ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2014, Article ID ama0187, 7 pages
ISSN 2307-7743
http://scienceasia.asia

SOME PROPERTIES OF STARLIKE FUNCTIONS WITH RESPECT TO (j, k) SYMMETRIC CONJUGATE POINTS

C. SELVARAJ, G. THIRUPATHI AND S. STELIN

ABSTRACT. In the present paper, we introduce new subclasses of starlike functions with respect to (j, k) symmetric conjugate points. Some interesting properties for these clases are obtained.

1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a, m)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots$.

Let

$$\mathcal{A}_m = \left\{ f \in \mathcal{H}, f(z) = z + a_{m+1}z^{m+1} + a_{m+2}z^{m+2} + \cdots, \right\} \tag{1.1}$$

and let $\mathcal{A} = \mathcal{A}_1$.

We denote $\mathcal{S}^*$ by the familiar subclass of $\mathcal{A}$ consisting of functions which are starlike in $\mathbb{U}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Also, a function $f \in \mathcal{A}$ is called strongly starlike of order α, $\alpha \in (0, 1]$ if

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z} \right)^{\alpha}.$$

The class of all such functions is denoted by $S_s^*(\alpha)$.

Let k be a positive integer and $j = 0, 1, 2, \ldots (k-1)$. A domain D is said to be (j, k)-fold symmetric if a rotation of D about the origin

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. $(2j, k)$ symmetric conjugate points, starlike, differential subordinations, integral operator.

©2014 Science Asia

through an angle $2\pi j/k$ carries D onto itself. A function $f \in \mathcal{A}$ is said to be $(j,\, k)$-symmetrical if for each $z \in \mathbb{U}$

$$f(\varepsilon z) = \varepsilon^{j} f(z), \tag{1.2}$$

where $\varepsilon = \exp(2\pi i/k)$. The family of $(j,\, k)$-symmetrical functions will be denoted by $\mathcal{F}_{k}^{j}$.

The class of $(j,\, k)$-symmetrical functions [3] was extended to the class $(j,\, k)$-symmetrical conjugate functions in [5]. For fixed positive integers j and k, let $f_{2j,\,k}(z)$ be defined by the following equality

$$f_{2j,\,k}(z) = \frac{1}{2k} \sum_{\nu=0}^{k-1} \left[\varepsilon^{-\nu j} f(\varepsilon^{\nu} z) + \varepsilon^{\nu j} \overline{f(\varepsilon^{\nu} \bar{z})} \right], \quad (f \in \mathcal{A}). \tag{1.3}$$

If ν is an integer, then the following identities follow directly from (1.3):

$$\begin{aligned} f'_{2j,\,k}(z) &= \frac{1}{2k} \sum_{\nu=0}^{k-1} \left[\varepsilon^{-\nu j+\nu} f'(\varepsilon^{\nu} z) + \varepsilon^{\nu j-\nu} \overline{f'(\varepsilon^{\nu} \bar{z})} \right], \\ f''_{2j,\,k}(z) &= \frac{1}{2k} \sum_{\nu=0}^{k-1} \left[\varepsilon^{-\nu j+2\nu} f''(\varepsilon^{\nu} z) + \varepsilon^{\nu j-2\nu} \overline{f''(\varepsilon^{\nu} \bar{z})} \right], \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} f_{2j,\,k}(\varepsilon^{\nu} z) &= \varepsilon^{\nu j} f_{2j,\,k}(z), \quad f_{2j,\,k}(z) = \overline{f_{2j,\,k}(\overline{z})}, \\ f'_{2j,\,k}(\varepsilon^{\nu} z) &= \varepsilon^{\nu j-\nu} f'_{2j,\,k}(z), \quad f'_{2j,\,k}(\overline{z}) = \overline{f'_{2j,\,k}(z)}. \end{aligned} \tag{1.5}$$

2. Definitions And Preliminaries

We define the following:

Definition 2.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{2j,\,k}$ if and only if it satisfies the condition

$$Re\left(\frac{zf'(z)}{f_{2j,k}(z)}\right) > 0, \quad z \in \mathbb{U}. \tag{2.1}$$

Definition 2.2. The function $f \in \mathcal{A}$ is called $\alpha - convex$, $\alpha \in R$ if

$$Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0, \qquad (z \in \mathbb{U}). \tag{2.2}$$

The class of all such functions is denoted by M_{α}.

