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MINIMUM COVERING ENERGY OF SOME THORNY GRAPHS

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Abstract: Thorn graphs are obtained by attaching pendent vertices to each of its vertices. The minimum covering energy of a graph is based on the minimum covering sets of a graph. In this paper minimum covering energies of some thorn graphs are computed.

1. Introduction

According to Huckel Molecular orbital method, the total π electron energy is sum of eigen values of the underlying molecular graph [3]. Motivated by HMO total molecular π electron energy, I.Gutman [4] conceived the *energy of a graph,* defined as the sum of absolute values of all the eigenvalues of a graph. There is variety of results available not only on energy, but also on bounds of eigen values etc [4 –7]. Apart from the adjacency matrix other matrices such as Incidence matrix [12], Laplacian Matrix [8], Distance Matrix [9] etc have been defined and corresponding energies are obtained. Recently Adiga, Gutman et al [1] defined the concept of *minimum covering energy* and obtained results on spectra as well as energy. In this paper we obtain spectra and energy of thorn graphs of a family of graphs.

All the graphs considered in this paper are finite, simple, undirected. Let G be such a graph of order n, with vertex set $V = \{v_1, v_2, \dots. v_n\}$and edge set E. A subset C of V is called a covering set of G if every edge of G is incident to at least one vertex of C.

Any covering set with minimum cardinality is called *minimum covering set.* Let C be a minimum covering set of a graph G. The minimum covering matrix is the n x n matrix $A_c(G) = (a_{ij})$, where,

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{Otherwise} \end{cases}$$

The characteristic polynomial of $A_c(G)$ is denoted by

$$f_m(G, \lambda) = \det(\lambda I - A_c(G))$$

The *minimum covering eigenvalues* of the graph G are the eigenvalues of $A_c(G)$. Since $A_c(G)$ is real symmetric, its eigenvalues are real numbers and we label them in non-

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increasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots\dots\dots \geq \lambda_n$. The *minimum covering energy* of G is then defined as $E_c(G) = \sum_{i=1}^{n} |\lambda_i|$. In [1] some basic properties of minimum covering energy are discussed and some upper and lower bounds for $E_c(G)$ are given. Also minimum covering energies of star graphs, complete graphs etc are obtained.

In what follows we consider a class of graphs constructed by attaching k new pendent vertices to each vertex of the underlying graph. These graphs are often referred to as *thorny graphs or thorn graphs* and have been much studied in the mathematical literature (see, for instance [2,11 ,13]. The thorny graph pertaining to the graph G will be denoted by G^{+k}. The spectrum of G^{+k} was determined in [7].

In this paper we compute the minimum covering energy of thorn graphs of all the graphs discussed in [1].

2. Minimum Covering Energy of Thorn Graphs

Let G be any connected graph of order n with vertex set $\{v_1, v_2, \dots\dots\dots v_n\}$.
The minimum covering set of G^{+k} is simply the vertex set of G if $k > 1$ and there are two minimum covering sets if $k = 1$. So we consider two cases.

Case 1: Let $k > 1$ then minimum covering set of G^{+k} is simply the vertex set of G. Thus if, $A_c(G)$ denotes the minimum covering matrix for G then minimum covering matrix for G^{+k} denoted by $A_c^*(G)$,obtained by making *all diagonal entries of $A_c(G)$ equal to* 1.With pertinent labeling of vertices, minimum covering matrix of G^{+k} has the form,

$A_c\left(G^{+k}\right) = \begin{bmatrix} O & A \\ A^T & A_C^*(G) \end{bmatrix}$ where A is a matrix of order $n\,k$ x n with

i^{th} column C_i having 1's in '$(i$-1) + 1'th to 'ik'th positions and rest 0's i=1,2,....n

The minimum covering polynomial of G^{+k} takes the form,

$$f_m(G^{+k}, \lambda) = \left|\lambda I - A_C\left(G^{+k}\right)\right| = \begin{vmatrix} \lambda I_{nk} & -A \\ -A^T & \lambda I - A_C^*(G) \end{vmatrix}$$

Using,

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M|\left|Q - PM^{-1}N\right|$$

Since $A^T A = k\, I_n$ we get,

$$f_m(G^{+k}, \lambda) = \lambda^{nk} \left| \frac{(\lambda^2 - k)}{\lambda} I - A_C^*(G) \right| \qquad (2.1)$$

Obviously from equation (2.1), the minimum covering polynomial for G^{+k} depends on the characteristic polynomial for $A_C^*(G)$.

