EPL distribution is given by,

$$M_X(t)=\frac{\alpha\beta^2}{\beta+1}\sum_{n=0}^{\infty}\frac{t^n}{n!}L_1(\alpha,\beta,\omega,n). \tag{3.10}$$

Proof. Consider,

$$M_X(t) = \frac{\alpha\beta^2\omega}{\beta+1}\int_0^\infty e^{tx}(1+x^\alpha)x^{\alpha-1}e^{-\beta x^\alpha}\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega-1}dx.$$

This can be simplified into,

$$\begin{aligned}
M_X(t)=\frac{\alpha\beta^2\omega}{\beta+1}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j+1}\binom{\omega-1}{i}\binom{i}{j}\binom{j+1}{k}\frac{(-1)^i\beta^j}{(\beta+1)^i}\\
\times\int_0^\infty x^{\alpha+\alpha k-1}e^{-(i+1)\beta x^\alpha}e^{tx}dx.
\end{aligned} \tag{3.11}$$

The proof of equation (3.10) is similar to the proof of Lemma 1, but without using the definition of the gamma function. Now consider,

$$\int_0^\infty x^{\alpha+\alpha k-1}e^{-(i+1)\beta x^\alpha}e^{tx}dx.$$

Using the series expansion, $e^{tx}=\sum_{n=0}^{\infty}\frac{t^nx^n}{n!}$, we have

$$\int_0^\infty x^{\alpha+\alpha k-1}e^{(-i\beta x^\alpha-\beta x^\alpha)}e^{tx}dx=\int_0^\infty\sum_{n=0}^{\infty}\frac{t^nx^n}{n!}x^{\alpha+\alpha k-1}e^{(-i\beta x^\alpha-\beta x^\alpha)}dx.$$

Consequently, the MGF of the EPL distribution reduces to

$$M_X(t)=\frac{\alpha\beta^2}{\beta+1}\sum_{n=0}^{\infty}\frac{t^n}{n!}L_1(\alpha,\beta,\omega,n). \tag{3.12}$$

$\square$

3.4. **Distribution of Order Statistics.** Order Statistics play a vital role in probability and statistics. In this section, we present the distribution of the order statistics for the EPL distribution. The pdf of the i^{th} order statistic is given by:

$$g_i(x) = \frac{n!g(x)}{(i-1)!(n-i)!}\left[G(x)\right]^{i-1}\left[1-G(x)\right]^{n-i}.$$

Using the series expansion,

$$(1-z)^{a-1} = \sum_{i=0}^{\infty}\binom{a-1}{i}(-1)^i z^i,$$

we have:

$$\begin{aligned}
g_i(x) &= \frac{n!}{(i-1)!(n-i)!}g(x)\sum_{j=0}^{\infty}\binom{n-i}{j}(-1)^j\left[G(x)\right]^{i+j-1} \\
&= \frac{\alpha\beta^2\omega n!}{(\beta+1)(i-1)!(n-i)!}x^{\alpha-1}(1+x^\alpha)e^{-\beta x^\alpha}\sum_{j=0}^{\infty}\binom{n-i}{j} \\
&\quad\times(-1)^j\left[1-\left(\frac{1+\beta+\beta x^\alpha}{\beta+1}\right)e^{-\beta x^\alpha}\right]^{\omega i+\omega j-1} \\
&= \frac{\alpha\beta^2\omega n! x^{\alpha-1}(1+x^\alpha)}{(\beta+1)(i-1)!(n-i)!}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{n-i}{j}\binom{\omega i+\omega j-1}{k} \\
&\quad\times(-1)^{j+k}e^{-\beta x^\alpha(k+1)}\left(\frac{1+\beta+\beta x^\alpha}{\beta+1}\right)^k
\end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha\beta^2\omega n! x^{\alpha-1}}{(\beta+1)(i-1)!(n-i)!}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{n-i}{j}\binom{\omega i+\omega j-1}{k}\binom{k}{l} \\
&\quad\times(-1)^{j+k}e^{-\beta x^\alpha(k+1)}\frac{\beta^l(1+x^\alpha)^{l+1}}{(\beta+1)^k} \\
&= \frac{\alpha\omega n!}{(i-1)!(n-i)!}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{l+1}\binom{n-i}{j}\binom{\omega i+\omega j-1}{k}\binom{k}{l}\binom{l+1}{m} \\
&\quad\times(-1)^{j+k}e^{-\beta x^\alpha(k+1)}\frac{\beta^{l+2}x^{\alpha m+\alpha-1}}{(\beta+1)^{k+1}}
\end{aligned}$$