In this paper we shall determine a sufficient condition for starlikeness with respect to $(2j, k)$ symmetric conjugate points. In addition, we find the images of certain subclasses of $\mathcal{S}_{2j,\,k}(\alpha)$ under the integral operator $I : A \to A$, $I(f) = F$ where,

$$F(z)=\frac{c+1}{(g(z))^{c}}\int_0^z f(t)(g(t))^{c-1}g'(t)\,dt, \tag{2.3}$$

$c\geq 0$ and $g\in\mathcal{A}$ is a given function. If we let $j=1$, then the class $\mathcal{S}_{2j,k}(\alpha)$ reduces to $\mathcal{S}_n^*$. The function class $\mathcal{S}_n^*$ was introduced by H.S.Al.Amiri in [1].

Lemma 2.1. [4] *Let $m\geq 1$ be an integer and*

$$p(z)=1+p_m z^m+p_{m+1}z^{m+1}+\cdots,\quad z\in\mathbb{U} \tag{2.4}$$

be analytic in $\mathbb{U}$. If the function p is not with positive real part in $\mathbb{U}$, then there is a point $z_0\in\mathbb{U}$ such that $p(z_0)=is$, $z_0p'(z_0)=t$, where s,t are real and $t\leq\frac{-m\left(1+s^2\right)}{2}$.

Lemma 2.2. [1] *If $f\in\mathcal{A}_m$ satisfies*

$$\left|\frac{f''(z)}{f'(z)}\right|\leq 1+\frac{m}{2},\qquad z\in\mathbb{U},$$

then for all $z\in\mathbb{U}$, $Re\frac{f(z)}{zf'(z)}>\frac{1}{2}$ and $\left|\left(\frac{zf'(z)}{f(z)}\right)-1\right|<1$.

Lemma 2.3. [2] *Let $\alpha\in(0,1]$. For $c=0$ suppose that $g\in S^*(1-\alpha)$, while $g\in M_{1/c}$, for $c>0$. If the function $f\in\mathcal{A}$ satisfies*

$$\frac{g(z)f'(z)}{g'(z)f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\alpha}$$

then the function F defined by (2.3) is also in $\mathcal{A}$, $\frac{F(z)}{z}\neq 0$ for $z\in\mathbb{U}$ and

$$\frac{g(z)F'(z)}{g'(z)F(z)}\prec\left(\frac{1+z}{1-z}\right)^{\alpha}$$

Lemma 2.4. [4] *Let $P(z)$ be analytic function in $\mathbb{U}$ with $ReP(z)>0$, $z\in\mathbb{U}$, and let h be a convex function in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ with $p(0)=h(0)$, then*

$$p(z)+P(z)zp'(z)\prec h(z)$$

implies

$$p(z)\prec h(z).$$

3. Main results

Theorem 3.1. *Let $f \in \mathcal{A}_m$, $m \geq 2$, and let n be a positive integer. If*

$$\left| \frac{f''(z)}{f'_{2j,k}(z)} \right| \leq \frac{m^2-1}{4m}, \qquad z \in \mathbb{U}, \tag{3.1}$$

where $f_{2j,k}(z)$ is defined by (1.3), then $f \in S_{2j,k}$.

Proof. From (1.4) and (3.1), we deduce that

$$\left| \frac{\varepsilon^{-\nu j+2\nu} f''(\varepsilon^{\nu} z)}{f'_{2j,k}(z)} \right| \leq \frac{m^2-1}{4m},$$

and

$$\left| \frac{\varepsilon^{\nu j-2\nu} \overline{f''(\varepsilon^{\nu} \overline{z})}}{f'_{2j,k}(z)} \right| \leq \frac{m^2-1}{4m}.$$

Let $\nu = 0, 1, 2, \ldots k-1$ in the above inequalities and summing them, we get

$$\left| \frac{f''_{2j,k}(z)}{f'_{2j,k}(z)} \right| \leq \frac{m^2-1}{4m}, \qquad z \in \mathbb{U}.$$

