ISSN: 2307-7743
www.scienceasia.asia

Asian J. Math. Appl. (2022) 2022:1

Perron Complementation On Linear Systems Involving M-Matrices

Jianhong Xu

School of Mathematical & Statistical Sciences, Southern Illinois University Carbondale, Carbondale, Illinois 62901, U.S.A.
Email(s): jhxu@siu.edu

Abstract
We propose in this paper a generalized Perron complementation method for uncoupling a consistent linear system which involves an irreducible, either singular or nonsingular, M-matrix. We show that this uncoupling arises naturally from a regular splitting, which also leads to an efficient iterative scheme for solving the linear system.

Keywords: Linear System; M-Matrix; Regular Splitting; Perron Complement; Iterative Method; Convergence.

Introduction

Linear systems involving M-matrices, i.e. whose coefficient matrices are M-matrices, arise in a wide variety of fields such as finite Markov chains and partial differential equations. Such linear systems, therefore, have been investigated extensively in literature. A majority of the solution methods are iterative, because many applications lead to large, often sparse, linear systems; see [5, 6, 9, 11, 22, 24, 28] for Jacobi, Gauss-Seidel, SOR, and some preconditioned or accelerated variants, and see [2, 7, 12, 18, 23] for multi-splitting and two-stage methods.

Another useful approach to dealing with large linear systems is divide and conquer methods. In particular, Meyer showed in [15] that the Perron eigenproblem can be uncoupled into smaller subproblems via the so-called Perron complementation. Such an eigenproblem is indeed a special case of a linear system which involves an M-matrix. The Perron complementation technique was later applied to the computation of mean first passage times for Markov chains [8] and group generalized inverses of certain Q-matrices [21].[1] However, the Perron complementation over linear systems featuring M-matrices has yet to be addressed.

In this paper, we show how Perron complementation emerges naturally on a linear system involving an M-matrix through some regular splitting. This results in an uncoupling of the linear system; furthermore, the regular splitting leads to an efficient iterative scheme for solving the linear system.

[1] See [24] for the definition of Q-matrices.

Received: 2021-5-28

Main Results

For any square matrix X, we shall denote by $\sigma(X)$ and $\rho(X)$ the spectrum and the spectral radius of X, respectively.

Throughout this paper, $A = [a_{i,j}]$ represents an $n \times n$ irreducible M-matrix. We mention here a few relevant characterizations of such a matrix, which are necessary for the subsequent development. First, there exists an irreducible nonnegative matrix B such that

$$A = rI - B, \tag{1}$$

where I is an identity matrix and $r \geq \rho(B)$; especially, A is nonsingular if and only if $r > \rho(B)$. Second, $\rho(B) > 0$ due to the celebrated Perron-Frobenius Theorem (see, for example, [16]). Third, $A \in \mathfrak{X}$, i.e. $a_{i,j} \leq 0$ for all $i \neq j$. Fourth, any (proper) principal submatrix of A is a nonsingular M-matrix. Last, the inverse of a nonsingular M-matrix is nonnegative. For detailed background material on M-matrices, we refer the reader to [1].

Consider now a consistent linear system which involves the above A in the form

$$Ax = b, \tag{2}$$

where $b \in \mathcal{R}(A)$, the range of A. In what follows, we shall focus on a new uncoupling algorithm, along with an effective iterative scheme, for solving the linear system (2).

We assume first that A is already expressed as in (1). Partition B into

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}, \tag{3}$$

where the diagonal blocks are all square, with $B_{1,1}$ being $k \times k$. According to [19], the generalized Perron complement of $B_{1,1}$ in B is defined to be

$$G = B_{2,2} + B_{2,1}(rI - B_{1,1})^{-1}B_{1,2}. \tag{4}$$

As shown in [14], G is an irreducible nonnegative matrix with $\rho(G) \leq \rho(B)$, where equality occurs if and only if $r = \rho(B)$. Thus, $F = rI - G$ is again an irreducible M-matrix, but of a smaller size; moreover, F is nonsingular exactly when A is so. The notion of Perron complementation turns out to play a key role in uncoupling the linear system (2).

We point out in passing that the generalized Perron complement can be similarly defined on any principal submatrix of B. For this reason, results obtained from the 2×2 partitioning of B can be extended to include an arbitrary principal submatrix of B or the case when B is partitioned in a general $m \times m$ block form with square diagonal blocks.

