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FEKETE-SZEGÖ PROBLEM FOR SOME SUBCLASSES OF COMPLEX ORDER RELATED TO SĂLĂGEAN OPERATOR

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ABSTRACT. In the present investigation,sharp upper bounds of $\left|a_3 - \mu a_2^2\right|$ for function $f(z)$ belonging to certain subclasses of $Re\left[1+\frac{1}{b}\left\{(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)-1\right\}\right] \succ 0$ are obtained.Also certain applications of the main results for subclasses of functions defined by convolution with a normalized analytic function are given.In particular,$Fekete - Szegö$ inequalities for certain classes of functions defined through fractional derivatives are obtained.

1. INTRODUCTION

We let $\mathcal{A}$ to denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in D = \{z \in \mathbb{C} : |z| < 1\}) \tag{1.1}$$

and $\mathbb{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. For two analytic functions f(z) given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their convolution (or Hadamard product) is defined to be the function $(f * g)(z)$ given by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$.

For two functions $f, g \in \mathcal{A}$,we say that the function $f(z)$ is *subordinate* to $g(z)$ in D and write $f \prec g$ or $f(z) \prec g(z) \quad (z \in D)$, if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in D)$,such that

$$f(z) = g(w(z)) \quad (z \in D).$$

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In particular , if the function g is univalent in D ,the above subordination is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$.

Throughout this paper, we assume that ϕ is an analytic univalent function with positive real part in D , $\phi(D)$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$, and $\phi'(0) > 0$.The Taylor's series expansion of such function is of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots with \quad B_1 > 0. \tag{1.2}$$

In the present investigation,we obtain Fekete-$Szeg\ddot{o}$ inequality for function in a more general class $\Re(\alpha, \phi)$ which we define below.We also give applications of our results to certain functions defined through Hadamard product and functions defined by fractional derivatives.

Definition 1.1. Let $\alpha \geq 0$. A function $f \in \mathcal{A}$ given by(1.1) is in the class $\Re(\alpha, \phi)$,if it satisfies

$$Re\left[1 + \frac{1}{b}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1\right\}\right] \succ 0 \tag{1.3}$$

Lemma 1.1. *If $p_1(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ is analytic function with positive real part in D,then*

$$\left|c_2 - v c_1^2\right| \leq \begin{cases} -4v + 2 & if \quad v \leq 0, \\ 2 & if \quad 0 \leq v \leq 1, \\ 4v - 2 & if \quad v \geq 1. \end{cases}$$

when $v < 0$ or $v > 1$, the equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations.If $0 < v < 1$,then the equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations.If $v = 0$,the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad , (0 \leq \lambda \leq 1)$$

or one of its rotations.If $v = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case $v = 0$.Also the above upper bound is sharp and it can be improved as follows:
when $0 < v < 1$,

$$\left|c_2 - v c_1^2\right| + v\left|c_1\right|^2 \leq 2 \quad (0 < v \leq 1/2)$$

and

$$\left|c_2 - vc_1^2\right| + (1-v)\left|c_1\right|^2 \le 2 \quad (1/2 < v \le 1).$$

Let a differential operator be defined Sălăgean [10] on a class of analytic functions of the form (1.1) as follows

$$D^0 f(z) = f(z), \qquad D^1 f(z) = Df(z) = zf'(z),$$

and in general

$$D^n f(z) = D\left(D^{n-1} f(z)\right), (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

We easily find that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^n \quad (n \in \mathbb{N}_0). \tag{1.4}$$

2. Fekete-SzegÖ Poroblem for the Function class $\Re(\alpha, \phi)$

By using Lemma 1.1 ,we prove the following Fekete-*Szegö* inequalities.

Theorem 2.1. *Let b be a non zero complex number.If $f(z)$ given by (1.1) belongs to $\mathcal{N}_n^b(\alpha,\phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{b}{3^n}\left[\frac{B_2}{1+2\alpha} - \frac{\mu B_1^2 b}{(1+\alpha)^2}\left(\frac{3}{4}\right)^n\right] & if \quad \mu \le \sigma_1, \\ \frac{bB_1}{3^n(1+2\alpha)} & if \quad \sigma_1 \le \mu \le \sigma_2, \\ -\frac{b}{3^n}\left[\frac{B_2}{1+2\alpha} - \frac{\mu B_1^2 b}{(1+\alpha)^2}\left(\frac{3}{4}\right)^n\right] & if \quad \mu \ge \sigma_2. \end{cases}$$

where

$$\sigma_1 := \frac{(1+\alpha)^2 (B_2 - B_1)}{b(1+2\alpha) B_1^2}\left(\frac{4}{3}\right)^n \quad , \qquad \sigma_2 := \frac{(1+\alpha)^2 (B_2 + B_1)}{b(1+2\alpha) B_1^2}\left(\frac{4}{3}\right)^n$$

The result is sharp.

