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EXTENSION OF SOME FIXED POINT THEOREMS ON WEAKLY CONTRACTIVE MAPS

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Abstract. In this paper we establish some fixed point theorems on weakly contractive maps. These results are the extension of the results of Alber and Guerre-Delabriere [1].

1. Introduction

In 1997, Alber and Guerre -Delabriere $[1]$ define weakly contractive maps. In 2001, Rhoades $[2]$ proved some theorems which extend the work of Alber and Guerre -Delabriere $[1]$ to arbitrary Banach spaces. In this paper now we extend the results of Rhoades $[2]$.

If X is an arbitrary Banach Space, then a self map T of X satisfies the Banach contraction principle, if there existe a constant K satisfying $0 \le K < 1$, such that, for each $x, y \in X$,

$$\|Tx - Ty\| \le K\|x - y\| \qquad \text{........ (1)}$$

Not only do maps satisfying (1) possess a unique fixed point, but the fixed point can be obtained by repeated iteration of T, beginning at any point x in X.

Inequality (1) can be written in the form

$$\|Tx - Ty\| \le \|x - y\| - q\|x - y\| \qquad \text{........ (2)}$$

where $q = a - K$.

The entension of (2) to what are called weakly contractive maps is a natural one. Let X be a Banach space, K a closed convex subset of X. A self map T of K is called weakly contractive if, for each $x, y \in K$,

$$\|Tx - Ty\| \le |x - y\| - \psi[\ \|x - y\|\ ] \qquad \text{........ (3)}$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that ψ is positive on $(0, \infty), \psi(o) = 0$ and $\lim_{t \to \infty} \psi(t) = \infty$. If K is bounded, then the infinity condition can be omitted.

Remark 1: Weakly contractive maps lie between those which satisfy Banach contradiction principle and contractive maps.

2. Preliminaries

It was shown in $[1]$ that, for Hilbert spaces, weakly contractive maps possess a unique fixed point without any additional assumptions, noted that the same is true, at least for

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uniformly smooth and uniformly convex Banach spaces. In 2001, Rhoades [2] showed that theorem remains true in arbitrary complete metric spaces and proved the following theorem:

Theorem A: Let (X, d) be a complete metric space, T a weakly contractive maps. Then T has a unique fixed point p in X.

In this paper we generalize the above theorem.

3. Main Results

We prove the following theorem:

Theorem 1: Let (X, d) be a complete metric space, S and T be two weakly contractive maps, i.e. for each x, y Є K,

$$\|Sx - Ty\| \le \|x - y\| - \psi[\|x - y\|] \qquad \text{........ (4)}$$

Where $\psi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that ψ is positive on $(0,\infty)$, ψ (0) = 0 and $\lim_{t\to\infty} \psi(0) = \infty$. Then S and T have an unique commen fixed point.

Poof: Let $x_0 \in X$ and define

$$x_{2n+1} = Sx_{2n} and x_{2n+2} = Tx_{2n+1}.$$

Then, from (4),

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\le d(x_{2n}, x_{2n+1}) - \psi[\ d(x_{2n}, x_{2n+1})\ ]. \end{aligned}$$

Set $\rho_n = d(x_{2n}, x_{2n+1})$. Then we have

$$\rho_{n+1} \le \rho_n - \psi(\rho_n) \le \rho_n . \qquad \text{......... (5)}$$

Therefore $\{\rho_n\}$ is a nonnegative nonincreasing sequence and hence possesses a limit $\rho^* \ge 0$. Suppose that $\rho^* > 0$. Since ψ is nondecreasing, therefore, we have, from (5), $\rho_{n+1} \le \rho_n - \psi(\rho^*)$.

Thus $\rho_{N+m} \le \rho_m - N\psi(\rho^*)$, a contradiction for N large enogh. Therefore $\rho^* = 0$.

