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Faber Polynomial Coefficient Estimates for a Certain Generalized Subclass of Bi-Univalent Functions Defined by Salagean Operator

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Abstract
The study of behaviour of coefficients of some analytic functions in a unit disk is interesting due to its wide application in the study of geometric function.The work will study the coefficient properties of a certain class of generalized Bi-univalent functions.The research work proved that application of Salagean differential operator on a subclass of generalized Bi-univalent function provided sharper inequalities,and contain several sets of Bi-univalent functions.

Keywords: Faber Polynomial Coefficient Estimates, Generalized Subclass of Bi-univalent Functions, Salagean Operator.

Introduction

Let A be the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$.Let S be the class of all functions in the normalized analytic function E which are univalent in E
Let $f \in A$.Salagean [1] introduced and studied the following operator:

$$D^0 f(z) = f(z) \tag{1.2}$$

$$D^1 f(z) = zf'(z) = Df(z) \tag{1.3}$$

$$D^j f(z) = D(D^{j-1} f(z)) \qquad (j \in \mathbb{N} := 1, 2, 3...) \tag{1.4}$$

In general if f is given by (1.1) ,then

$$D^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n, \quad (j \in \mathbb{N}_0 = \mathbb{N} \cup 0), \tag{1.5}$$

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with $D^j f(z) = 0$.
It is well known every function $f \in S$ has an inverse f^{-1},which is defined by $f^{-1}(f(z)) = z$, $z \in E$ and $f(f^{-1}(w)) = w$ $(|w| < r_0(f) : r_0(f) \geq \frac{1}{4})$.
Obviously, the inverse function $g = f^{-1}$ is given by

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots \\ &=: w + \sum_{n=2}^{\infty} A_n w^n \end{aligned} \tag{1.6}$$

We define p to be the class of functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ that are analytic in E and satisfy the condition $\Re(p(z)) > 0$ in E.By the Caratheodory Lemma see [2].
A function $f \in A$ is said to be bi-univalent in E if both $f \in S$ and $g \in f^{-1} \in S$. Let Σ be the class of bi-univalent functions in E given by (1.1).The class of bi-univalent functions was introduced and studied by Lewin [3], where it was proved that $|a_2| < 1.51$.Brannan and Clunie [4] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [5] proved that $|a_2| \leq \frac{4}{3}$.Brannan and Taha [6] also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficient $|a_2|$ and $|a_3|$.For a brief history and interesting examples of functions in the class Σ [7].The work of Srivastava et al [7] resucitated the investigation of different subclasses of the bi-univalent function.Other works include Frasin and Aouf [8],Xu et al [9, 10],Hayami and Owa [11],Porwal and Darus [12],Ukeje and Nwachukwu [13] and others.
The main purpose of this work is to use the Faber polynomial expansions for a modified subclass of analytic generalized bi-univalent functions to determine estimates for the general coefficient bounds $|a_n|$.

The Class $N_\Sigma(j, \alpha, \lambda, \delta)$

Let us recall the class of generalized bi-univalent function introduced by Bulut [14, 15]

Defhnition 1.

$\lambda \geq 1$ and $\delta \geq 0$,a function $f \in \sum$ given by (1.1) is said to be in the class $N_{\sum}(j, \alpha, \lambda, \delta)$ if the following conditions are satisfied:

$$\Re\left((1-\lambda)\frac{D^j f(z)}{z} + \lambda(D^j f(z))' + \delta z((D^j f(z))''\right) > \alpha \tag{2.1}$$

and

$$\Re\left((1-\lambda)\frac{D^j g(w)}{w} + \lambda(D^j g(w))' + \delta w((D^j g(w))''\right) > \alpha \tag{2.2}$$

Remarks:the following special cases of Definition 1 are as follows:
(i) when $\delta = 0$,the class becomes a special case of the class $\mu_\Sigma(j, \alpha, \lambda)$ studied by Bulut [15],that is $\mu_\Sigma(j, \alpha, \lambda) = N_{\sum}(j, \alpha, \lambda, 0)$

$$\Re\left((1-\lambda)\frac{D^j f(z)}{z}) + \lambda(D^j f(z))''\right) > 0 \tag{2.3}$$

and

$$\Re\left((1-\lambda)\frac{D^j g(w)}{w}) + \lambda(D^j g(w))''\right) > 0 \tag{2.4}$$