For any square matrix X, $X = M - N$, with M being nonsingular, is called a regular splitting when $M^{-1} \geq 0$ and $N \geq 0$; see, for example, [27]. Partition A in conformity with (3) as

$$A = \begin{bmatrix} rI - B_{1,1} & -B_{1,2} \\ -B_{2,1} & rI - B_{2,2} \end{bmatrix}.$$

Let

$$M = \begin{bmatrix} rI - B_{1,1} & 0 \\ -B_{2,1} & rI \end{bmatrix} \quad \text{and} \quad N = M - A = \begin{bmatrix} 0 & B_{1,2} \\ 0 & B_{2,2} \end{bmatrix}. \tag{5}$$

It can be easily seen that $N \geq 0$ and

$$M^{-1} = \begin{bmatrix} (rI - B_{1,1})^{-1} & 0 \\ r^{-1}B_{2,1}(rI - B_{1,1})^{-1} & r^{-1}I \end{bmatrix} \geq 0,$$

showing that (5) defines a (nontrivial) regular splitting of A. This regular splitting leads to the following uncoupling of the linear system (2).

THEOREM 1 *Let the linear system (2) be partitioned in accordance to (3), with* $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$*. Suppose that* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *is a solution to (2). Then* x_1 *and* x_2 *satisfy*

$$Fx_2 = c_2, \tag{6}$$

where $F = rI - G$*, with* G *being the generalized Perron complement (4) of* $B_{1,1}$ *in* B*, and*

$$c_2 = B_{2,1}(rI - B_{1,1})^{-1}b_1 + b_2,$$

and

$$(rI - B_{1,1})x_1 = B_{1,2}x_2 + b_1. \tag{7}$$

Proof: By the preceding regular splitting of A, (2) is equivalent to

$$(I - M^{-1}N)x = M^{-1}b.$$

From (5), we further obtain

$$\begin{bmatrix} I & -(rI - B_{1,1})^{-1}B_{1,2} \\ 0 & I - r^{-1}Q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (rI - B_{1,1})^{-1}b_1 \\ r^{-1}B_{2,1}(rI - B_{1,1})^{-1}b_1 + r^{-1}b_2 \end{bmatrix},$$

which yields (6) and (7). □

In the sequel, we shall refer to (6) and (7) as the reduced and the companion linear systems, respectively, for (2). Note that (6) is also a linear system involving an irreducible M-matrix.

In particular, on setting $b = 0$ and $r = \rho(B)$, the linear system (2) translates into the (right) Perron eigenproblem

$$Bx = \rho(B)x.$$

Accordingly, Theorem 1 reduces to

COROLLARY 1 [15, Theorem 2.1] *Given the irreducible nonnegative matrix* B *as partitioned in (3) with its Perron eigenvector* x *being conformably partitioned as* $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$*, it follows that*

$$Gx_2 = \rho(B)x_2.$$

Theorem 1 suggests that the solution to (2) can be computed in two separate phases — first on the $(n-k) \times (n-k)$ reduced linear system (6), and then on the $k \times k$ companion linear system (7). However, one concern regarding this uncoupling approach comes from its complexity. To address this concern, we estimate as follows the amount of work for the case when $k \approx n/2$, assuming that the solution method is LU factorization. [2]

[2] See [4, 10] for LU factorizations of nonsingular or irreducible singular M-matrices.

(1) To form the generalized Perron complement G, we first solve

$$Y(rI - B_{1,1}) = B_{2,1}$$

by an LU factorization, which requires $2k^3/3 + 2k^2(n-k)$ operations, and we then calculate $YB_{1,2}$, which adds an extra $2k^2(n-k)$ operations.

(2) The LU factorization on the reduced linear system (6) needs $2(n-k)^3/3$ operations.

(3) The matrix Y and the LU factorization of $rI - B_{1,1}$ are stored to be used in the right-hand side of the reduced linear system (6) and in the solution of the companion linear system (7), respectively.

