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MODELLING SOLID WASTE GENERATION USING LEFKOVITCH MATRIX

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Abstract: Population projection models provide a vital tool for planning, not only in the provision of social services but also in various other fields such as conservation strategies for endangered plants and animal species. As the name implies, these models have traditionally been used in ecological and population studies. In this article, the Lefkovitch matrix is used to predict solid waste generation by linking population growth with solid waste generation. The results indicate gradual population growth results in gradual increase in solid waste generation rate. Variability analysis shows that survival rates of the first three groups have significant effect on the population growth rate and so solid waste generation rate. The results are based on the solid waste characterisation and quantification study carried out in Dar es Salaam, Tanzania.

1. Introduction

Mathematical modelling provides an insight into a problem by establishing mathematical relationships among the variables and parameters involved. Several attempts at modelling solid waste generation have been made (Najm *et al.*., 2004; Benitez *et al.*., 2008; Dyson and Chang, 2005; Eleyan *et al.* 2013; Senzige *et al.*, 2014). According to Beigl *et al.* (2008), the differences in previous attempts at modelling solid waste were in the size of the area considered, choice of independent variables, waste streams considered and techniques employed. The techniques that have been in use are least square regression (Benitez *et al.*., 2008); linear programming models (Najm *et al.*., 2004); system dynamics modelling (Dyson and Chang, 2005, Eleyan *et al.*. 2013) and deterministic compartmental models (Senzige *et al.*., 2014).

Apart from the differences in the techniques employed there were also differences in the streams addressed. For example, the works of Daskalopoulos *et al.* (1998), Christiansen and Fischer (1999) and Skovgaard *et al.* (2005) addressed materials constituting the waste while the works of Leao *et al.* (2001), Beigl *et al.* (2004) and Dyson and Chang (2004) dealt will solid collection and disposal systems. Modelling of fractions of household waste (comingled or residual waste) is accredited to the works of Abu Qdais *et al.* (1997) and Denison *et al.* (1996).

Despite the various attempts at modelling solid waste, nowhere in literature, the Leslie (1945) or Lefkovitch (1965) matrix models have been used to model solid waste generation. The models are normally used in population ecology and demographic studies (Leslie, 1945; Lefkovitch, 1965; Crouse *et al.* 1987; Lo *et al.* 1995; Shea and Kelly, 1998; Godfray and Rees, 2002; Carslake *et al.* 2008; Coulson *et al.* 2010; Jonzen *et al.* 2010;

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Tuljapurkar, 2010; Akamine and Suda, 2011 and Kajin *et al.* 2012). In this work, the author uses the already established strength of the Leslie/Lefkovitch matrix models in modelling population growth to predict solid waste generation as there is a direct link between population growth and solid waste generation rates (Senzige *et al.* 2014). The author demonstrates how population parameters affect solid waste generation through variability and sensitivity analysis.

2. Development of the Model

Model development is split into two sections. The first part is on the model for population dynamics where the Lefkovitch matrix is used. The second part is modelling solid waste generation and treatment based on population dynamics. The Lefkovitch (1965) matrix model is a stage structured model based on fecundity and survivorship of discrete stage-classes in a population. In Figure 1six stage-structured classes A_1, A_2, A_3, A_4, A_5 and A_6 is presented.