From the minimum covering matrix of K_n, $K_{m,n}$ $K_{l\times 2}$ (cocktail party graph) and S^0_n (crown graph which is the complete bipartite graph $K_{n,n}$ with the horizontal edges removed) we obtain $A_c^*(K_n)$ $A_c^*(K_{m.n})$ $A_c^*(K_{l\times 2})$ and $A_c^*(S^0_n)$ consequently giving $\left|\lambda I - A^*(K_n)\right|$ $\left|\lambda I - A^*(K_{m,n})\right|$ $\left|\lambda I - A^*(K_{l\times 2})\right|$ and $\left|\lambda I - A^*(S^0{}_n)\right|$. Finally replacing λ by $\frac{\lambda^2-k}{\lambda}$ we get the desired minimum covering polynomial as per equation (2.1).

Theorem2.1: Let K_n be a complete graph of order n then minimum covering energy the thorn graph $K_n{}^{+k}$ is given by, $2\sqrt{k}\,(n-1)+\sqrt{n^2+4k}$

Proof: For K_n, $A_c^*(K_n) = J$ (matrix of all 1's) and hence the characteristic polynomial is, $\left|\lambda I - A_c^*(K_n)\right| = \lambda^{n-1}(\lambda - n)$ replacing λ by $\frac{\lambda^2-k}{\lambda}$ we get,

$$\left|\lambda I - A_c\left(K_n{}^{+k}\right)\right| = \lambda^{nk}\left(\frac{\lambda^2-k}{\lambda}\right)^{n-1}\left(\frac{\lambda^2-k}{\lambda} - n\right)$$

On simplifying results into,

$$f_m[(K_n)^{+k},\lambda] = \left|\lambda I - A_c\left(K_n{}^{+k}\right)\right| = \lambda^{nk-n}(\lambda^2-k)^{n-1}(\lambda^2 - n\lambda - k)$$

Therefore minimum covering eigen values are

$$\pm\sqrt{k}\ (n-1)\ times,\ \frac{n\pm\sqrt{n^2+4k}}{2}\ and\ 0\ (nk-n)\ times$$

Consequently $E_c\ [(K_n)^{+k}] = 2\sqrt{k}\,(n-1)+\sqrt{n^2+4k}$

Theorem2.2: Let $K_{m,n}$ be a complete bipartite graph of order $m+n$ then minimum covering polynomial of the thorn graph $K_{m,n}{}^{+k}$ is given by,

$$f_m[(K_{m,n})^{+k},\lambda] = \lambda^{(m+n)k-m-n}(\lambda^2-\lambda-k)^{m+n-2}[\lambda^4 - 2\lambda^3 - (2k+n^2-1)\lambda^2 + 2k\lambda + k^2] \quad (2.2)$$

Proof: On similar lines.

Corollary 2.3: Putting $m=1$ we get star $K_{1,n}$ with minimum covering polynomial as,

$$f_m[(K_{1,n})^{+k},\lambda] = \lambda^{(n+1)(k-1)}(\lambda^2-\lambda-k)^{n-1}[\lambda^4 - 2\lambda^3 - (2k+n^2-1)\lambda^2 + 2k\lambda + k^2]$$

Further when $k=n$

$$f_m[(K_{1,n})^{+n},\lambda] = \lambda^{(n+1)(n-1)}(\lambda^2-\lambda-n)^{n-1}[\lambda^4 - 2\lambda^3 - (2n+n^2-1)\lambda^2 + 2n\lambda + n^2]$$

$$= \lambda^{(n^2-1)}(\lambda^2-\lambda-n)(\lambda+n)(\lambda-1)[\lambda^2-(n+1)\lambda-n]$$

So for $n=k$ the minimum covering energy becomes,

$$E_c\ [(K_{1,n})^{+n}] = (n-1)\sqrt{4n+1} + \sqrt{(n+1)^2+4n} + n + 1$$

Theorem 2.4: If $K_{l\times 2}$ denotes the cocktail party graph of order $2l$ then, minimum covering energy of $(K_{l\,x2})^{+k}$ is, $(2l-1)\sqrt{4k+1}+\sqrt{(2l-1)^2+4k}$

Proof: On similar lines

Theorem 2.5: If S_n^0 denotes the crown graph of order $2n$ then, minimum covering energy of $(S_n^0)^{+k}$ is given by,

$$(2n-1)(\sqrt{k+1}+\sqrt{k})+\sqrt{(n-2)^2+4k}+\sqrt{n^2+4k}$$

Proof: On similar lines

Case 2: Let $k=1$ then G^{+1} has precisely **two** minimum covering sets namely $V(G)$ and the **new set of pendent vertices.** Here we discuss both cases.

Case 2.1: Consider $V(G)$ as a minimum covering set of G^{+}. The minimum covering matrix of G^{+1} has the form,

$$A_c(G)=\begin{bmatrix} I & I \\ I & A(G) \end{bmatrix}$$

where I is the identity matrix of order n and $A(G)$ is the adjacency matrix of G.