Fix $\in > 0$ and chose N so that

$$d(x_N, x_{N+1}) \le \min\left\{\frac{\in}{2}, \psi(\frac{\in}{2})\right\}.$$

We wish to show that S is a map of the closed ball $B(x_N, \in)$. Let $x \in B(x_N, \in)$.

Case 1. $d(x, x_N) \le \frac{\in}{2}$.

$$d(Sx, x_N) \le d(Sx, Tx_N) + d(Tx_N, x_N)$$

$$\leq d(x, x_N) - \psi[\ d(x, x_N)\ ] + d(x_{N+1}, x_N)$$
$$< \frac{\in}{2} + \frac{\in}{2} = \in.$$

Case 2. $\frac{\in}{2} < d(x, x_N) \leq \in$. Then $\psi[d(x, x_N)] \geq \psi(\frac{\in}{2})$.

Therefore,

$$d(Sx, Tx_N) \leq d(x, x_N) - \psi[d(x, x_N)] + d(x_{N+1}, x_N)$$
$$\leq d(x, x_N) - \psi(\frac{\in}{2}) + \psi(\frac{\in}{2})$$
$$\leq d(x, x_N) \leq \in.$$

Since S is a self map of $B(x_N, \in)$, it follows that each $x_{2n} \in B(x_N, \in)$ for n>N. Since $\in$ is arbitrary, $\{x_{2n}\}$ is Cauchy; hence convergent. The continuity of S implies that the limit is a fixed point. Similarly T is a self map of $B(x_N, \in)$, it follows that each $x_{2n} \in B(x_N, \in)$ for n>N. The continuity of T implies that T has a fixed point. Hence S and T have a common fixed point. In order to show the uniqueness of fixed point z, let $w(w \neq z)$ be another common fixed point of S and T. Then Using (4), we have

$$d(z, w) = d(Sz, Tw)$$
$$\leq d(z, w) - \psi[d(z, w)]$$
$$< d(z, w)$$

a contradiction. It follows that z = w. This completes the proof of the theorem.

Remark 2: If we put S = T in theorem 1 we get Theorem 1 of Rhoades [2].

Theorem 3.1of [1] remains true in Banach spaces we state the following theorem which is the generalization of theorem 3.1 of [1].

Theorem 2: Let S and T be two weakly contractive selfmaps of a closed convex subset K of a Banach space X. Then the iterative process $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ converges strongly to fixed point, with the following error estimate,

$$\|x_{2n} - P\| \leq \phi^{-1}[\phi(\|x_1 - p\|)] - (n-1)$$

where ϕ is defined by the antiderivative.

$$\Phi(t) = \int \frac{dt}{\psi(t)}, \Phi(0) = 0,$$

and Φ^{-1} is the inverse of Φ.

We shall now investigate the convergence of other iterative procedures applied to S and T. The Mann iterative scheme is defined by

$$\left.\begin{array}{l} x_0 \in X, x_{2n+1} = (1-\alpha_{2n})x_{2n} + \alpha_{2n}Sx_{2n} \\ x_{2n+2} = (1-\alpha_{2n+1})x_{2n+1} + \alpha_{2n+1}Tx_{2n+1} \end{array}\right\} \qquad \ldots\ldots (6)$$

where $0 \le \alpha_{2n} \le 1$ & $0 \le \alpha_{2n+1} \le 1$ for each n.

REFERENCES

[1] Ya.I.Alber and S.Guerre-Delabriere, Principles of weakly contractive maps in Hilbert Spaces, Birkhauser Verlag, Basel 98, 1997, 7-22.

[2] B.E.Rhoades, some theorems on weakly contractive maps, Nonlinear Analysis 47(2001), 2683-2693.

[3] N. K. Singh and R. P. Dubey. Extension of a fixed point theorem of Rhoades, J. Ind. Acad. Maths. 31(2), 2009, 577-580.

[4] N. K. Singh and R. P. Dubey. 2010. Common Maximal Elements in Banach Space, J. Ind. Acad. Maths. 32 (2), 2010, 419-426.

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