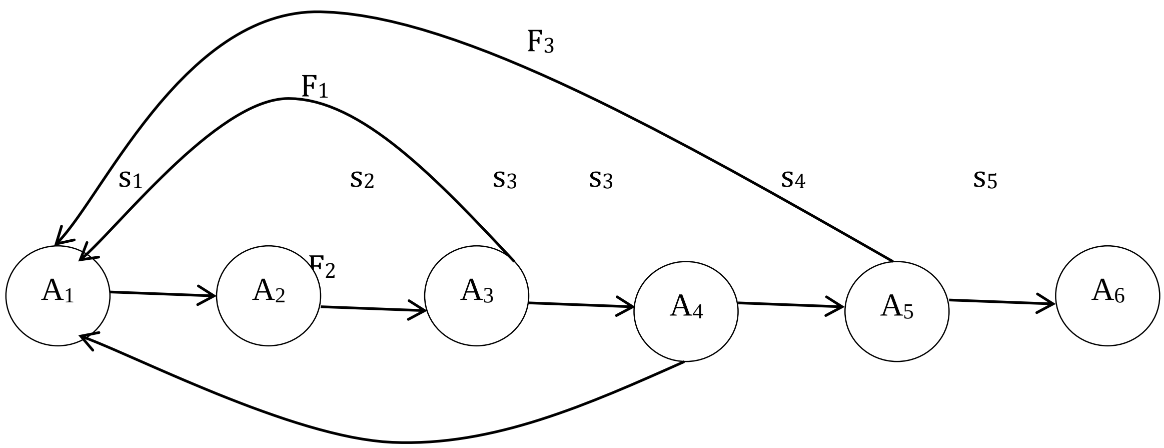

Figure1: Life cycle graph for six-class stage-structured population

From the model we derive the system of equations in eq. (1)

$$\left.\begin{aligned} A_1(t+1) &= F_1A_3(t) + F_2A_4(t) + F_3A_5(t) \\ A_2(t+1) &= s_1A_1(t) \\ A_3(t+1) &= s_2A_2(t) \\ A_4(t+1) &= s_3A_3(t) \\ A_5(t+1) &= s_4A_4(t) \\ A_6(t+1) &= s_5A_5(t) \end{aligned}\right\} \quad (1)$$
,

where A_1 represents infant (age 0-4) group, A_2 represents children (age 5-14) group, A_3 represents adolescents (15-19), A_4 represents young adults (age 20-34), A_5 represents adults (age 35-49) A_6 represents elderly (age 50 and above). The s_1, s_2, s_3, s_4 and s_5 are survival rates, that is, the chances that individuals in group A_1 will survive and cross to age group A_2, A_2 to A_3, A_3 to A_4, A_4 to A_5 and A_5 to A_6 respectively while F_1, F_2 and F_3 are reproductive (fecundity) rates for groups A_3, A_4 and A_5 respectively. It is assumed that A_1, A_2 and A_6 group are non-reproductive due to age. In this way, population growth can be projected using the Lefkovitch (1965) population matrix model.

When written in matrix form, the system of equations in (1) becomes:

$$\begin{bmatrix} A_1(t+1) \\ A_2(t+1) \\ A_3(t+1) \\ A_4(t+1) \\ A_5(t+1) \\ A_6(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & F_1 & F_2 & F_3 & 0 \\ s_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_5 & 0 \end{bmatrix} \begin{bmatrix} A_1(t) \\ A_2(t) \\ A_3(t) \\ A_4(t) \\ A_5(t) \\ A_6(t) \end{bmatrix} \tag{2}$$

Written in compact form, equation (2) becomes,

$$A(t+1) = LA(t) \tag{3}$$

Where $L = \begin{bmatrix} 0 & 0 & F_1 & F_2 & F_3 & 0 \\ s_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_5 & 0 \end{bmatrix}$ is the Lefkovitch Matrix

It can also be shown from equation (3) that at time $t+r$ we have,

$$A(t+r) = L^r A(t) \tag{4}$$

2.1.Determining the vital rates

Solid waste generation is a function of population, among other factors. Therefore, in order to forecast solid waste generation as a function of population growth, we need to determine age-specific rates of survival and fertility (s_i and F_i) of the population in question. In this case, the survival and fertility rates of the population of Dar es Salaam which is the study area are used. The survival and fertility rates are collectively known as vital rates. It is by using these rates the model determines population growth rate, projected estimates of the population size and age distribution over a specific period of time. Estimates of survival and fertility rates for each stage class over the population projection time are estimated using data from the 2002 and 2012 population and housing census in Table 1.

