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UNSTEADY FLOW OF VISCOELASTIC FLUID DUE TO IMPULSIVE MOTION OF PLATE

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ABSTRACT. New exact solutions for unsteady magneto-hydro dynamic (MHD) flows of a generalized second-grade fluid due to a impulsive motion of the plate have been derived. The generalized second-grade fluid saturates the porous space. Fractional derivative is used in the governing equation. The analytical expressions for velocity and shear stress fields have been obtained by using Laplace transform technique for the fractional calculus. The obtained solutions are expressed in series form in terms of Fox H-functions.

1. Introduction

During the past several decades, the Navier-Stokes equations were intensively studied in the literature. However, these equations are unable to predict the behaviors of many real fluids encountered in numerous industrial applications such as processing of polymers, pharmaceuticals, personal care products, food products, inks, and so forth. Therefore, it has been now recognized in industrial and technological applications that non-Newtonian fluids are more appropriate than Newtonian fluids. Non-Newtonian fluids are a broad class of fluids in which the relation connecting the shear stress and shear rate is non linear and hence there is no universal constitutive model available that can alone predict the behavior of all non-Newtonian fluids. Hence, it is necessary to study the behavior of non-Newtonian fluids in order to obtain a thorough understanding and improve their utilization in various manufacturing processes. Rivlin and Ericksen [1] introduced a subclass of non-Newtonian fluids known as second-grade fluid for which a possibility exist to obtain the exact solution. Exact solutions of second-grade fluid for start-up flows have been investigated by Bandelli [2] using integral transform technique. Tan [3] discussed the flow

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of suddenly moved flat plate in a generalized second-grade. Exact solutions of a generalized second grade fluid corresponding to the oscillatory flow between two cylinders have been achieved by Mahmood et al. [4]. Tripathy [5] discussed peristaltic motion of a generalized second grade fluid passing through a cylindrical tube. Tan [6] obtained solutions for unsteady motions between two parallel plates of the generalized second grade fluid.

In the last few decades the study of fluid motions through porous medium have received much attention due to its importance not only to the field of academic but also to the industry. Such motions have many applications in many industrial and biological processes such as food industry, irrigation problems, oil exploitation, motion of blood in the cardiovascular system [7], chemistry and bio-engineering, soap and cellulose solutions and in biophysical sciences where the human lungs are considered as a porous layer. etc. Unsteady MHD flows of viscoelastic fluids passing through porous space are of considerable interest. In the last few years alot of work has been done on MHD flow, see [8-12] and reference therein.

Recently, the fractional derivative [13] approach is proving to be an important tool for considering the behaviors of such types of fluids. Many researchers investigated different problems using fractional derivative technique regarding such fluids. In their works, the integer order time derivatives in the constitutive models for generalized second-grade fluids are replaced by the Riemann-Liouville fractional derivatives. Alot of work has been done on fractional derivatives during the last few years. Here we mention only those contributions which regards with the viscoelastic type fluids [13-20] and the references therein.

According to the authors informations upto yet no study has been done on the MHD flow of generalized second-grade fluid induced by impulsive motion of the plate flowing through a porous space. Hence, our main objective in this note is to make a contribution in this regard. We take an incompressible MHD flow passing through porous space of a generalized second-grade fluid. Laplace transform method has been used for the fractional calculus to obtained exact solutions for the profiles of velocity field and the corresponding shear stress. The obtained solutions satisfies all the imposed initial and boundary conditions are expressed in terms of Fox-H function.