.

Now using the series expansion,

$$e^{-\beta x^\alpha(k+1)} = \sum_{p=0}^{\infty}\frac{\beta^p x^{\alpha p}(k+1)^p}{p!},$$

we have:

$$g_i(x) = \frac{\alpha\omega n!}{(i-1)!(n-i)!}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{m=0}^{l+1}\sum_{p=0}^{\infty}\binom{n-i}{j}\binom{\omega i+\omega j-1}{k}\binom{k}{l}\binom{l+1}{m}$$
$$\times\frac{(-1)^{j+k}\beta^{l+p+2}x^{\alpha(1+m+p)-1}(k+1)^p}{p!(\beta+1)^{k+1}}.$$

4. Mean Deviations, Lorenz and Bonferroni Curves

In this section, we present the mean deviation about the mean, the mean deviation about the median, Lorenz and Bonferroni curves. Bonferroni and Lorenz curves are income inequality measures that are also useful and applicable to other areas including reliability, demography, medicine and insurance. The mean deviation about the mean and mean deviation about the median are defined by

$$D(\mu) = \int_0^\infty \mid x-\mu \mid g(x)dx.$$

and

$$D(M) = \int_0^\infty \mid x-M \mid g(x)dx.$$

respectively, where $\mu = E(X)$ and $M = Median(X) = G^{-1}(1/2)$ is the median of G. These measures $D(\mu)$ and $D(M)$ can be calculated using the relationships:

$$D(\mu) = 2\mu G(\mu) - 2\mu + 2\int_\mu^\infty xg(x)dx = 2\mu G(\mu) - 2\int_0^\mu xg(x)dx, \tag{4.1}$$

and

$$D(M) = -\mu + 2\int_M^\infty xg(x)dx = \mu - 2\int_0^M xg(x)dx. \tag{4.2}$$

Now using Lemma 2, we have

$$D(\mu) = 2\mu G(\mu) - 2\mu + \frac{2\alpha\beta^2\omega}{\beta+1}L_2(\alpha,\beta,\omega,1,\mu)$$

and

$$D(M) = -\mu + \frac{2\alpha\beta^2\omega}{\beta+1}L_2(\alpha,\beta,\omega,1,M).$$

Lorenz and Bonferroni curves are given by

$$L(G(x)) = \frac{\int_0^x tg(t)dt}{E(X)}, \quad \text{and} \quad B(G(x)) = \frac{L(G(x))}{G(x)}, \tag{4.3}$$

or

$$L(p) = \frac{1}{\mu}\int_0^q tg(t)dt, \quad \text{and} \quad B(p) = \frac{1}{p\mu}\int_0^q tg(t)dt, \tag{4.4}$$

respectively, where $q = G^{-1}(p)$. Now using the Lemma 2, we can re-write Lorenz and Bonferroni curves as

$$\begin{aligned} B(p) &= \frac{1}{p\mu}\int_0^q tg(t)dt \\ &= \frac{1}{p\mu}\left[\int_0^\infty xg(x)dx - \int_q^\infty xg(x)dx\right] \\ &= \frac{1}{p\mu}\left[\mu - \frac{\alpha\beta^2\omega}{\beta+1}L_2(\alpha,\beta,\omega,1,q)\right]. \end{aligned}$$

and

$$\begin{aligned} L(p) &= \frac{1}{\mu}\int_0^q tg(t)dt \\ &= \frac{1}{\mu}\left[\int_0^\infty xg(x)dx - \int_q^\infty xg(x)dx\right] \\ &= \frac{1}{\mu}\left[\mu - \frac{\alpha\beta^2\omega}{\beta+1}L_2(\alpha,\beta,\omega,1,q)\right]. \end{aligned}$$