The formula for computing the survival rate is:

$s_i = \dfrac{l(i)}{l(i-1)}$, where l is the number of individuals in a given stage class at a given time and I is an integer and the age specific fertility rate (ASFR) is given by the formula:

$$ASFR = \frac{Number\ of\ live births\ to\ women\ in the\ specific age\ group}{Number\ of\ women\ in the\ same\ age\ group} x1000$$

Table1: Computation of Vital rates

Ages at 2002 Census	Ages at 2012 Census	2002 Female population	2012 Female population	No. of live births (2012 census)	Survival rates	Fertility rates
0-4	10-14	285386	262555	0	0.92	0
5-14	15-24	490940.	436937	0	0.89	0
15-19	25-29	324910	281047	101,177	0.865	0.36
20-34	30-44	941198	775547	589,416	0.824	0.76
35-49	45-59	425233	330831	117,446	0.778	0.355
50+	60+	225262	151838	0	0	0

The stage-structured matrix model (Lefkovitch, 1965) was employed basically for three reasons. First, for humans; the vital rates are better predicted by social status rather than age. Secondly, the data are based on census and so are not fine-grained to generate age-specific vital rates (Alberts and Altmann, 2003) that are normally required in Leslie (1945) matrix model to give good results. Thirdly, the Lefkovitch (1965) matrix model provides more flexibility in terms of stage durations – the stage durations do not have to be equal to projection interval as is the case in the Leslie (1945) matrix model. However, regardless of duration of the stage-classes, vital rates are calculated based on the same projection interval (Caswell, 2001).

3. Results and Discussion

Substituting the vital rates in equation (3) the Lefkovitch matrix is given by;

$$L = \begin{bmatrix} 0 & 0 & 0.360 & 0.760 & 0.355 & 0 \\ 0.920 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.890 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.865 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.824 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.778 & 0 \end{bmatrix}$$

And by using Maple (Appendix 2), the Lefkovitch matrix above has the following eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1.010129096$, $\lambda_3 = -0.0040270666 + 0.8556497155i$, $\lambda_4 = -0.0040270666 - 0.8556497155i$, $\lambda_5 = -0.5010375003 + 0.1705835670i$ and $\lambda_6 = -0.5010375003 - 0.1705835670i$.

Surely; the dominant eigenvalue is $\lambda = 1.01012909$ which implies that the population grows at 1% and solid waste generation increases with increasing population growth. The solid waste dynamics are influenced by the solid waste generation and treatment rate as shown in Figure 2. λ is the population growth rate which determines the amount of solid waste generation and r is the solid waste treatment rate.

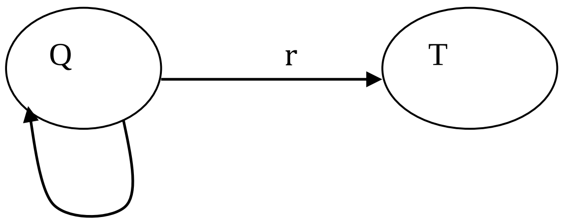

Figure1: Life cycle graph for solid waste generation and treatment

From Figure 2 we have,

$$\left.\begin{aligned} Q(t+1) &= \lambda Q(t) \\ T(t+1) &= rQ(t) \end{aligned}\right\} \qquad (5),$$

which in matrix form can be written as

$$\begin{bmatrix} Q(t+1) \\ T(t+1) \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} Q(t) \\ Q(t) \end{bmatrix} \qquad (6)$$

From equation (6), it is clear that:

$$Q(t+1) = \lambda Q(t)$$

$$Q(t+2) = \lambda Q(t+1)$$

$$= \lambda^2 Q(t)$$

.

.

$$.Q(t+10) = \lambda^{10} Q(t)$$

.

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$$Q(t+n) = \lambda^n Q(t) \qquad (7)$$

Similarly,

$$T(t+n) = r^n Q(t) \qquad (8)$$

Now let β be the stable solid waste generation- treatment rate; then β can be obtained by solving the characteristic equation $\beta^2 - (\lambda + r)\beta + \lambda r = 0$ which gives $\beta = \lambda \ or \ \beta = r$.

However for a solid waste free equilibrium $\lambda Q(t) = rQ(t)$ implying that $\lambda = r$; that is, all waste generated is treated. By treatment here, we mean all means, techniques, technologies, methods and methodologies geared towards elimination of solid waste in the environment. These include collection, reuse, recycling, composting, incineration, land filling and even natural decay.