2. Governing equations

The equation of continuity and momentum of MHD flow passing through porous space is given by:(Tan and Masuoka [6])

$$\nabla \cdot \mathbf{V} = 0;\ \rho(\frac{d\mathbf{V}}{dt}) = div\mathbf{T} - \sigma\beta_o^2\mathbf{V} + \mathbf{R}, \tag{2.1}$$

where **V**=(u,v,w) represents velocity vector, electrical conductivity and density of the fluid are represented by σ and ρ respectively, B_0 is the magnitude of a uniform magnetic field, material time derivative is denoted by d/dt, Cauchy stress tensor is represented by **T**, and **R** is the Darcy's resistance of the porous space.
For an incompressible and unsteady generalized second-grade fluid the cauchy stress tensor **T** is given as [7]:

$$\mathbf{T} = \mathbf{S} - p\mathbf{I};\ \mathbf{S} = \mu\mathbf{W}_1 + \alpha_1\mathbf{W}_2 + \alpha_2\mathbf{W}_1^2, \tag{2.2}$$

where **S** and p**I** represents the extra stress tensor and the indeterminate spherical stress, the dynamic viscosity is denoted by μ, normal stress moduli are represented by α_1 and α_2 and the kinematic tensors are $\mathbf{W}_1$ and $\mathbf{W}_2$ defined as

$$\mathbf{W}_1 = \mathbf{L} + \mathbf{L}^T,\ \mathbf{W}_2 = D_t^\beta + \mathbf{W}_1\mathbf{L} + \mathbf{L}^T\mathbf{W}_1 \tag{2.3}$$

where **L** is the velocity gradient and D_t^β represents the operator for fractional differentiation whose order is β and is based on the Riemann-Liouville definition [13],

$$D_a^b[g(a)] = \frac{1}{\Gamma(1-b)}\frac{d}{da}\int_0^a \frac{g(t)}{(a-t)^b}dt,\ 0 \le b < 1 \tag{2.4}$$

where Gamma function is denoted by $\Gamma(\cdot)$. Model for ordinary second-grade fluid can be obtained by putting $\beta = 1$. For the compatibility of this model with thermodynamics it is required that the material moduli should obey the following conditions

$$\alpha_1 + \alpha_2 = 0,\ 0 \le \alpha_1\ and\ \mu \ge 0. \tag{2.5}$$

For the second-grade fluid the Darcy's resistance satisfies the following equation:

$$\mathbf{R} = -\frac{\phi}{\kappa}(1 + \alpha_1\frac{\partial}{\partial t})\mathbf{V} \tag{2.6}$$

where k > 0 and $\phi(0 < \phi < 1)$ are the permeability and the porosity of the porous medium. For the following problem we consider the velocity field and an extra stress of the form

$$\mathbf{V} = (u(y,t), 0, 0),\ \mathbf{S} = S(y,t). \tag{2.7}$$

where u is the velocity taken in the x-direction. Substituting Eq.(7) into Eq.(2) and taking into account the initial condition

$$S(y,0)=0,\ y>0, \tag{2.8}$$

the fluid being at rest up to the time $t=0$, we get

$$S_{xy}=(\mu+\alpha_1 D_t^{\beta})\partial_y u(y,t), \tag{2.9}$$

where $S_{yy}=S_{zz}=S_{xz}=S_{yz}=0$, and $S_{xy}=S_{yx}$. The balance of linear momentum in the absence of body forces and pressure gradient is given as:

$$\partial_y S_{xy}-\sigma B_0^2 u-\frac{\mu\phi}{\kappa}(1+\alpha_1\frac{\partial}{\partial t})=\rho\partial_t u(y,t), \tag{2.10}$$

By putting S_{xy} from Eq. (9) into (10), we find the governing equation under the form

$$\rho\partial_t u(y,t)=(\mu+\alpha_1 D_t^{\beta})\partial_y^2 u(y,t)-\sigma B_0^2 u(y,t)-\frac{\phi}{\kappa}(\mu+\alpha_1\frac{\partial}{\partial t})u(y,t), \tag{2.11}$$

3. Statement of the problem

We take an unsteady incompressible flow of homogenous and electrically conducting second-grade fluid bounded by a rigid plate at $y=0$. The plate is taken normal to y-axis and the fluid saturates the porous medium $y>0$. The electrically conducting fluid is stressed by a uniform magnetic field $\mathbf{B}_0$ parallel to the y axis, while the induced magnetic field is neglected by choosing a small magnetic Reynolds number. Initially, both the plate and the fluid are at rest, and after time t=0, it is suddenly set into motion by translating the flate plate in its plane, with a constant velocity A. The initial and boundary conditions of velocity field are:

$$\begin{gathered} u(y,0)=0;\ y>0, \\ u(0,t)=A;\ t>0, \\ u(y,t),\ \partial_y u(y,t)\to 0\ as\ y\to\infty\ and\ t>0. \end{gathered} \tag{3.1}$$