The above two-phase approach, therefore, requires a total of

$$t = 2k^3/3 + 2(n-k)^3/3 + 4k^2(n-k) = 2n^3/3 - 2k(n-k)(n-2k)$$

operations. In particular, $t \le 2n^3/3$ when $k \le n/2$. This explains that, in view of efficiency, we should choose a k-value not exceeding $n/2$. Consequently, the two-phase method can be expected to be less costly than a direct LU factorization on the entire linear system.

The two-phase solution process may be deployed recursively on reduced linear systems. To compute x_2 from (6), for example, we may partition G into 2×2 blocks similar to (3), then set up and solve the reduced and the companion linear systems for (6).

We remark that the reduced linear system (6) is also suitable for computing selected entries in the solution to the linear system (2). Let α be a nonempty (proper) subset of $\langle n \rangle = \{1, \ldots, n\}$, arranged in ascending order. The generalized Perron complement on $\langle n \rangle \backslash \alpha$ in $\langle n \rangle$ is similarly defined as

$$G = B[\alpha, \alpha] + B[\alpha, \langle n \rangle \backslash \alpha](rI - B[\langle n \rangle \backslash \alpha, \langle n \rangle \backslash \alpha])^{-1} B[\langle n \rangle \backslash \alpha, \alpha], \tag{8}$$

where $B[\alpha, \beta]$ is the principal submatrix of B on rows α and columns β. A similar notation is adopted for vectors, i.e. $x[\alpha]$ represents the entries of x indexed by α. Then, according to Theorem 1, we arrive at

COROLLARY 2 *Given α, a nonempty subset of $\langle n \rangle = \{1, \ldots, n\}$, and G as in (8), the entries $x[\alpha]$ in the solution to (2) can be determined by*

$$(rI - G)x[\alpha] = b[\alpha] + B[\alpha, \langle n \rangle \backslash \alpha](rI - B[\langle n \rangle \backslash \alpha, \langle n \rangle \backslash \alpha])^{-1} b[\langle n \rangle \backslash \alpha].$$

In addition to the uncoupling of the linear system (2), the regular splitting in (5) provides a way of computing the solution to (2) by iterations. This approach applies to subsequent reduced linear systems as well if the two-phase process is used recursively. We proceed next to such an iterative scheme and the issue of its convergence.

The iterative formula on (2) induced naturally via the regular splitting in (5) is given by

$$x^{(i+1)} = Hx^{(i)} + c, \quad i = 0, 1, \ldots, \tag{9}$$

where $H = M^{-1}N$ and $c = M^{-1}b$. It is known that H is convergent, i.e. $\lim_{i\to\infty} H^i = 0$, if and only if A is nonsingular [27, Theorem 3.29]. In this case, $\rho(H) < 1$; moreover, the asymptotic rate of convergence is given by $-\ln \rho(H)$. The conclusion below indicates that the iteration (9) converges faster on reduced linear systems than on the entire linear system (2).

THEOREM 2 *Suppose that A is nonsingular. Let H_A be the iteration matrix which is induced by the regular splitting (5) on the linear system (2). Let H_F be the iteration matrix which is induced by any subsequent regular splitting of the same type on the reduced linear system (6). Then*

$$-\ln \rho(H_F) > -\ln \rho(H_A).$$

Proof: From (5), we have

$$H_A = \begin{bmatrix} 0 & (rI - B_{1,1})^{-1}B_{1,2} \\ 0 & r^{-1}G \end{bmatrix}.$$

Hence $\rho(H_A) = r^{-1}\rho(G) < 1$.

On the other hand, we partition G into 2×2 blocks as

$$\begin{bmatrix} G_{1,1} & G_{1,2} \\ G_{2,1} & G_{2,2} \end{bmatrix},$$

where $G_{1,1}$ is $m \times m$ such that $1 \le m < n-k$. Using a similar regular splitting as in (5) but on the reduced linear system (6), we obtain

$$H_F = \begin{bmatrix} 0 & (rI - G_{1,1})^{-1}G_{1,2} \\ 0 & r^{-1}K \end{bmatrix},$$

where K is the generalized Perron complement of $G_{1,1}$ in G. This implies $\rho(H_F) = r^{-1}\rho(K) < 1$.

Hence the conclusion follows due to the fact $\rho(K) < \rho(G)$. □

With the nonsingular case being clarified, we now turn to the scenario when A is singular.