If $\lambda Q(t) > rQ(t) \Rightarrow \lambda > r \; or \; \frac{\lambda}{r} > 1$ then, solid waste generation will increase with increasing population, otherwise; it will decrease with treatment provided

$\lambda Q(t) < rQ(t) \Rightarrow \lambda < r \; or \; \frac{\lambda}{r} < 1$

3.1. Variability Analysis

Variability analysis is intended to tell us how the population growth rate and therefore solid waste generation rate varies with a small change in the vital rates.

Table1: Variability of λ due to a 0.005 increase in vital rates

Elements	$\nabla\lambda$	Percentage change
l_{13}	0.001025	0.101511
l_{14}	0.000877	0.086858
l_{15}	0.000715	0.07082
l_{21}	0.001401	0.138693
l_{32}	0.001448	0.143358
l_{43}	0.001064	0.105333
l_{54}	0.000309	0.030541
l_{65}	0	0

Table 2 shows how the population growth rate (the dominant eigenvalue) changes as each of the vital rates increases by 0.005. The elements l_{13} to l_{15} are fertility rates for age groups A_3 to A_5 respectively while elements l_{21} to l_{65} are survival rates for groups A_1 to A_5. It is clear that the survival of the first three age groups has significant effect on the population growth rate and so more important as far as solid waste generation is concerned. So, any change in the survival rates of these groups will result in the change of solid waste generation rates. It is surprising however that the survival of the fifth group has no effect of the population growth rate.

4. Numerical results

In this section graphical representation of equations (7)-(8) is presented. Figure 3 depicts solid waste generation when λ=1. That is the population is stable; it is neither increasing nor decreasing. The solid waste generation remains constant. That is if the population is not growing there is no increase in solid waste generation.