4. Calculation of Velocity field

Employing the non-dimensional quantities

$$u^* = \frac{u}{U},\ y^* = \frac{yU}{\nu},\ t^* = \frac{tU^2}{\nu},\ \alpha^* = \frac{\alpha_1 U^2}{\rho\nu^2}, A^* = \frac{A}{U} \tag{4.1}$$

$$\tau = \frac{S}{\rho U^2},\ K = \frac{\kappa U^2}{\phi\nu^2},\ M^2 = \frac{\sigma\nu B_0^2}{\rho U^2},$$

The dimensionless mark * is omitted here for simplicity. Thus, the governing equations of dimensionless motion becomes

$$\partial_t u(y,t) = (1+\alpha D_t^\beta)\partial_y^2 u(y,t) - \frac{1}{K}(1+\alpha\frac{\partial}{\partial t})u(y,t) - M^2 u(y,t), \tag{4.2}$$

$$\tau(y,t) = (1+\alpha D_t^\beta)\partial_y u(y,t) \tag{4.3}$$

with the given conditions as

$$u(y,0) = 0;\ y > 0,$$

$$u(0,t) = A;\ t > 0, \tag{4.4}$$

$$u(y,t),\ \partial_y u(y,t) \to 0\ as\ y \to \infty,\ and\ t > 0.$$

First we will apply the Laplace transform to eq (14) and using the Laplace transform formula for sequential fractional derivatives [21]

$$\bar{u}(y,t) = \int_o^\infty u(y,t)e^{-st}dt,\ s \geq 0, \tag{4.5}$$

Taking into the account the corresponding initial and boundary conditions (16), we get the following differential equation

$$\partial_y^2 \bar{u}(y,q) - \left(\frac{1+\alpha q}{K(1+\alpha q^\beta)} + \frac{q+M^2}{1+\alpha q^\beta}\right)\bar{u}_(y,q) = 0,\ s \geq 0, \tag{4.6}$$

$$\bar{u}(0,q) = \frac{A}{q};\ t > 0,$$

$$\bar{u}(y,q),\ \partial_y \bar{u}(y,q) \to 0\ as\ y \to \infty,\ and\ q > 0. \tag{4.7}$$

The solution of Eq.(18) satisfying the boundary conditions (19) is of the following form:

$$\bar{u}(y,q) = \frac{A}{q} exp\left(-y\sqrt{\frac{1}{K(1+\alpha q^\beta)}((1+\alpha q)+K(q+M^2))}\right) \tag{4.8}$$

To get the analytical solution for velocity field and to avoid difficult calculations of contour integrals and residues, we will apply the discrete inverse Laplace transform method [21], but first we have to expressed Eq. (20) in series form as

$$\bar{u}(y,q) = \sum_{e_1=0}^{\infty}\sum_{f_1=0}^{\infty}\sum_{g_1=0}^{\infty}\sum_{h_1=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{A(-1)^{e_1+f_1+g_1+h_1+r+s}\alpha^{h_1+r+s}M^{2g_1}y^{e_1}}{e_1!f_1!g_1!h_1!r!s!K^{e_1/2-f_1}q^{-f_1-h_1-\beta r-s+1}} \times \frac{\Gamma(f_1-e_1/2)\Gamma(g_1-f_1)\Gamma(h_1+f_1)\Gamma(r+e_1/2)\Gamma(s-e_1/2)}{\Gamma(e_1/2)\Gamma(-e_1/2)\Gamma(e_1/2)\Gamma(f_1)\Gamma(-f_1)} \tag{4.9}$$