Since $(m^2-1)/4m \leq 1+m/2$, then Lemma 2.2 can be applied to $f_{2j,k}$ to deduce, in particular, $\frac{f_{2j,k}(z)}{z} \neq 0$ for $z \in \mathbb{U}$.
Let

$$p(z) = \frac{zf'(z)}{f_{2j,k}(z)},$$

to show that $Rep(z) > 0$. Since f and $f_{2j,k}$ are in $\mathcal{A}_m$, so p has the form (2.4) for $m \geq 1$. In addition,

$$\frac{zf''(z)}{f_{2j,k}(z)} = \frac{f_{2j,k}(z)}{zf'_{2j,k}(z)} \left[zp'(z) + p(z) \left(\frac{zf'_{2j,k}(z)}{zf_{2j,k}(z)} - 1 \right) \right].$$

Assume p is not with positive real part in $\mathbb{U}$. Then by Lemma2.1, there is a point $z_0 \in \mathbb{U}$ such that $p(z_0) = is$, $z_0p'(z_0) = t$, and $t \leq$

$-m\left(1+s^{2}\right)/2$. Using Lemma2.2 for $f_{2j,\,k}$, we obtain

$$\begin{aligned}\left|\frac{z_0 f''\left(z_0\right)}{f'_{2j,\,k}\left(z_0\right)}\right| &\geq \frac{1}{2}\left|t+is\left(\frac{z_0 f'_{2j,\,k}\left(z_0\right)}{f_{2j,\,k}\left(z_0\right)}-1\right)\right| \\ &\geq \frac{1}{2}\left(|t|-|s|\right) \\ &\geq \frac{1}{2}\left(\frac{m\left(1+s^{2}\right)}{2}-|s|\right) \\ &\geq \frac{m^{2}-1}{4m},\end{aligned} \tag{3.2}$$

which contradicts the hypothesis(3.1). Hence $f \in S_{2j,\,k}$. □

Theorem 3.2. *Suppose* $\alpha \in (0,1]$, $c \geq 0$ *and* $n \geq 1$ *is an integer. Let* $g \in S_{2j,\,k}(1-\alpha)$ *be a function with the power series expansion*

$$g\left(z\right)=z+g_1 z^{n+1}+g_2 z^{2n+1}+\ldots,$$

$z \in \mathbb{U}$, *where all the coefficients* g_j *are real. In addition, suppose that* $g \in M_{1/c}$ *for* $c>0$. *Conider the integral operator* $I : A \to A$, $I\left(f\right)=F$, *where* F *is given by (2.3). If*

$$\frac{g\left(z\right)f'\left(z\right)}{g'\left(z\right)f_{2j,\,k}\left(z\right)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \tag{3.3}$$

then

$$\frac{g\left(z\right)F'\left(z\right)}{g'\left(z\right)F_{2j,\,k}\left(z\right)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$$

where $f_{2j,\,k}$ *and* $F_{2j,\,k}$ *are the functions associated with* f *and* F *as given by (1.3), respectively.*

Proof. From(2.3) one can easily write,

$$F\left(z\right)=\frac{c+1}{\left(g\left(z\right)/z\right)^{c}}\int_0^1 f\left(xz\right)\left(g\left(xz\right)/xz\right)^{c-1}g'\left(xz\right)x^{c-1}dx.$$

From the expansion form of $g\left(z\right)$, it folllows that

$$\frac{1}{2k}\varepsilon^{-\nu j}F\left(\varepsilon^{\nu}z\right)=\frac{c+1}{\left(g\left(z\right)/z\right)^{c}}\int_0^1 \frac{1}{2k}\varepsilon^{-\nu j}f\left(\varepsilon^{\nu}xz\right)\left(g\left(xz\right)/xz\right)^{c-1}g'\left(xz\right)x^{c-1}dx,$$

and

$$\frac{1}{2k}\varepsilon^{\nu j}\overline{F\left(\varepsilon^{\nu}\overline{z}\right)}=\frac{c+1}{\left(g\left(z\right)/z\right)^{c}}\int_0^1 \frac{1}{2k}\varepsilon^{\nu j}\overline{f\left(\varepsilon^{\nu}x\overline{z}\right)}\left(g\left(xz\right)/xz\right)^{c-1}g'\left(xz\right)x^{c-1}dx.$$

Let $\nu=0,1,2,\ldots k-1$ in the above equations, from (1.3) and summing them, we get $F_{2j,\,k}=I\left(f_{2j,\,k}\right)$. Replacing z by $\varepsilon^{\nu}z$ and then by $\varepsilon^{\nu}\overline{z}$,