For a square matrix X, denote $\delta(X) = \max\{|\lambda| : \lambda \in \sigma(X), \lambda \neq 1\}$. The index of X, $\mathrm{ind}(X)$, is defined to be the smallest nonnegative integer m such that $\mathrm{rank}(X^{m+1}) = \mathrm{rank}(X^m)$. The convergence of (9) can then be characterized as:

LEMMA 1 ([1, p. 198]) *The iteration (9) converges for any initial $x^{(0)}$ if*

(1) $\delta(H) < 1$ *and*

(2) $\mathrm{ind}(I - H) = 1$.

A matrix H satisfying the two conditions in Lemma 1 is said to be semiconvergent; besides, $\lim_{k\to\infty} H^k$ exists. With the regular splitting of A in (5), we show next that $H = M^{-1}N$ is indeed semiconvergent when A is singular.

THEOREM 3 *For the splitting M and N as defined in (5), when A is singular, $H = M^{-1}N$ is semiconvergent.*

Proof: According to (5), we have

$$H = \begin{bmatrix} 0 & (rI - B_{1,1})^{-1}B_{1,2} \\ 0 & r^{-1}G \end{bmatrix},$$

where $r = \rho(G)$. Thus $\sigma(H) = \sigma(r^{-1}G) \cup \{0\}$. The conclusion $\delta(H) < 1$ follows from the irreducibility of the generalized Perron complement G.

On the other hand,

$$I - H = \begin{bmatrix} I & -(rI - B_{1,1})^{-1}B_{1,2} \\ 0 & I - r^{-1}G \end{bmatrix}.$$

By [17, Lemma 1], $\mathrm{ind}(I - r^{-1}G) = 1$. In addition, from [3, Theorem 7.7.2],

$$\max\{\mathrm{ind}(I), \mathrm{ind}(I - r^{-1}G)\} \le \mathrm{ind}(I - H) \le \mathrm{ind}(I) + \mathrm{ind}(I - r^{-1}G),$$

which yields $\operatorname{ind}(I - H) = 1$. □

Lemma 3 and Theorem 1 guarantee that the iteration formula (9) converges to a solution of the linear system (2) when A is singular. In the proof of Theorem 1, we observe that the asymptotic rate of convergence $-\ln \delta(H)$ (see [1]) depends on a subdominant eigenvalue of G, i.e. $\delta(G)$. Considering a similar situation as in Theorem 2, we comment that, in general, $\delta(K)$ may not be smaller than $\delta(G)$. However, in the special case when A is also a Q-matrix with zero row sums, it is shown in [20] that

$$Z(K) \le Z(G),$$

where $Z(\cdot)$ is the coefficient of ergodicity [25] which bounds subdominant eigenvalues; hence the asymptotic rate of convergence can be expected not to degenerate much as the solution process is implemented using reduced linear systems.

There is one remaining question which concerns the choice of r and B given the linear system (2), as the uncoupling and the iteration processes both hinge on these quantities. To answer this question, we have

THEOREM 4 *Let $A = [a_{i,j}]$, which is an $n \times n$ irreducible singular M-matrix. Set $r = \max_i a_{i,i}$. Then*

(1) $a_{i,i} > 0$ for all i.

(2) $B = rI - A$ is irreducible and nonnegative.

(3) $r = \rho(B)$

Proof: By [1, Theorem 6.4.16], there exists some $x > 0$ such that $Ax = 0$. Therefore, for any i,

$$a_{i.i}x_i + \sum_{j \ne i} a_{i,j}x_j = 0.$$

It is clear that $a_{i,i} \ge 0$. If $a_{i,i} = 0$, then $a_{i,j} = 0$ for all $j \ne i$, which is a contradiction to the irreducibility of A.