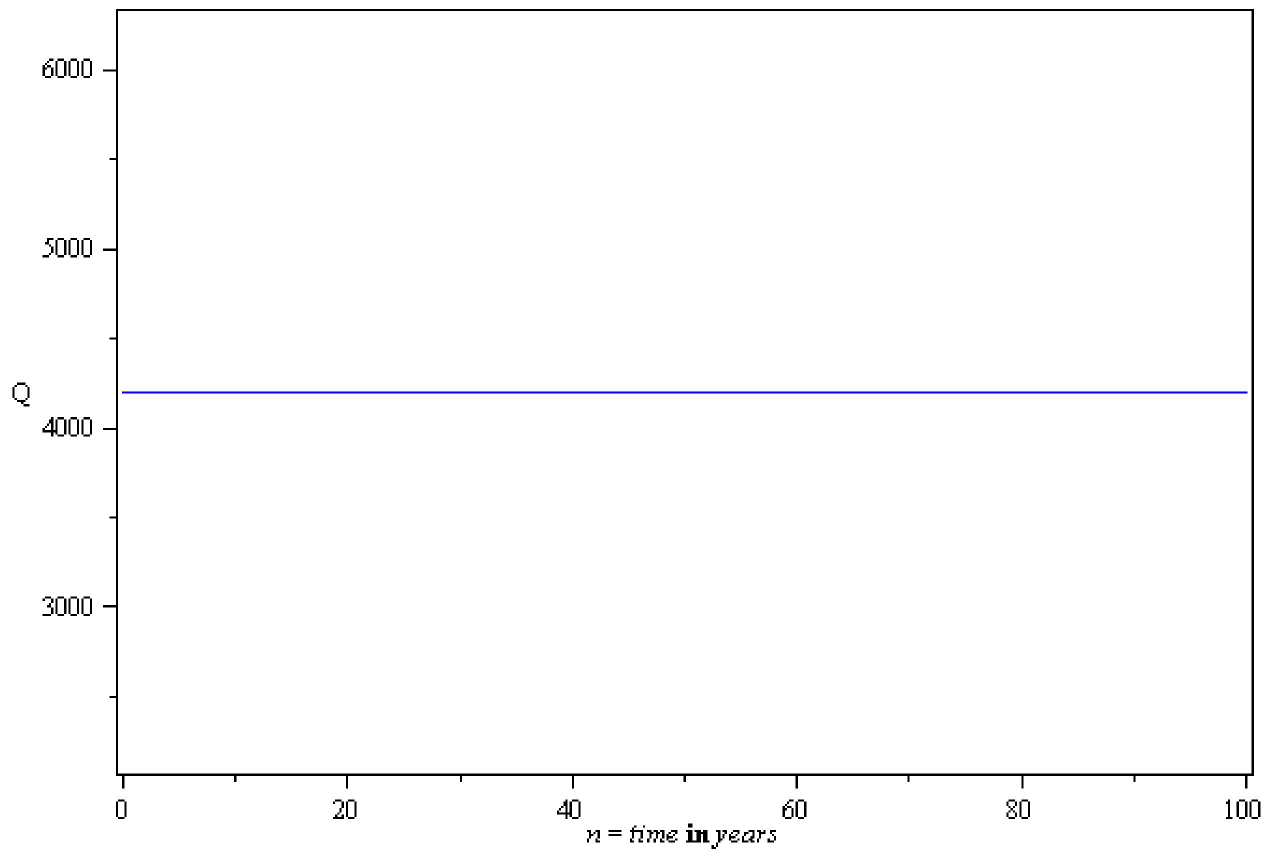

Figure1: Solid waste generation rate when λ=1

Figure 4 presents solid waste generation when the population is growing at 1% (that is λ =1.01). We note that solid waste generation increases with time as population increases. In this case, treatment has to be employed in order to achieve a solid waste free environment.