$\nu = 0, 1, 2, \dots k-1$ in (3.3) and using the relations (1.4) and (1.5) and also the fact that

$$g\left(\varepsilon^{\nu} z\right) = \varepsilon^{\nu} g\left(z\right), g\left(\varepsilon^{\nu} \overline{z}\right) = \varepsilon^{\nu} \overline{g\left(z\right)}, g'\left(\varepsilon^{\nu} z\right) = g'\left(z\right), g'\left(\varepsilon^{\nu} \overline{z}\right) = \overline{g'\left(z\right)}.$$

We deduce the relation

$$\frac{g\left(z\right) f'_{2j,k}\left(z\right)}{g'\left(z\right) f_{2j,k}\left(z\right)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

Applying Lemma(2.3) to the above, we get

$$arg\left(\frac{G\left(z\right) zF'_{2j,k}\left(z\right)}{F_{2j,k}\left(z\right)+c}\right) < \frac{\alpha\pi}{2}, \tag{3.4}$$

where

$$G\left(z\right) = g\left(z\right)/zg'\left(z\right).$$

Let

$$P\left(z\right) = G\left(z\right)\left(\frac{G\left(z\right) zF'_{2j,k}\left(z\right)}{F_{2j,k}\left(z\right)+c}\right)^{-1}. \tag{3.5}$$

From (3.4) and the fact that $g \in S_{2j,k}\left(1-\alpha\right)$, one can easily deduce from (3.5) that

$$ReP(z) > 0.$$

Let

$$p\left(z\right) = \frac{g\left(z\right) F'\left(z\right)}{g'\left(z\right) F_{2j,k}\left(z\right)}.$$

Lemma(2.3) shows that $p(z)$ is analytic in $\mathbb{U}$. Hence multiplication of (2.3) by g^{c} and differentiating the new equation we obtain

$$G\left(z\right) zF'\left(z\right) + cF(z) = (c+1)f(z) \tag{3.6}$$

and

$$G\left(z\right) zF'_{2j,k}\left(z\right) + cF_{2j,k}(z) = (c+1)f_{2j,k}(z) \tag{3.7}$$

Substituting in (3.6), $G(z)F'(z) = p(z)F_{2j,k}(z)$ then differentiating the new equation and using (3.7) to get

$$\begin{aligned} p\left(z\right) + P\left(z\right) zp'\left(z\right) &= \frac{g\left(z\right) f'\left(z\right)}{g'\left(z\right) f_{2j,k}\left(z\right)} \\ &\prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \end{aligned} \tag{3.8}$$

where $P(z)$ is given by (3.5) with $ReP(z) > 0$. Applying Lemma (2.4) to (3.8), we get

$$Rep(z) = Re\frac{g\left(z\right) F'\left(z\right)}{g'\left(z\right) F_{2j,k}\left(z\right)} > 0.$$

□

References

[1] Al-Amiri, Hassoon; Coman, Dan; Mocanu, Petru T. Some properties of starlike functions with respect to symmetric-conjugate points. Internat. J. Math. Math. Sci. 18 (1995), no. 3, 469–474.

[2] Coman, Dan. On a generalized type of the Briot-Bouquet differential subordination. Mathematica (Cluj) 35(58) (1993), no. 1, 15–19.

[3] Karthikeyan, Kadhavoor Ragavan. Some classes of analytic functions with respect to (j, k)-symmetric points. ROMAI J. 9 (2013), no. 1, 51–60.

[4] Miller, Sanford S.; Mocanu, Petru T. General second order differential inequalities in the complex plane. Seminar of geometric function theory, 96–114, Preprint, 82-4, Univ. "Babes,-Bolyai", Cluj-Napoca, 1983

[5] Selvaraj.C, K.R.Karthikeyan, and G.Thirupathi, Multivalent functions with respect to symmetric conjugate points, Punjab University Journal of Mathematics, 46(1), (2014) pp. 1-8.

C. Selvaraj, Presidency College, Chennai-600 005, Tamilnadu, India

G. Thirupathi, R.M.K.Engineering College, R.S.M.Nagar, Kavaraipettai-601 206, Tamilnadu, India

S. Stelin, Tagore Engineering College, Rathinamangalam, Vandalur -601 127, Tamilnadu, India