(ii) when $j=\delta=0$,we obtain the bi-univalent function class
$N_{\Sigma}(0,\alpha,\lambda,0)=\mu_{\Sigma}(0,\alpha,\lambda)$ which in turn equal to the class introduced by Frasin and Aouf [8].The class of functions $f\in\Sigma$ satisfying

$$\Re\left((1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\right)>\alpha$$

and

$$\Re\left((1-\lambda)\frac{g(w)}{w}+\lambda g'(w)\right)>\alpha$$

where $0\leq\alpha<1$:$\lambda\geq 1$;$w\in E$ and $g=f^{-1}$ is defined by (1.6).
(iii) When $j=0=\delta$ and $\lambda=1$,then we have the bi-univalent function class

$$N_{\Sigma}(0,\alpha,1,0)=\mu_{\Sigma}(0,\alpha,1)$$

Obviously,is the class studied by Bulut [14] and which in turn equivalent to the class introduced by Srivastava etal [7],with $f\in\Sigma$ satisfying $\Re\left(f'(z)\right)>\alpha$ and
satisfying $\Re\left(g'(w)\right)>\alpha$ where $0\leq\alpha<1:z,w\in E$ and $g=f^{-1}$ is defined by (1.6).

Coefficient Estimates

Using the Faber polynomial expansion of functions $f\in A$ of the form (1.1),the coefficients of its inverse map $g=f^{-1}$ expressed as [16]:

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\ldots)w^n, \tag{3.1}$$

where

$$\begin{aligned}K_{n-1}^{-n}&=\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1}+\frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3\\&+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4\\&+\frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}[a_5+(-n+2)a_3^2]\\&+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6+(-2n+5)a_3a_4]+\sum_{k\geq 7}a_2^{n-k}V_k\end{aligned} \tag{3.2}$$

such that V_k with $7\leq k\leq n$ is a homogeneous polynomial in the variables $a_2,a_3,\ldots,a_n$, [17].In particular,the first three terms of K_{n-1}^{-n} are

$$\begin{aligned}K_1^{-2}&=-2a_2\\K_2^{-3}&=3(2a_2^2-a_3)\\K_3^{-4}&=-4(5a_2^3-5a_2a_3+a_4).\end{aligned} \tag{3.3}$$

In general,for any $p\in\mathbb{N}$,an expansion of K_n^p is as,[16].

$$K_n^p=pa_n+\frac{p(p-1)}{2}D_n^2+\frac{p!}{(p-3)!3!}D_n^3+\cdots+\frac{p!}{(p-n)!n!}D_n^n \tag{3.4}$$

where $D_n^p=D_n^p(a_2,a_3,\ldots)$, and by [18],

$$D_n^m(a_1,a_2,\ldots,a_n)=\sum_{\infty}^{n=1}\frac{m!}{i_1!\ldots i_n!}a_1^{i_1}\ldots a_n^{i_n}$$

while $a_1 = 1$,the sum is taken over all non-negative integers $i_1, \ldots, i_n$ satisfying

$$i_1 + i_2 + \ldots + i_n = m$$
$$i_1 + 2i_2 + \ldots + ni_n = n$$

It is clear that $D_n^n(a_1, a_2, \ldots, a_n) = a_1^n$.

Lemma 1.0

For $\lambda \geq 1$, $\delta \geq 0$, $j \geq 0$ and $0 \leq \alpha < 1$,let $f \in N_\Sigma(j, \alpha, \lambda, \delta)$ of the form (1.1) then

$$\begin{aligned} &(1-\lambda)\frac{D^j f(z)}{z} + \lambda\left(D^j f(z)\right)' + \delta z\left(D^j f(z)\right)'' \\ &= 1 + \sum_{n=2}^{\infty} a_n n^j \left(1 + \lambda(n-1) + n(n-1)\delta\right) z^{n-1} \\ &= 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \ldots, a_n) z^{n-1} \end{aligned} \tag{3.5}$$

The next theorem will introduce the upper bound for the coefficients $|a_n|$ of analytic bi-univalent functions in the class $N_\Sigma(j, \alpha, \lambda, \delta)$.