If $f(z) \in \Re(\alpha,\phi)$, then there exists a Schwarz functions $w(z)$ analytic in D with
$w(0) = 0$ and $|w(z)| < 1 \quad (z \in D)$,such that

$$1 + \frac{1}{b}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1\right\} = \phi(w(z)) \tag{2.1}$$

Define the function p_1 by

$$p_1 = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{2.2}$$

since $w(z)$ is a Schwarz function,we see that $\Re(p_1(z)) > 0\,(z \in D)$ and $p_1(0) = 1$.Now, defining the function $p(z)$ by

(2.3)

$$p(z) = 1 + \frac{1}{b}\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1\right\} = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$$

we find from (2.1) and (2.2) that

$$p(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{2.4}$$

Thus by using (2.2) in (2.4), we obtain

$$b_1 = \frac{1}{2}B_1 c_1 \qquad b_2 = \frac{1}{2}B_1\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{1}{4}B_2 c_1^2.$$

Also, from (2.3) we obtain

$$b_1 = \frac{1}{b}(1+\alpha)\,2^n a_2 \qquad b_2 = \frac{1}{b}(1+2\alpha)\,3^n a_3.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{3^n 2(1+2\alpha)}\left[c_2 - v c_1^2\right]. \tag{2.5}$$

where

$$v := \frac{1}{2}\left[1 - \frac{B_2}{B_1} + \frac{b\mu B_1(1+2\alpha)}{(1+\alpha)^2}\left(\frac{3}{4}\right)^n\right].$$

Our result now follows by an application of lemma 1.1 .To show that the bounds are sharp,we define the functions $K_{\phi_n}^{\alpha} \quad (n = 2, 3, 4, \ldots)$ by

$$1 + \frac{1}{b}\left\{(1-\alpha)\frac{\left[K_{\phi_n}^{\alpha}(z)\right]}{z} + \alpha\left[K_{\phi_n}^{\alpha}\right]'(z) - 1\right\} = \phi\left(z^{n-1}\right),$$

$$K_{\phi_n}^{\alpha}(0) = 0 = \left[K_{\phi_n}^{\alpha}\right]'(0) - 1$$

and the function F_λ^α and $G_\lambda^\alpha \quad (0 \le \lambda \le 1)$ by

$$1 + \frac{1}{b}\left\{(1-\alpha)\frac{\left[F_\lambda^\alpha(z)\right]}{z} + \alpha\left[F_\lambda^\alpha\right]'(z) - 1\right\} = \phi\left(z^{n-1}\right),$$

$$F_\lambda^\alpha(0) = 0 = \left[F_\lambda^\alpha\right]'(0) - 1$$

$$1+\frac{1}{b}\left\{(1-\alpha)\frac{[G_\lambda^\alpha(z)]}{z}+\alpha\,[G_\lambda^\alpha]'(z)-1\right\}=\phi\left(z^{n-1}\right),$$

$$G_\lambda^\alpha(0)=0=[G_\lambda^\alpha]'(0)-1$$

Clearly the functions $K_{\phi_n}^\alpha$, F_λ^α and $G_\lambda^\alpha \in \Re(\alpha,\phi)$.Also we write $K_\phi^\alpha := K_{\phi_2}^\alpha$.

If $\mu \le \sigma_1$ or $\mu \ge \sigma_2$, then the equality holds if and only if f is K_ϕ^α or one of its rotations.When $\sigma_1 \le \mu \le \sigma_2$, the equality holds if and only if f is $K_{\phi_3}^\alpha$ or one of its rotations.If $\mu = \sigma_1$, then the equality holds if and only if f is F_λ^α or one of its rotations.If $\mu = \sigma_2$, then the equality holds if and only if f is G_λ^α or one of its rotations.