Partition A as

$$A = \begin{bmatrix} A_{1,1} & u \\ v^T & w \end{bmatrix},$$

where $w = a_{n,n}$ and, without loss of generality, $w = \max_i a_{i,i}$. Due to the singularity of A,

$$w = v^T A_{1,1}^{-1} u. \tag{10}$$

Obviously, $B = wI - A \ge 0$ and it is irreducible. It remains to show that $w = \rho(B)$. Note that

$$B = \begin{bmatrix} wI - A_{1,1} & -u \\ -v^T & 0 \end{bmatrix},$$

which implies that w is indeed an eigenvalue of B; in addition, on letting $y = \begin{bmatrix} -A_{1,1}^{-1}u \\ 1 \end{bmatrix}$, we see that $By = wy$. Since $y \ge 0$ but $y \ne 0$, it follows, according to [1, Theorem 2.1.4], that y must be an eigenvector associated with $\rho(B)$, i.e. $w = \rho(B)$. □

When A is nonsingular, the conclusions (1) and (2) in Theorem 4, together with $r > \rho(B)$, follow directly from [1, Theorem 6.2.3]. However, the case involving an irreducible singular M-matrix does not seem to be stated explicitly in literature except for the case of irreducible Q-matrices.[3] Our proof here is different in that no Q-matrix argument is needed.

[3] In fact, the redundant condition $a_{i,i} > 0$ appears in some existing work; see [2, p. 309].

Algorithm and Examples

We begin with a description of the uncoupling algorithm for computing a solution to the linear system (2). The situation of a two-level uncoupling is illustrated here. The first level is used to construct the reduced and the companion linear systems, while the second level is used to carry out the iterative process for solving the reduced linear system.

Write the linear system (2) as $A^{(1)}x = b^{(1)}$, with $A^{(1)} = [a_{i,j}^{(1)}]$ and $b^{(1)}$ being partitioned conformably as

$$A^{(1)} = \begin{bmatrix} A_{1,1}^{(1)} & A_{1,2}^{(1)} \\ A_{2,1}^{(1)} & A_{2,2}^{(1)} \end{bmatrix} \text{ and } b^{(1)} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \end{bmatrix}.$$

Choose $r = \max_i a_{i,i}^{(1)}$. Set $B^{(1)} = rI - A^{(1)}$, partitioned accordingly as

$$\begin{bmatrix} B_{1,1}^{(1)} & B_{1,2}^{(1)} \\ B_{2,1}^{(1)} & B_{2,2}^{(1)} \end{bmatrix}.$$

Next, formulate $B^{(2)}$, the generalized Perron complement of $B_{1,1}^{(1)}$ in $B^{(1)}$:

$$B^{(2)} = B_{2,2}^{(1)} + B_{2,1}^{(1)}(rI - B_{1,1}^{(1)})^{-1}B_{1,2}^{(1)}.$$

This leads to the reduced linear system for (2):

$$A^{(2)}x = b^{(2)}, \tag{11}$$

where $A^{(2)} = rI - B^{(2)}$ and $b^{(2)} = B_{2,1}^{(1)}(rI - B_{1,1}^{(1)})^{-1}b_1^{(1)} + b_2^{(1)}$.

Continuing, partition $B^{(2)}$ and $b^{(2)}$ in conformity as

$$\begin{bmatrix} B_{1,1}^{(2)} & B_{1,2}^{(2)} \\ B_{2,1}^{(2)} & B_{2,2}^{(2)} \end{bmatrix} \text{ and } b^{(2)} = \begin{bmatrix} b_1^{(2)} \\ b_2^{(2)} \end{bmatrix}.$$

Similarly, formulate $B^{(3)}$, the generalized Perron complement of $B_{1,1}^{(2)}$ in $B^{(2)}$:

$$B^{(3)} = B_{2,2}^{(2)} + B_{2,1}^{(2)}(rI - B_{1,1}^{(2)})^{-1}B_{1,2}^{(2)}.$$

The iteration procedure on (11) can then be expressed as

$$x^{(i+1)} = H^{(2)}x^{(i)} + c^{(2)}, \quad i = 0, 1, \ldots, \tag{12}$$

where

$$H^{(2)} = \begin{bmatrix} 0 & (rI - B_{1,1}^{(2)})^{-1}B_{1,2}^{(2)} \\ 0 & r^{-1}B^{(3)} \end{bmatrix}$$

and

$$c^{(2)} = \begin{bmatrix} (rI - B_{1,1}^{(2)})^{-1}b_1^{(2)} \\ r^{-1}B_{2,1}^{(2)}(rI - B_{1,1}^{(2)})^{-1}b_1^{(2)} + r^{-1}b_2^{(2)} \end{bmatrix}.$$