Theorem 1.0

For $\lambda \geq 1$, $\delta \geq 0$, $j \geq 0$ and $0 \leq \alpha < 1$, Let $f \in N_\Sigma(j, \alpha, \lambda, \delta)$ be given by (1.1) if $a_k = 0$,$(2 \leq k \leq n-1)$,then

$$|a_n| \leq \frac{2(1-\alpha)}{\left[1 + (n-1)\lambda + n(n-1)\delta\right] n^j} \tag{3.6}$$

Proof: For the function $f \in N_\Sigma(j, \alpha, \lambda, \delta)$ of the form (1.1),we have expansion (2.4) and for the inverse map $g = f^{-1}$,considering (1.6) one gets

$$\begin{aligned} &\left((1-\lambda)\frac{D^j g(w)}{w} + \lambda(D^j g(w))' + \delta w(D^j g(w))''\right) \\ &= 1 + \sum_{n=2}^{\infty} A_n n^j \left(1 + \lambda(n-1) + n(n-1)\delta\right) w^{n-1} \\ &= 1 + \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3, \ldots, A_n) w^{n-1} \end{aligned} \tag{3.7}$$

with

$$A_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) \tag{3.8}$$

On the other hand,since $f \in N_\Sigma(j, \alpha, \lambda, \delta)$ and $g = f^{-1} \in N_\Sigma(j, \alpha, \lambda, \delta)$,by definition,there exist two positive real-part functions:

$$P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and

$$q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$$

where $\Re(P(z)) > 0$ and $\Re(q(w)) > 0$ in E so that

$$\begin{aligned}&(1-\lambda)\frac{D^j f(z)}{z}+\lambda\left(D^j f(z)\right)'+\delta z\left(D^j f(z)\right)''\\&=1+(1-\alpha)\sum_{n=1}^{\infty}K_n^1(c_1,c_2,\ldots,c_n)z^n\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}&(1-\lambda)\frac{D^j g(w)}{w}+\lambda\left(D^j g(w)\right)'+\delta w\left(D^j g(w)\right)''\\&=1+(1-\alpha)\sum_{n=1}^{\infty}K_n^1(d_1,d_2,\ldots,d_n)w^n\end{aligned} \tag{3.10}$$

Note that,by the Caratheodory Lemma [2]. $|c_n| \le 2$ and $|d_n| \le 2$,$(n \in \mathbb{N})$. Comparing,the corresponding coefficients of (3.5) and (3.9) for any $n \ge 2$,yields

$$[1+(n-1)\lambda+n(n-1)\delta]a_n n^j = (1-\alpha)K_{n-1}^1(c_1,c_2,\ldots,c_{n-1}) \tag{3.11}$$

and similarly,(3.7) and (3.10) gives

$$[1+(n-1)\lambda+n(n-1)\delta]A_n n^j = (1-\alpha)K_{n-1}^1(d_1,d_2,\ldots,d_{n-1}) \tag{3.12}$$

Since $a_k = 0$,$(2 \le k \le n-1)$ and $A_n = -a_n$,so (3.11) and (3.12) becomes:

$$[1+(n-1)\lambda+n(n-1)\delta]a_n n^j = (1-\alpha)C_{n-1} \tag{3.13}$$

$$-[1+(n-1)\lambda+n(n-1)\delta]a_n n^j = (1-\alpha)d_{n-1} \tag{3.14}$$

Now taking the absolute values of (3,13) and (3.14) yields

$$\begin{aligned}|a_n| &\le \frac{(1-\alpha)c_{n-1}}{[1+(n-1)\lambda+n(n-1)\delta]n^j} = \frac{(1-\alpha)d_{n-1}}{[1+(n-1)\lambda+n(n-1)\delta]n^j}\\&\le \frac{2(1-\alpha)}{[1+(n-1)\lambda+n(n-1)\delta]n^j}\end{aligned} \tag{3.15}$$

which is the required result.
We now consider the following corollaries which are consequences of the above theorem.

COROLLARY I.