5. Some Measures of Uncertainty

In this section, we present Shannon entropy [10],[11], as well as the Rényi entropy, [9] for the EPL distribution. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

5.1. **Shannon Entropy.** Shannon entropy is defined to be

$$H\left[g(X;\alpha,\beta,\omega)\right] = E\left[-\log(g(X;\alpha,\beta,\omega))\right].$$

Thus we have

$$\begin{aligned} H\left[g(X;\alpha,\beta,\omega)\right] &= \log\left[\frac{\beta+1}{\alpha\beta^2\omega}\right] - E\left[\log(1+X^\alpha\right] - (\alpha-1)E\left[\log(X)\right] + \beta E\left[X^\alpha\right] \\ &- (\omega-1)E\left[\log\left\{1-\left(1+\frac{\beta X^\alpha}{1+\beta}\right)e^{-\beta X^\alpha}\right\}\right] \\ &= \log\left[\frac{\beta+1}{\alpha\beta^2\omega}\right] + \frac{\alpha\beta^2\omega}{\beta+1}\Bigg[\beta L_1(\alpha,\beta,\omega,\alpha) \\ &+ \sum_{q=1}^\infty \frac{(-1)^q}{q} L_1(\alpha,\beta,\omega,q\alpha) \\ &+ (\alpha-1)\sum_{p=1}^\infty\sum_{a=0}^\infty \frac{(-1)^a}{p} L_1(\alpha,\beta,\omega,a) \\ &+ (\omega-1)\sum_{t=1}^\infty\sum_{s=0}^\infty\sum_{k=0}^\infty \frac{1}{t}\binom{t}{s}\frac{(-\beta t)^k}{k!}\frac{\beta^s}{(\beta+1)^s} \\ &\times L_1(\alpha,\beta,\omega,\alpha(s+k))\Bigg]. \end{aligned} \tag{5.1}$$

5.2. **Rényi Entropy.** Rényi entropy is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty [g(x;\alpha,\beta,\omega)]^v dx\right), \quad v \neq 1, v > 0. \tag{5.2}$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Note that by using the series expansion in equation (3.1), we have

$$\begin{aligned}\int_0^\infty g^v(x)dx &= \left(\frac{\alpha\beta^2\omega}{1+\beta}\right)^v \sum_{j=0}^{\infty}\sum_{k=0}^{v}\sum_{r=0}^{j}(-1)^j\binom{v\omega-v}{j}\binom{v}{k}\binom{j}{r}\frac{\beta^r}{(1+\beta)^r}\\ &\times \int_0^\infty x^{r\alpha+v\alpha+k\alpha-v}e^{-\beta(j+v)x^\alpha}dx.\end{aligned}$$

Now, let $y = \beta(j+v)x^\alpha$, then

$$\int_0^\infty x^{r\alpha+v\alpha+k\alpha-v}e^{-\beta(j+v)x^\alpha}dx = \frac{\Gamma(r+v+k-(\frac{v-1}{\alpha}))}{\alpha[\beta(j+v)]^{r+v+k-(\frac{v-1}{\alpha})}}.$$

Consequently, Rényi entropy is given by

$$\begin{aligned}I_R(v) &= \frac{1}{1-v}\log\left[\left(\frac{\alpha\beta^2\omega}{1+\beta}\right)^v \sum_{j=0}^{\infty}\sum_{k=0}^{v}\sum_{r=0}^{j}(-1)^j\binom{v\omega-v}{j}\binom{v}{k}\binom{j}{r}\frac{\beta^r}{(1+\beta)^r}\right.\\ &\times \left.\frac{\Gamma(r+v+k-(\frac{v-1}{\alpha}))}{\alpha[\beta(j+v)]^{r+v+k-(\frac{v-1}{\alpha})}}\right],\end{aligned}$$

for $v \neq 1$, $v > 0$.