Finally, iterate (12) until for some i, $\|x^{(i+1)} - x^{(i)}\|/\|x^{(i+1)}\|$ is within a prescribed tolerance, where $\|\cdot\|$ denotes some vector norm. Let $x_2 = x^{(i+1)}$. Then a solution to (2) is given by $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, with

$$x_1 = (rI - B_{1,1}^{(1)})^{-1}(B_{1,2}^{(1)}x_2 + b_1^{(1)}).$$

Based on the analysis following Corollary 1, it is advisable to choose both $B_{1,1}^{(1)}$ and $B_{1,1}^{(2)}$ to be approximately $1/3$ of the size of $A^{(1)}$. In a similar fashion, when a three-level uncoupling is used, we choose all $B_{1,1}^{(\cdot)}$ to be approximately $1/4$ of the size of $A^{(1)}$.

To demonstrate the above algorithm, we now provide two examples. In both examples, $\|\cdot\|_\infty$ is adopted, the tolerance is 10^{-6}, and the initial approximation is set to be a zero vector. The computation is carried out in Matlab.

Example 1. Consider the following $n \times n$ irreducible nonsingular M-matrix [9]:

$$A = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_1 & \cdots & \\ \alpha_3 & 1 & \alpha_1 & \ddots & \ddots & \ddots & \vdots \\ \alpha_2 & \alpha_3 & \ddots & \ddots & \ddots & \ddots & \alpha_1 \\ \alpha_1 & \ddots & \ddots & \ddots & \ddots & \ddots & \alpha_3 \\ \alpha_3 & \ddots & \ddots & \ddots & \ddots & \ddots & \alpha_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \alpha_1 \\ & \cdots & \alpha_3 & \alpha_1 & \alpha_2 & \alpha_3 & 1 \end{bmatrix},$$

with $\alpha_1 = -1/n$, $\alpha_2 = -1/(n+1)$, and $\alpha_3 = -1/(n+2)$. The right-hand side of the linear system (2) is generated in such a way that the solution is $x = [1, 2, \ldots, n]^T$. We summarize in Table 1 the results from a three-level implementation, which indicate that, in terms of iteration numbers, the three-level algorithm is on a par with the adaptive Gauss-Seidel (AGS) method in [26], yet it performs better as compared with the Gauss-Seidel method and the modified Gauss-Seidel method in [5] — see a comparison in [9].

n	number of iterations (three-level)	number of iterations (AGS)
20	37	35
30	49	50
50	77	79
100	145	148

Table 1: The iteration numbers from a three-level algorithm on Example 1 are reported here. The data from the AGS method are quoted from [9].

Example 2. This example arises from the discretization of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 0.5\frac{\partial u}{\partial x} = f(x, y), \quad (x, y) \in \Omega = (0, 1) \times (0, 1),$$

with periodic boundary conditions [13]. For this example, the matrix A is given by, in an $m \times m$ block form,

$$A = \begin{bmatrix} D & -I & & & & -I \\ -I & D & -I & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -I & D & -I \\ -I & & & & -I & D \end{bmatrix},$$

where D, with $\alpha_+ = 1 + 0.5/m$ and $\alpha_- = 1 - 0.5/m$, is an $m \times m$ matrix

$$D = \begin{bmatrix} 4 & -\alpha_+ & & & & -\alpha_- \\ -\alpha_- & 4 & -\alpha_+ & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha_- & 4 & -\alpha_+ \\ -\alpha_+ & & & & -\alpha_- & 4 \end{bmatrix}.$$

Thus A is an $m^2 \times m^2$ irreducible singular M-matrix. We choose the right-hand side vector in the linear system (2) by $b = Ax$ using a random vector x. The number of iterations from, again, a three-level implementation and the residual $\|Ax^* - b\|_\infty$, with x^* being the computed solution to (2), are summarized in Table 2.

m	number of iterations (three-level)	residual
5	14	5.67×10^{-7}
10	55	1.38×10^{-6}
15	110	1.85×10^{-6}

Table 2: The iteration numbers and the residuals here are obtained from a three-level algorithm on Example 2.

References

[1] A. Berman and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, 1994.

[2] R. Bru, R. Cantó, and J.-J. Climent, On M-multisplittings of singular M-matrices with applications to Markov chains, Numer. Linear Algebra Appl. 5 (1998), 299-311.