For $\lambda \ge 1$, $0 \le \alpha < 1$,let the function $f \in N_\Sigma(0,\alpha,\lambda,\delta)$ be given by equation (1.1) if $a_k = 0$ $(2 \le k \le n-1)$ then

$$|a_n| \le \frac{2(1-\alpha)}{[1+(n-1)\lambda+n(n-1)\delta]} \qquad (n \ge 4) \tag{3.16}$$

Remark : Observe that when $j = 0$ we obtain the class of bi-univalent function $N_\Sigma(\alpha,\lambda,\delta)$ see [14].

COROLLARY II.

For $\lambda \ge 1$, $0 \le \alpha < 1$,let the function $f \in N_\Sigma(j,\alpha,\lambda,0)$ be given by equation (1.1) if $a_k = 0$ $(2 \le k \le n-1)$ then

$$|a_n| \le \frac{2(1-\alpha)}{[1+(n-1)\lambda]n^j} \qquad (n \ge 4) \tag{3.17}$$

Remark: It is obvious that setting $\delta = 0$ will yield the results of Jahangiri etal and Bulut,see [14, 20].

COROLLARY III.
For $\lambda \geq 1, 0 \leq \alpha < 1$,let the function $f \in N_\Sigma(0, \alpha, \lambda, 0)$ be given by equation (1.1) if $a_k = 0$ $(2 \leq k \leq n-1)$ then

$$|a_n| \leq \frac{2(1-\alpha)}{1+(n-1)\lambda} \qquad (n \geq 4) \tag{3.18}$$

Remark: Note that when $j = \delta = 0$,we obtain the earlier results in [15, 19].

COROLLARY IV.
For $\lambda \geq 1, 0 \leq \alpha < 1$,let the function $f \in N_\Sigma(j, \alpha, 1, \delta)$ be given by equation (1.1) if $a_k = 0$ $(2 \leq k \leq n-1)$ then

$$|a_n| \leq \frac{2(1-\alpha)}{[n+n(n-1)\delta]n^j} \qquad (n \geq 4) \tag{3.19}$$

Setting $\lambda = 1$ in (3.15) will yield the results of Bulut,Sivasubramanian etal [14, 21].

Theorem 2

For $\lambda \geq 1, 0 \leq \alpha < 1$,let the function $f \in N_\Sigma(j, \alpha, \lambda, \delta)$ be given by equation (1.1) if $a_k = 0$ $(2 \leq k \leq n-1)$ then

$$|a_2| \leq \min\left\{\sqrt{\frac{2(1-\alpha)}{(1+2\lambda+6\delta)3^j}}, \frac{2(1-\alpha)}{(1+\lambda+2\delta)2^j}\right\} \tag{3.20}$$

$$|a_3| \leq \frac{2(1-\alpha)}{[1+2\lambda+6\delta]3^j} \tag{3.21}$$

$$|a_3 - 2a_2^2| \leq \frac{2(1-\alpha)}{[1+2\lambda+6\delta]3^j}$$

Proof:
when $n = 2$,and $n = 3$ in (3.11) and (3.12) respectively,we get

$$a_2[1+\lambda+2\delta]2^j = (1-\alpha)c_1 \tag{3.22}$$

$$a_3[1+2\lambda+6\delta]3^j = (1-\alpha)c_2 \tag{3.23}$$

with the inverses

$$-a_2[1+\lambda+2\delta]2^j = (1-\alpha)d_1 \tag{3.24}$$

$$2a_2^2[1+2\lambda+6\delta]3^j - a_3[1+2\lambda+6\delta]3^j = (1-\alpha)d_2 \tag{3.25}$$

from (3.20) and (3.22) followed by application of Caratheodory lemma will yield

$$|a_2| = \frac{(1-\alpha)c_1}{[1+\lambda+2\delta]2^j} = \frac{2(1-\alpha)d_1}{[1+\lambda+2\delta]2^j} \leq \frac{2(1-\alpha)}{[1+\lambda+2\delta]2^j}. \tag{3.26}$$

Now adding (3.23) and (3.25),one gets

$$\begin{aligned} 2[1+2\lambda+6\delta]3^j a_2^2 &= (1-\alpha)c_2 + (1-\alpha)d_2 \\ &= (1-\alpha)(c_2+d_2) \end{aligned} \tag{3.27}$$

hence, using Caratheodory lemma we get

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{[1+2\lambda+6\delta]3^j}}$$