6. Maximum Likelihood Estimation

In this section, the maximum likelihood estimates of the EPL parameters α, β and ω are presented. The log-likelihood of a single observation x of X from the EPL distribution is

$$\begin{aligned}\log(g(x)) &= \log(\alpha) + 2\log(\beta) + \log(\omega) - \log(1+\beta) + \log(1+x^\alpha) + (\alpha-1)\log(x)\\ &- \beta x^\alpha + (\omega-1)\log\left[1-\left(1+\frac{\beta x^\alpha}{\beta+1}\right)exp(-\beta x^\alpha)\right].\end{aligned} \tag{6.1}$$

The partial derivatives of $\log(g(x))$ with respect to the parameters α, β and ω are:

$$\begin{aligned}\frac{\log(g(x))}{\partial\alpha} &= \frac{1}{\alpha} + \log(x) + \frac{x^\alpha\log(x)}{1+x^\alpha} - \beta x^\alpha\log(x)\\ &= -(\omega-1)\frac{(e^{-\beta x^\alpha}\frac{\beta x^\alpha\log(x)}{\beta+1} - (1+\frac{\beta x^\alpha}{\beta+1})e^{-\beta x^\alpha}(\beta x^\alpha\log(x)))}{1-(1+\frac{\beta x^\alpha}{\beta+1})e^{-\beta x^\alpha}},\end{aligned}$$

$$\begin{aligned}\frac{\log(g(x))}{\partial\beta} &= \frac{2}{\beta} - \frac{1}{\beta+1} - x^\alpha\\ &= -(\omega-1)\frac{(e^{-\beta x^\alpha}(\frac{x^\alpha}{\beta+1} - \frac{\beta x^\alpha}{(\beta+1)^2}) - (1+\frac{\beta x^\alpha}{\beta+1})e^{-\beta x^\alpha}x^\alpha)}{1-(1+\frac{\beta x^\alpha}{\beta+1})e^{-\beta x^\alpha}},\end{aligned}$$

and

$$\frac{\partial \log(g(x))}{\partial \omega} = \frac{1}{\omega} + \log\left[1 - \left(1 + \frac{\beta x^{\alpha}}{\beta+1}\right)e^{-\beta x^{\alpha}}\right].$$

The total log-likelihood based an a random sample $x_1, x_2, ..., x_n$, of size n is $\ell = \sum_{i=1}^{n} \log(g(x_i)) = \sum_{i=1}^{n} \ell_i$. The elements of the score vector are:

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \frac{x_i^{\alpha}\log(x_i)}{1+x_i^{\alpha}} - \beta \sum_{i=1}^{n} x_i^{\alpha}\log(x_i) \\ &- (\omega-1)\sum_{i=1}^{n} \frac{(e^{-\beta x_i^{\alpha}} \frac{\beta x_i^{\alpha}\log(x_i)}{\beta+1} - (1+\frac{\beta x_i^{\alpha}}{\beta+1})e^{-\beta x_i^{\alpha}}(\beta x_i^{\alpha}\log(x_i)))}{1-(1+\frac{\beta x_i^{\alpha}}{\beta+1})e^{-\beta x_i^{\alpha}}}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= \frac{2n}{\beta} - \frac{n}{\beta+1} - \sum_{i=1}^{n} x_i^{\alpha} \\ &- (\omega-1)\sum_{i=1}^{n} \frac{(e^{-\beta x_i^{\alpha}}(\frac{x_i^{\alpha}}{\beta+1} - \frac{\beta x_i^{\alpha}}{(\beta+1)^2}) - (1+\frac{\beta x_i^{\alpha}}{\beta+1})e^{-\beta x_i^{\alpha}}x_i^{\alpha})}{1-(1+\frac{\beta x_i^{\alpha}}{\beta+1})e^{-\beta x_i^{\alpha}}}, \end{aligned} \tag{6.3}$$