[3] S. L. Campbell and C. D. Meyer, Generalized Inverses of Linear Transformations, Dover Publications, 1991.

[4] M. Fiedler and V. Ptak, On matrices with nonpositive off-diagonal elements and positive principal minors, Czech. Math. J. 12 (1962), 382–400.

[5] A. D. Gunawardena, S. K. Jain, and L. Snyder, Modified iterative methods for consistent linear systems, Linear Algebra Appl. 154–156 (1991), 123–143.

[6] A. Hadjidimos, D. Noutsos, and M. Tzoumas, More on modifications and improvements of classical iterative schemes for M-matrices, Linear Algebra Appl. 364 (2003), 253–279.

[7] J. P. Kavanagh and M. Neumann, Consistency and convergence of the parallel multisplitting method for singular M-matrices, SIAM J. Matrix Anal. Appl. 10 (1989), 210–218.

[8] S. Kirkland, M. Neumann, and J. Xu, A divide and conquer approach to computing the mean first passage matrix for Markov chains via Perron complement reductions, Numer. Linear Algebra Appl. 8 (2001), 287–295.

[9] T. Kohno, H. Kotakemori, H. Niki, and M. Usui, Improving the modified Gauss-Seidel method for Z-matrices, Linear Algebra Appl. 267 (1997), 113-123.

[10] I. Kuo, A note on factorizations of singular M-matrices, Linear Algebra Appl. 16 (1977), 217–220.

[11] W. Li and W. Sun, Modified Gauss-Seidel type methods and Jacobi type methods for Z-matrices, Linear Algebra Appl. 317 (2000), 227–240.

[12] W. Li, W. Sun, and K. Liu, Parallel multisplitting iterative methods for singular M-matrices, Numer. Linear Algebra Appl. 8 (2001), 181–190.

[13] L. Lin, Y. Wei, and N. Zhang, Convergence and quotient convergence of iterative methods for solving singular linear equations with index one, Linear Algebra Appl. 430 (2009), 1665–1674.

[14] L. Lu, Perron complement and Perron root, Linear Algebra Appl. 341 (2002), 239–248.

[15] C. D. Meyer, Uncoupling the Perron eigenvector problem, Linear Algebra Appl. 114/115 (1989), 69–94.

[16] C. D. Meyer, Matrix Analysis and Applied Linear Algebra, SIAM, 2000.

[17] C. D. Meyer and M. W. Stadelmaier, Singular M-matrices and inverse positivity, Linear Algebra Appl. 22 (1978), 139–156.

[18] V. Migallón and J. Penadés, Block two-stage methods for singular systems and Markov chains, Numer. Linear Algebra Appl. 3 (1996), 413–426.

[19] M. Neumann, Inverse of Perron complements of inverse M-matrices, Linear Algebra Appl. 313 (2000), 163–171.

[20] M. Neumann and J. Xu, On the stability of the computation of the stationary probabilities of Markov chains using Perron complements, Numer. Linear Algebra Appl. 10 (2003), 603–618.

[21] M. Neumann and J. Xu, A parallel algorithm for computing the group inverse via Perron complementation, Electron. J. Linear Algebra, 13 (2005), 131–145, .

[22] M. Neumann and R. J. Plemmons, Convergent nonnegative matrices and iterative methods for consistent linear systems, Numer. Math. 31 (1978), 265–279.

[23] D. P. O'Leary and R. E. White, Multi-splittings of matrices and parallel solution of linear systems, SIAM J. Alg. Disc. Meth. 6 (1985), 630–640.

[24] D. J. Rose, Convergent regular splittings for singular M-matrices, SIAM J. Alg. Disc. Meth. 5 (1984), 133–144.

[25] E. Seneta, Nonnegative Matrices and Markov Chains, 2nd ed., Springer, 1981.

[26] M. Usui, H. Niki, and T. Kohno, Adaptive Gauss-Seidel method for linear systems, Intl. J. Comput. Math. 51 (1994), 119–125.

[27] R. S. Varga, Matrix Iterative Analysis, 2nd ed., Springer-Verlag, 2000.

[28] L. Wang and Y. Song, Preconditioned AOR iterative methods for M-matrices, J. Comput. Appl. Math. 226 (2009), 114–124.