The minimum covering polynomial is then,

$$f_m(G^{+1},\lambda)=\begin{vmatrix} (\lambda-1)I & -I \\ -I & \lambda I-A(G) \end{vmatrix}$$

$$=\left|(\lambda-1)I\right|\left|\lambda I-A(G)-\frac{I}{\lambda-1}\right|$$

$$f_m(G^{+1},\lambda)=(\lambda-1)^n\left|\frac{(\lambda^2-\lambda-1)}{\lambda-1}I-A(G)\right| \qquad (2.3)$$

Thus knowing the adjacency polynomial the minimum covering polynomial can be easily obtained from equation (2.3).

Theorem 2.6 : The minimum covering energy of thorn graph of a complete graph $(K_n)^{+1}$ is, $2(n-1)\sqrt{n}+n$

Proof: Using the adjacency polynomial of K_n we have from equation (2.3)

$$f_m(K_n^{+1},\lambda)=(\lambda-1)^n\left(\frac{\lambda^2-\lambda-1}{\lambda-1}+1\right)^{n-1}\left[\frac{\lambda^2-\lambda-1}{\lambda-1}-(n-1)\right]$$

$$=(\lambda^2-2)^{n-1}[\lambda^2-n\lambda+(n-2)]$$

Equating to zero we get eigen values and adding their absolute values the theorem follows.

Theorem 2.7: The minimum covering energy of $(K_{m,n})^{+1}$ is given by,

$$(2n-2)\sqrt{5}+\sqrt{(1-\sqrt{mn})^2+4(\sqrt{mn}+1)}+(1+\sqrt{mn})$$

Proof: On similar lines

Corollary 2.8: When $m = n$, the minimum covering energy of $(K_{n,n})^{+1}$ will be,

$$2(n-1)\sqrt{5} + \sqrt{n^2 + 2n + 3} + n + 1$$

Theorem 2.9: The minimum covering energy for thorn graph of a cocktail party graph $K_{l\times 2}$ is given by,

$$l\sqrt{5} + (l-1)\sqrt{13} + (2l-1)$$

Proof :

Theorem 2.10: If S_n^0 denotes the crown graph of order $2n$ then, minimum covering energy of $(S_n^0)^{+1}$ is,

$$2(n-1)(1+\sqrt{2}) + n + \sqrt{n^2 + 4}$$

Proof :On similar lines

Case 2.2: Consider set of pendent vertices as minimum covering set of G^{+1}.The minimum covering matrix of G^{+1} has the form,

$$A_c(G) = \begin{bmatrix} O & I \\ I & A^*(G) \end{bmatrix}$$

where I is the identity matrix of order n and $A^*(G)$ is the adjacency matrix of G having all diagonal entries 1.

The minimum covering polynomial is then,

$$f_m(G^{+1}, \lambda) = \begin{vmatrix} \lambda I & -I \\ -I & \lambda I - A^*(G) \end{vmatrix}$$

$$= |\lambda I| \left| \lambda I - A^*(G) - \frac{I}{\lambda} \right|$$

$$f_m(G^{+1}, \lambda) = \lambda^n \left| \frac{(\lambda^2 - 1)}{\lambda} I - A^*(G) \right| \qquad (2.4)$$

Thus knowing the modified adjacency polynomial the minimum covering polynomial can be easily obtained from equation (2.4).

Theorem 2.11: The minimum covering energy of $(K_n)^{+1}$ is given by

$$2(n-1) + \sqrt{n^2 + 4}$$

Proof : Since $A^*(K_n) = J$

$$f_m(K_n^{+1}, \lambda) = \lambda^n \left| (\frac{\lambda^2 - 1}{\lambda}) I - J \right|$$

$$= \lambda^n \left| \frac{(\lambda^2 - 1)}{\lambda} I - I - K_n \right| = \lambda^n \left| \frac{(\lambda^2 - \lambda - 1)}{\lambda} I - K_n \right|$$

$$\therefore\ f_m(K_n{}^{+1}, \lambda) = (\lambda^2 - 1)^{n-1}[\lambda^2 - n\lambda - 1]$$

Equating to zero we get eigen values and adding their absolute values t he theorem follows

Theorem 2.12: The minimum covering spectrum of thorn graph $(K_{m,n})^{+1}$ is,

$$(m + n - 2)\sqrt{5} + \sqrt{mn} + 1 + \sqrt{mn + 2\sqrt{mn} + 5}$$

Proof: On similar lines

Theorem 2.13 :For cocktail party graph $K_{l\text{x}2}$ the minimum covering energy of

$(K_{l\,\text{x}2})^{+1}$ is, $(2l - 1)\sqrt{5} + \sqrt{4l^2 - 4l + 5}$

Proof: On similar lines

Theorem 2.14: If $S_n{}^0$ denotes the crown graph of order $2n$ then,

minimum covering energy of $(S_n{}^0)^{+1}$ is,$2(n - 1)(1 + \sqrt{2}) + \sqrt{n^2 + 4} + \sqrt{n^2 - 4n + 8}$

Proof: On similar lines

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