and

$$\frac{\partial \ell}{\partial \omega} = \frac{n}{\omega} + \sum_{i=1}^{n} \log\left[1 - \left(1 + \frac{\beta x_i^{\alpha}}{\beta+1}\right)e^{-\beta x_i^{\alpha}}\right]. \tag{6.4}$$

The maximum likelihood estimates, $\hat{\mathbf{\Theta}}$ of $\mathbf{\Theta} = (\alpha, \beta, \omega)$ are obtained by solving the nonlinear equations $\frac{\partial \ell}{\partial \alpha} = 0$, $\frac{\partial \ell}{\partial \beta} = 0$, and $\frac{\partial \ell}{\partial \omega} = 0$. These equations are not in closed form and must be solved via iterative methods such as Newton-Raphson method.

6.1. **Asymptotic Confidence Intervals.** In this section, we present the asymptotic confidence intervals for the parameters of the EPL distribution. The expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\mathbf{\Theta}} = (\hat{\alpha}, \hat{\beta}, \hat{\omega})^T$ be the maximum likelihood estimate of $\mathbf{\Theta} = (\alpha, \beta, \omega)^T$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Theta}} - \mathbf{\Theta}) \xrightarrow{d} N_3(\mathbf{0}, I^{-1}(\mathbf{\Theta}))$, where $I(\mathbf{\Theta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Theta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$, that is $J(\hat{\mathbf{\Theta}})$. The multivariate normal distribution $N_3(\mathbf{0}, J(\hat{\mathbf{\Theta}})^{-1})$, where the mean vector $\mathbf{0} = (0, 0, 0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

The likelihood ratio (LR) test can be used to compare the fit of the EPL distribution with its sub-models for a given data set. In fact to test $\omega = 1$, the LR statistic $\lambda = 2[\ln L(\hat{\alpha}, \hat{\beta}, \hat{\omega}) - \ln L(\tilde{\alpha}, \tilde{\beta}, 1)]$, where $\hat{\alpha}, \hat{\beta}$, and $\hat{\omega}$ are the unrestricted estimates, and $\tilde{\alpha}$ and $\tilde{\beta}$ are the restricted estimates. The LR test rejects the null hypothesis H_0 if $\lambda > \chi^2_{\eta}$, where χ^2_{η} denotes the upper $100\eta\%$ point of the χ^2 distribution with 1 degree of freedom.

7. Applications

In this section, the EPL distribution is applied to real data in order to illustrate the usefulness and applicability of the model. We fit the density functions of the exponentiated power Lindley (EPL),power Lindley (PL), exponentiated Lindley (EL) and Lindley (L) distributions. These models are also compared to the exponentiated Weibull (EW) and Weibull (W) distributions. Estimates of the parameters of the distributions, standard errors (in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC) are given in Table 1 for the first data set, and Table 2 for the second data set.

The first data set is a subset of the breast feeding study from the National Longitudinal Survey of Youth, the complete data set is available in Klein, J.P., Moeschberger, M.L., Survival Analysis: Techniques for Censoring and Truncated Data, 2nd ed., Springer-Verlag New York, Inc., New York (2003). The data set considered here consists of the times to weaning 927 children of white-race mothers who choose to breast feed their children. The duration of the breast feeding was measured in weeks.

The second data set is the Cancer Patients data. The data represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang, (2003). It consists of the observations listed below: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Estimates of the parameters of EPL distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC) and Bayesian Information Criterion (BIC) are given in Table 8.1 for the first data set and in Table 8.2 for the second data set.