This result in addition to inequality in (3.26) gives the required estimate on the coefficient $|a_2|$ as claimed in (3.20).
To obtain the bound on the coefficient $|a_3|$,we subtract (3.25) from (3.23) to get

$$2a_3[1+2\lambda+6\delta]3^j - 2a_2^2[1+2\lambda+6\delta]3^j = (1-\alpha)c_2 - (1-\alpha)d_2$$

further simplification will give

$$|a_3| = \frac{(1-\alpha)(c_2-d_2)}{[1+2\lambda+6\delta]3^j} + a_2^2 \tag{3.28}$$

Substituting the value of a_2^2 from (3.22) gives

$$a_3 = \frac{(1-\alpha)^2 c_1^2}{[1+\lambda+2\delta]^2.2^{2j}} + \frac{(1-\alpha)(c_2-d_2)}{[1+2\lambda+6\delta]3^j}$$

applying Caratheodory Lemma we have

$$|a_3| \leq \frac{4(1-\alpha)^2}{[1+\lambda+2\delta]^2.2^{2j}} + \frac{2(1-\alpha)}{[1+2\lambda+6\delta]3^j}. \tag{3.29}$$

Now,substituting the value of a_2^2 from (3.27) into (3.28) will yield

$$= \frac{(1-\alpha)(c_2-d_2)}{2[1+2\lambda+6\delta]3^j} + \frac{(1-\alpha)(c_2+d_2)}{2[1+2\lambda+6\delta]3^j}$$

which reduces to

$$a_3 = \frac{(1-\alpha)c_2}{2[1+2\lambda+6\delta]3^j}$$

on application of Caratheodory Lemma gives

$$|a_3| \leq \frac{2(1-\alpha)}{[1+2\lambda+6\delta]3^j}$$

which gives the required result.
Consequently,from (3.25) one can claim with the help of Caratheodory Lemma that

$$|a_3 - 2a_2^2| = \frac{(1-\alpha)|-d_2|}{[1+2\lambda+6\delta]3^j} \leq \frac{2(1-\alpha)}{[1+2\lambda+6\delta]3^j}$$

which ends the proof.We now consider the following corollary as consequence of setting $j = 0$ in theorem 2.Clearly,the result is an improvement on the result of S.Bulut.
COROLLARY V. [14] For $\lambda \geq 1$, $0 \leq \alpha < 1$,let the function $f \in B_\Sigma(\alpha,\lambda)$ then

$$|a_2| \leq \begin{cases} \sqrt{\dfrac{2(1-\alpha)}{1+2\lambda+6\delta}} & 0 \leq \alpha < 1 - \dfrac{(1+\lambda+2\delta)^2}{2(1+2\lambda+6\delta)} \\[2ex] \dfrac{2(1-\alpha)}{1+\lambda+2\delta} & 1 - \dfrac{(1+\lambda+2\delta)^2}{2(1+2\lambda+6\delta)} \leq \alpha < 1 \end{cases}$$

$$|a_3| \leq \frac{2(1-\alpha)}{1+2\lambda+6\delta}, \qquad |a_3 - 2a_2^2| \leq \frac{2(1-\alpha)}{1+2\lambda+6\delta}$$

In same manner,if we set $j = 0$, $\delta = 0$, $\lambda = 1$ In Theorem 2 we obtain the following corollary.
COROLLARY VI [15] For $0 \le \alpha < 1$,let the function $f \in \mu_\Sigma(\alpha)$ be given by (1.1).Then one has the following

$$|a_2| \le \begin{cases} \sqrt{\frac{2(1-\alpha)}{3}}, & 0 \le \alpha < \frac{1}{3} \\ 1-\alpha, & \frac{1}{3} \le \alpha < 1 \end{cases}$$

and

$$|a_3| \le \frac{2(1-\alpha)}{3}$$

Remark. The estimates obtained for $|a_2|$ and $|a_3|$ clearly proved that corollary vi is an improvement of the estimates given by Srivastava etal,Bulut [7, 15].

Conclusion

Further variations of the parameters continues to yield results encountered in literatures previously which simply suggest that the differential operator used in this research generated several functions of same class.

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