Now apply the discrete inverse Laplace transform to Eq. (22), we get

$$u(y,t) = \sum_{e_1=0}^{\infty}\sum_{f_1=0}^{\infty}\sum_{g_1=0}^{\infty}\sum_{h_1=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{A(-1)^{e_1+f_1+g_1+h_1+r+s}t^{-f_1-h_1-\beta r-s}\alpha^{h_1+r+s}}{e_1!f_1!g_1!h_1!r!s!\Gamma(-f_1)K^{e_1/2-f_1}} \times \frac{\Gamma(f_1-e_1/2)\Gamma(g_1-f_1)\Gamma(h_1+f_1)\Gamma(r+e_1/2)\Gamma(s-e_1/2)M^{2g_1}y^{e_1}}{\Gamma(e_1/2)\Gamma(-e_1/2)\Gamma(e_1/2)\Gamma(f_1)\Gamma(-f_1-h_1-\beta r-s+1)} \tag{4.10}$$

To get Eq. (23) in a more compact form we use Fox H-function [13],

$$u(y,t) = A\sum_{e_1=0}^{\infty}\sum_{f_1=0}^{\infty}\sum_{g_1=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{e_1+f_1+g_1+r+s}M^{2g_1}y^{e_1}t^{-f_1-\beta r-s+1}\alpha^{r+s}}{e_1!f_1!g_1!r!s!K^{e_1/2-f_1}} \times H^{1,5}_{5,7}\left[\frac{\alpha}{t}\left|\begin{matrix}(1-f_1+e_1/2,0),(1-g_1+f_1,0),(1-f_1,1),(1-s+e_1/2,0),\\(1-r-e_1/2,0).\\(1-e_1/2,0),(1-f_1,0),(1+f_1,0),(0,1),(1+e_1/2,0),\\(1-e_1/2,0),(f_1+\beta r+s,-1).\end{matrix}\right.\right] \tag{4.11}$$

To obtain Eq. (24), the following Fox H-function property has been used,

$$H^{1,s}_{s,t+1}\left[-\sigma\left|\begin{matrix}(1-a_1,A_1),...,(1-a_s,A_s)\\(1,0),(1-b_1,B_1),...,(1-b_t,B_t)\end{matrix}\right.\right] = \sum_{r=0}^{\infty}\frac{\Gamma(a_1+A_1r)...\Gamma(a_s+A_sr)}{r!\Gamma(b_1+B_1r)...\Gamma(b_t+B_tr)}\sigma^r.$$

5. Calculation of Shear Stress

To get the shear stress first we apply Laplace transform on Eq. (15), we get

$$\bar{\tau}(y,q) = (1+\alpha q^{\beta})\partial_y\bar{u}(y,q), \tag{5.1}$$

Substituting $\bar{u}(y,q)$ from eq. (20), we get

$$\bar{\tau}(y,t) = -\frac{A(1+\alpha q^{\beta})}{q} exp(-\sqrt{B}y)\sqrt{B}. \tag{5.2}$$

where

$$B = \frac{(1+\alpha q) + K(q+M^2)}{K(1+\alpha q^{\beta})}$$

To get a more compact form of $\bar{\tau}_(y,q)$, we write eq. (26) in series form as

(5.3)

$$\bar{\tau}(y,q) = \sum_{e_1=0}^{\infty}\sum_{f_1=0}^{\infty}\sum_{g_1=0}^{\infty}\sum_{h_1=0}^{\infty}\sum^{**}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{A(-1)^{e_1+f_1+g_1+h_1+\zeta_1+r+s+1}}{e_1!f_1!g_1!h_1!i_1!j_1!k_1!l_1!m_1!r!s!}$$
$$\times \frac{\alpha^{h_1+k_1+l_1+m_1+r+s}\Gamma(f_1-e_1/2)\Gamma(g_1-f_1)\Gamma(h_1+f_1)\Gamma(r+e_1/2)\Gamma(s-e_1/2)}{\Gamma(e_1/2)\Gamma(-e_1/2)\Gamma(e_1/2)\Gamma(f_1)\Gamma(-f_1)K^{e_1/2-f_1-i_1+1/2}}$$
$$\times \frac{M^{2g_1+2j_1}y^{e_1}\Gamma(i_1-1/2)\Gamma(j_1-i_1)\Gamma(k_1+i_1)\Gamma(l_1-1/2)\Gamma(m_1-1/2)}{\Gamma(1/2)\Gamma(1/2)\Gamma(1/2)\Gamma(i_1)\Gamma(-i_1)q^{-f_1-h_1-i_1-k_1-\beta l_1-m_1-\beta r-s+1}}$$