Plots of the fitted densities and the histogram of the data are given in Figure 8.1 for the breastfeeding data, and Figure 8.2 for the cancer patients data.

For the breasfeeding data, the LR statistic of the hypothesis H_0: $PL(\alpha,\beta,1)$ against H_a: $EPL(\alpha,\beta,\omega)$, is $\lambda = 7021.2 - 6981.3 = 39.9$. The p-value is $2.67 \times 10^{-10} < 0.001$. Therefore, we reject H_0 in favor of H_a. Thus the EPL distribution performs better than the PL distribution. There is a significant difference between EPL and EL distributions with $\lambda = 33.9$ and p-value$=5.80 \times 10^{-9} < 0.001$. Thus, reject H_0: EL vs H_a :EPL in favor of H_a. A test of H_0: L vs H_a : EL shows that $\lambda = 199.5$ and p-value$=2.68 \times 10^{-45}$. Thus, we reject H_0 in favor of H_a. To test the hypothesis H_0: $EPL(\alpha,\beta,\omega)$ against H_a: $EW(\alpha,\beta,\omega)$, we have $\lambda = 6981.3 - 6978.4 = 2.9$. The p-value is $0.089 > 0.05$. Thus we fail to reject H_0, and conclude that there is no significant difference between the EPL and EW distributions. There is a significant difference between EPL and W with the LR statistic $\lambda = 30.9$ and p-value$=2.7 \times 10^{-8}$. Thus the EPL distribution is preferred to the Weibull distribution. The values of the statistics AIC, AICC and BIC show that the EPL distribution is a “better” fit for the breast-feeding data.

For the Cancer Patients data set, the LR statistic for the hypothesis H_0: $PL(\alpha, \beta, 1)$ against H_a: $EPL(\alpha, \beta, \omega)$, is $\lambda = 5.8$. The p-value is $0.016 < 0.05$. Therefore, there is a significant difference between PL and EPL distributions. There is also a significant difference between EPL and EL distributions where $\lambda = 11.7$. with a p-value of $6.25 \times 10^{-4} < 0.001$. There is a significant difference between EL and L distributions with $\lambda = 6.5$ and p-value$=0.011 < 0.05$. A test of H_0: $EPL(\alpha, \beta, \omega)$ vs $H_a : EW(\alpha, \beta, \omega)$ shows that $\lambda = 0.5$, and p-value$=0.480 > 0.05$, that there is no significant difference between the two distributions. Based on the values of $-2 \log L$ the EPL distribution fits the cancer patients data the best. However, the values of the statistics AIC, AICC and BIC are smaller for the EPL distribution and show that the EPL distribution is a "better" fit for the Cancer Patients data.

8. Concluding Remarks

We have presented and developed the mathematical properties of a new class of distributions called the Exponentiated Power Lindley (EPL) distribution including the hazard and reverse hazard functions, moments, conditional moments, entropies, mean deviations, Lorenz and Bonferroni curves, Fisher information and maximum likelihood estimates. Applications of the proposed model to real data in order to demonstrate the usefulness and applicability of the class of distributions are also presented.

Table 8.1. Estimates of Models for Breastfeeding Data

	Estimates			Statistics			
Model	α	β	ω	$-2 \log L$	AIC	$AICC$	BIC
EPL(α, β, ω)	0.4186	1.0830	4.4536	6981.3	6987.3	6987.3	7001.8
	(0.0430)	(0.2180)	(1.3189)				
PL$(\alpha, \beta, 1)$	0.7349	0.2565	1	7021.2	7025.2	7025.2	7034.9
	(0.0158)	(0.0128)					
EL$(1, \beta, \omega)$	1	0.0837	0.5579	7073.4	7077.4	7077.4	7087.0
		(0.0033)	(0.0242)				
L$(1, \beta, 1)$	1	0.1172	1	7272.9	7274.9	7274.9	7279.7
		(0.0027)					
EW(α, β, ω)	0.5099	0.5625	3.7864	6978.4	6984.4	6984.4	6998.9
	(0.062)	(0.1687)	(1.1000)				
W$(\alpha, \beta, 1)$	0.9610	0.07017	1	7012.2	7016.2	7016.2	7025.8
	(0.0241)	(0.0059)					