Remark 2.1. If $\sigma_1 \le \mu \le \sigma_2$, then,in view of Lemma 1.1 , Theorem 2.1 can be improved. Let σ_3 be given by

$$\sigma_3 := \frac{(1+\alpha)^2 B_2}{b(1+2\alpha)B_1^2}\left(\frac{4}{3}\right)^n.$$

Let $f \in \Re(\alpha,\phi)$.If $\sigma_1 \le \mu \le \sigma_3$, then

$$\left|a_3-\mu a_2^2\right|+\frac{1}{b(1+2\alpha)B_1^2}\left(\frac{4}{3}\right)^n\left[(1+\alpha)^2(B_2-B_1)\left(\frac{3}{4}\right)^n\right.$$

$$\left.+\mu b(1+2\alpha)B_1^2\left(\frac{4}{3}\right)^n|a_2|^2\le\frac{bB_1}{3^n(1+2\alpha)}.\right.$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\left|a_3-\mu a_2^2\right|+\frac{1}{b(1+2\alpha)B_1^2}\left(\frac{4}{3}\right)^n\left[(1+\alpha)^2(B_2+B_1)\left(\frac{3}{4}\right)^n\right.$$

$$\left.-\mu b(1+2\alpha)B_1^2\left(\frac{4}{3}\right)^n|a_2|^2\le\frac{bB_1}{3^n(1+2\alpha)}.\right.$$

For $\phi(z) = (1+Cz)/(1+Dz)$, $-1 \le D < C \le 1$.Theorem 2.1 leads to the following results:

Corollary 2.2. *Let* $-1 \le D < C \le 1$.*If* $f \in \Re(\alpha,(1+Cz)/(1+Dz))$,*then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{b(D-C)}{3^n(1+2\alpha)}\left[D-\frac{b\mu(D-C)(1+2\alpha)}{2(1+\alpha)^2}\left(\frac{3}{4}\right)^n\right] & if\ \mu \le -\frac{2}{b}\left[\frac{(1+D)(1+\alpha)^2}{(1+2\alpha)(C-D)}\left(\frac{4}{3}\right)^n\right], \\ \frac{b(C-D)}{3^n(1+2\alpha)} \quad if\ -\frac{2}{b}\left[\frac{(1+D)(1+\alpha)^2}{(1+2\alpha)(C-D)}\left(\frac{4}{3}\right)^n\right] \le \mu \le \frac{2}{b}\left[\frac{(1-D)(1+\alpha)^2}{(1+2\alpha)(C-D)}\left(\frac{4}{3}\right)^n\right], & \\ -\frac{b(D-C)}{3^n(1+2\alpha)}\left[D-\frac{b\mu(D-C)(1+2\alpha)}{2(1+\alpha)^2}\left(\frac{3}{4}\right)^n\right] & if\ \mu \le \frac{2}{b}\left[\frac{(1-D)(1+\alpha)^2}{(1+2\alpha)(C-D)}\left(\frac{4}{3}\right)^n\right]. \end{cases}$$

For $\phi(z) = (1+z)/(1-z)$,Theorem 2.1 leads to the following results:

Corollary 2.3. *Let* $-1 \le D < C \le 1$.*If* $f \in \Re(\alpha, (1+z)/(1-z))$,*then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{2b}{3^n(1+2\alpha)}\left[1-\frac{\mu b(1+2\alpha)}{(1+\alpha)^2}\left(\frac{3}{4}\right)^n\right] & if \quad \mu \le 0, \\ \frac{2b}{3^n(1+2\alpha)} & if \quad 0 \le \mu \le \frac{2(1+\alpha)^2}{b(1+2\alpha)}\left(\frac{4}{3}\right)^n, \\ -\frac{2b}{3^n(1+2\alpha)}\left[1-\frac{\mu b(1+2\alpha)}{(1+\alpha)^2}\left(\frac{3}{4}\right)^n\right] & if \quad \mu \ge \frac{2(1+\alpha)^2}{b(1+2\alpha)}\left(\frac{4}{3}\right)^n. \end{cases}$$

For $C = 1-2\beta$ with $0 \le \beta < 1$ and $D = -1$, Corollary 2.2 reduces to the following result:

Corollary 2.4.