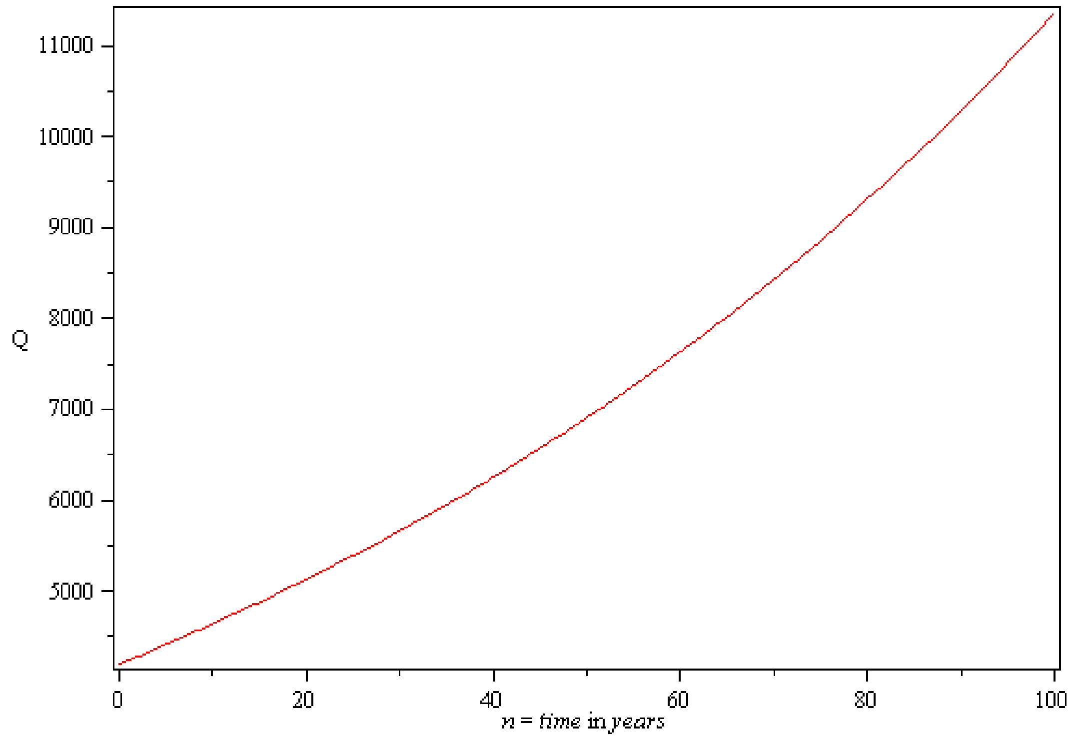

Figure2: solid waste generation λ=1.01

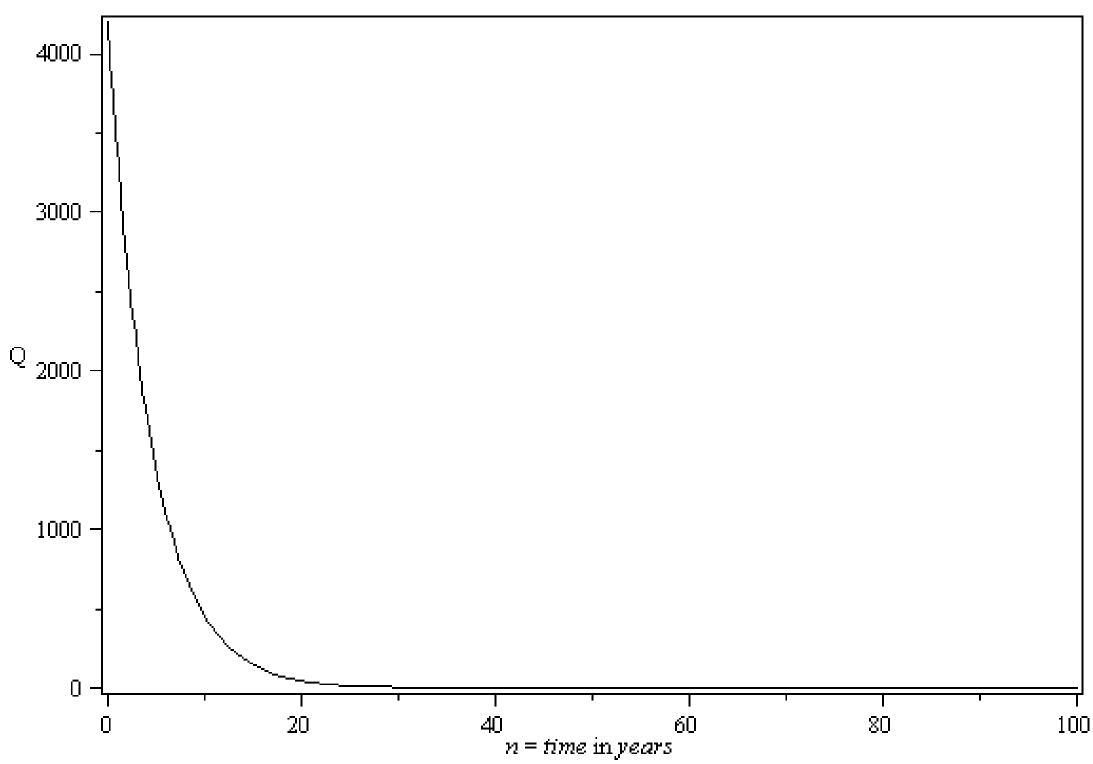

Figure3: solid waste generation when λ=0.8

Figure 5 depicts a picture about solid waste generation when the population is decreasing that is when $\lambda<1$ (0.8 in this case). A population growth rate less than one indicate that the population in question is decreasing. It implicitly means that the solid waste generation rate should also decrease as the population decreases. In the event of the population extinction (a most unlikely scenario) solid waste generated will be zero.

5. Conclusion

The value of $\lambda = 1.01012$ implies that a gradual increase in the population with time is expected. This also implies a gradual increase in the solid waste generation, and hence treatment of solid waste must be intensified on yearly bases in order to reduce the menace. Solid waste free equilibrium can be achieved when the population growth rate λ equals the solid waste treatment rate r. In the event $\lambda > r$ solid waste accumulation increases with increasing population while if $\lambda < r$ solid waste generation decreases with decreasing population and due to treatment. The variability analysis also indicates that the survival of the first three age groups has significant effect on the population growth rate and so more important as far as solid waste generation is concerned. So, any change in survival rates of these groups will result in the change of solid waste generation rates.

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