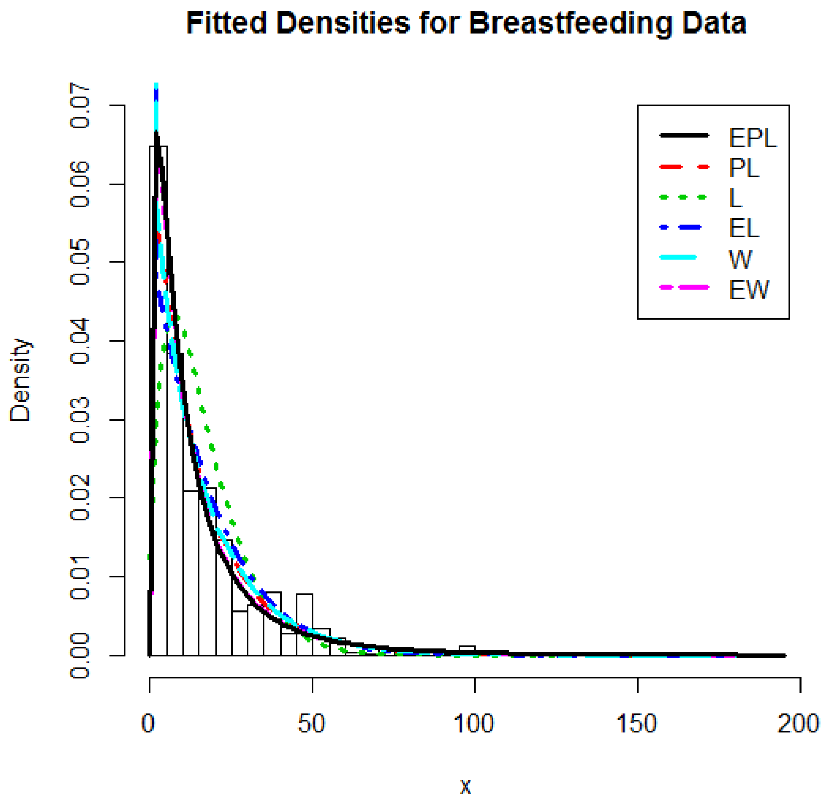

FIGURE 8.1. Plot of the fitted densities for the Breastfeeding Data

TABLE 8.2. Estimates of Models for Cancer Patients Data

	Estimates			Statistics			
Model	α	β	ω	$-2\log L$	AIC	$AICC$	BIC
EPL(α, β, ω)	0.5663	0.8191	2.7684	820.9	826.9	827.1	835.4
	(0.1017)	(0.3116)	(1.2903)				
PL($\alpha, \beta, 1$)	0.8302	0.2943	1	826.7	830.7	830.8	836.4
	(0.0472)	(0.0370)					
EL($1, \beta, \omega$)	1	0.1649	0.7336	832.6	836.6	836.7	842.3
		(0.0166)	(0.0912)				
L($1, \beta, 1$)	1	0.1960	1	839.1	841.1	841.1	843.9
		(0.0123)					
EW(α, β, ω)	0.6544	0.4537	2.7960	821.4	827.4	827.6	835.9
	(0.1347)	(0.2399)	(1.2635)				
W($\alpha, \beta, 1$)	1.0478	0.0939	1	828.2	832.2	832.3	837.9
	(0.0676)	(0.0191)					