where

$$\sum^{**} = \sum_{i_1=0}^{\infty}\sum_{j_1=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{l_1=0}^{\infty}\sum_{m_1=0}^{\infty},$$
$$\zeta_1 = i_1+j_1+k_1+l_1+m_1,$$

Taking the inverse Laplace of eq.(27), we get

(5.4)

$$\tau(y,t) = \sum_{e_1=0}^{\infty}\sum_{f_1=0}^{\infty}\sum_{g_1=0}^{\infty}\sum_{h_1=0}^{\infty}\sum^{**}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{A(-1)^{e_1+f_1+g_1+h_1+\zeta_1+r+s+1}}{e_1!f_1!g_1!h_1!i_1!j_1!k_1!l_1!m_1!r!s!}$$
$$\times \frac{\alpha^{h_1+k_1+l_1+m_1+r+s}\Gamma(f_1-e_1/2)\Gamma(g_1-f_1)\Gamma(h_1+f_1)\Gamma(r+e_1/2)\Gamma(s-e_1/2)}{\Gamma(e_1/2)\Gamma(-e_1/2)\Gamma(e_1/2)\Gamma(f_1)\Gamma(-f_1)K^{e_1/2-f_1-i_1+1/2}}$$
$$\times \frac{M^{2g_1+2j_1}y^{e_1}\Gamma(i_1-1/2)\Gamma(j_1-i_1)\Gamma(k_1+i_1)\Gamma(l_1-1/2)\Gamma(m_1-1/2)}{\Gamma(1/2)\Gamma(1/2)\Gamma(1/2)\Gamma(i_1)\Gamma(-i_1)\Gamma(-f_1-h_1-i_1-k_1-\beta l_1-m_1-\beta r-s+1)}$$
$$\times\, t^{-f_1-h_1-i_1-k_1-\beta l_1-m_1-\beta r-s}$$

Finally, using Fox H-function to get the stress field as,

(5.5)

$$\tau(y,t) = \sum_{e_1=0}^{\infty}\sum_{f_1=0}^{\infty}\sum_{g_1=0}^{\infty}\sum^{**}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{A(-1)^{e_1+f_1+g_1+\zeta_1+r+s+1} y^{e_1}}{e_1!f_1!g_1!i_1!j_1!k_1!l_1!m_1!r!s!M^{-2g_1-2j_1}} \times \frac{t^{-f_1-i_1-k_1-\beta l_1-m_1-\beta r-s}}{\alpha^{-k_1-l_1-m_1-r-s} K^{e_1/2-f_1-i_1+1/2}}$$

$$H^{1,10}_{10,12}\left[\frac{\alpha}{t}\left|\begin{array}{c} (-i_1+3/2,0),(1-j_1+i_1,0),(1-k_1-i_1,0),(1-l_1+1/2,0),\\ (1-f_1,1),(1-s+e_1/2,0),(1-r-e_1/2,0),(1-f_1+e_1/2,0),\\ (-m_1+3/2,0),(1-g_1+f_1,0).\\ (1/2,0),(1-i_1,0),(1+i_1,0),(1-e_1/2,0),(1-f_1,0),(0,1),\\ (1+f_1,0),(1-e_1/2,0),(1/2,0),(1/2,0),(1+e_1/2,0)\\ (f_1+i_1+k_1+\beta l_1+m_1+\beta r+s,-1).\end{array}\right.\right]$$

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