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \frac{2b(1-\beta)}{3^n(1+2\alpha)}\left[1-\frac{\mu b(1-\beta)(1+2\alpha)}{(1+\alpha)^2}\left(\frac{3}{4}\right)^n\right] & if \quad \mu \le 0, \\ \frac{2b(1-\beta)}{3^n(1+2\alpha)} & if \quad 0 \le \mu \le \frac{2(1+\alpha)^2}{b(1-\beta)(1+2\alpha)}\left(\frac{4}{3}\right)^n, \\ -\frac{2b(1-\beta)}{3^n(1+2\alpha)}\left[1-\frac{\mu b(1-\beta)(1+2\alpha)}{(1+\alpha)^2}\left(\frac{3}{4}\right)^n\right] & if \quad \mu \ge \frac{2(1+\alpha)^2}{b(1-\beta)(1+2\alpha)}\left(\frac{4}{3}\right)^n. \end{cases}$$

3. Application to Functions Defined by Fractional Derivatives

In order to introduce the class $\Re^{\lambda}(\alpha, \phi)$, we need the following:

Definition 3.1. see ([2, 3] , see also [7, 8]).Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin.The *fractional derivative of* f of order λ is defined by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \leq \lambda < 1),$$

where the multiplicity of $(z-\zeta)^{\lambda}$ is removed by requiring that $log\,(z-\zeta)$ is real for $z-\zeta > 0$. Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [3] introduced the operator $\Omega^{\lambda} : \mathcal{A} \to \mathcal{A}$ defined by

$$\left(\Omega^{\lambda} f\right)(z) = \Gamma(2-\lambda)\, z^{\lambda} D_z^{\lambda} f(z) \quad (\lambda \neq 2, 3, 4, \ldots).$$

The class $\Re^{\lambda}(\alpha, \phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^{\lambda} f \in \Re(\alpha, \phi)$. Note that $\Re^{\lambda}(\alpha, \phi)$ is the special case of the class $\Re^{g}(\alpha, \phi)$ when

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\,\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n. \tag{3.1}$$

Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0)$.Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \Re^{g}(\alpha, \phi)$ if and only if $(f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \Re(\alpha, \phi)$,we obtain the coefficient estimate for functions in the class $\Re^{g}(\lambda, \phi)$, from the corresponding estimate for functions in the class $\Re(\lambda, \phi)$.Applying Theorem 2.1 for the function $(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots$,we get the following theorem after an obvious change of the parameter μ:

Theorem 3.1. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$.If $f(z)$ given by (1.1) belongs to $\Re^{g}(\alpha, \phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{b}{g_3}\left[\frac{B_2}{1+2\alpha} - \frac{\mu b g_3 B_1^2}{(1+\alpha)^2 g_2^2}\right] & if \quad \mu \leq \sigma_1, \\ \frac{bB_1}{g_3(1+2\alpha)} & if \quad \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{b}{g_3}\left[\frac{B_2}{1+2\alpha} - \frac{\mu b g_3 B_1^2}{(1+\alpha)^2 g_2^2}\right] & if \quad \mu \geq \sigma_2. \end{cases}$$

where

$$\sigma_1 := \frac{g_2^2 (1+\alpha)^2 (B_2 - B_1)}{b g_3 (1+2\alpha) B_1^2} \quad , \quad \sigma_1 := \frac{g_2^2 (1+\alpha)^2 (B_2 + B_1)}{b g_3 (1+2\alpha) B_1^2}$$

The result is sharp.

Since

$$\left(\Omega^{\lambda} f\right)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n \quad ,$$

we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \tag{3.2}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \tag{3.3}$$

For g_2 and g_3 given by (3.2) and (3.3) , Theorem 3.1 reduces to the following:

Theorem 3.2. *Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$.If $f(z)$ given by (1.1) belongs to $\Re^{\lambda}(\alpha, \phi)$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{b(2-\lambda)(3-\lambda)}{6}\left[\frac{B_2}{1+2\alpha} - \frac{\mu b 3(2-\lambda)B_1^2}{2(1+\alpha)^2(3-\lambda)}\right] & if \quad \mu \leq \sigma_1, \\ \frac{b(2-\lambda)(3-\lambda)B_1}{6(1+2\alpha)} & if \quad \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{b(2-\lambda)(3-\lambda)}{6}\left[\frac{B_2}{1+2\alpha} - \frac{\mu b 3(2-\lambda)B_1^2}{2(1+\alpha)^2(3-\lambda)}\right] & if \quad \mu \geq \sigma_2. \end{cases}$$

where

$$\sigma_1 := \frac{2(3-\lambda)(1+\alpha)^2(B_2 - B_1)}{3b(2-\lambda)(1+2\alpha)B_1^2} \quad , \qquad \sigma_2 := \frac{2(3-\lambda)(1+\alpha)^2(B_2 + B_1)}{3b(2-\lambda)(1+2\alpha)B_1^2}$$

The result is sharp.

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