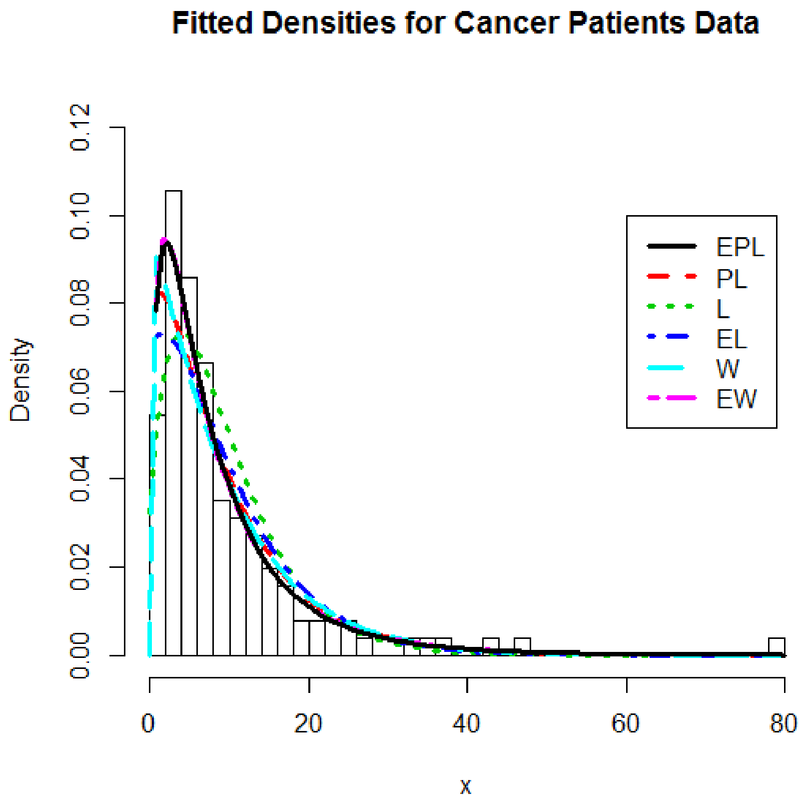

FIGURE 8.2. Plot of the fitted densities for Cancer Patients Data

References

[1] A. Asgharzedah, H.S. Bakouch and H. Esmaeli, Pareto Poisson-Lindley Distribution with Applications, Journal of Applied Statistics, 40(8)(2013).

[2] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth, On the Lambert W function. Advances in Computational Mathematics, 5(1996), 329-359.

[3] M.E. Ghitany, B. Atieh and S. Nadarajah, Lindley Distribution and Its Applications, Mathematics and Computers in Simulation, 78(4)(2008), 493-506.

[4] M.E. Ghitany, D.K. Al-Mutairi, N. Balakrishnan and L.J. Al-Enezi, Power Lindley distribution and associated inference, Computational Statistics and Data Analysis, 64(2013), 20-33.

[5] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series, and Products, seventh ed. Academic Press, New York, 2007.

[6] J.P. Klein, M.L. Moeschberger, Survival Analysis: Techniques for Censoring and Truncated Data, 2nd ed., Springer-Verlag New York, Inc., New York, 2003.

[7] D.V. Lindley, Fiducial distributions and Bayes theorem, Journal of the Royal Statistical Society, Series B 20(1958), 102-107.

[8] S. Nadarajah, H.S. Bakouch and R. Tahmasbi A Generalized Lindley Distribution, Sankhya B, 73(2011), 331-359.

[9] A. Rényi, On measures of entropy and information, Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, vol. I, University of California Press, Berkeley, (1961), 547-561.

[10] E.A. Shannon, A Mathematical Theory of Communication, The Bell System Technical Journal, 27(10)(1948), 379-423.

[11] E.A. Shannon, A Mathematical Theory of Communication, The Bell System Technical Journal, 27(10)(1948), 623-656.

[12] M. Shaked and J.G. Shanthikumar, Stochastic Orders and Their Applications, Academic Press, New York, 1994.

[13] M. Sankaran, The Discrete Poisson-Lindley Distribution, Biometrics, 26(1)(1970), 145-149.

Department of Mathematics, Indiana University of Pennsylvania, PA 15705, United States