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Results of Semigroup of Linear Operator Generating a Parabolic Equations $\mathscr{L}^p(\Omega)$ Theory

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Abstract
This paper present results of ω-order preserving partial contraction mapping generating a parabolic equation of $\mathscr{L}^p(\Omega)$ theory. The theory of semigroups of linear operators has approximation theory, ergodic theory and many others. In this paper we will restrict our attention to the application which are related to the solution of initial value problems for partial differential equations. We show that A generates a semigroup of linear operator which is closed and analytic on $\mathscr{L}^p(\Omega)$. We also deduced that operator A is associated with strong elliptic operator $A(x, D)$.

Keywords: ω-OCP_n, Analytic, C_0-semigroup, Elliptic.

Introduction

Let us consider the initial value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = Au(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{1.1}$$

in the sense of the Banach space X. The solution X thus obtained may actually be a classical solution of the initial value problem (1.1). If this is the case, it is usually proved by regularity provided by the abstract theory. Suppose Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$ and let $\mathscr{L}^p(\Omega)$ be a Banach space with $1 \leq p \leq \infty$ since we wish to obtain optimal regularity results. Assume $1 < p < \infty$ and let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Let

$$A(x, D)_u = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u \tag{1.2}$$

be a strong elliptic differential operator in Ω. The operator

$$A^*(x, D) = \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} D^\alpha (\overline{a_\alpha(x) u}) \tag{1.3}$$

is called the formal adjoint of $A(x, D)$. Assume X is a Banach space, $X_n \subseteq X$ is a finite set, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and A is a generator of

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C_0-semigroup. This paper consist of results of ω-order preserving partial contraction mapping generating a parabolic equations $\mathscr{L}^p(\Omega)$ theory. Agmon *et al.* [1], approximated some boundary problems for solutions of elliptic partial differential equation. Akinyele *et al.* [2], generated a continuous time Markov semigroup of linear operators and also in [3], Akinyele *et al.*, proved some perturbation results of the infinitesimal generator in the semigroup of the linear operator. Balakrishnan [4], introduced an operator calculus for infinitesimal generators of semigroup. Banach [5], established and introduced the concept of Banach spaces. Brezis and Gallouet [6], obtained nonlinear Schrodinger evolution equation. Chill and Tomilov [7], presented some resolvent approach to stability operator semigroup. Davies [8], obtained linear operators and their spectra. Engel and Nagel [9], deduced one-parameter semigroup for linear evolution equations. Omosowon *et al.* [10], established some analytic results of semigroup of linear operator with dynamic boundary conditions, and also in [11], Omosowon *et al.*, obtained dual Properties of ω-order Reversing Partial Contraction Mapping in Semigroup of Linear Operator. Omosowon*et al.* [12], generated a regular weak*-continuous semigroup of linear operators. Pazy [13], presented asymptotic behavior of the solution of an abstract evolution and some applications and also in [14], obtained a class of semi-linear equations of evolution. Pr*ü*ss [15], showed some semilinear evolution equations in Banach spaces. Rauf and Akinyele [16], established ω-order preserving partial contraction mapping and obtained its properties, also in [17], Rauf *et al.*, presented some results of stability and spectra properties on semigroup of linear operator. Vrabie [18], proved some results of C_0-semigroup and its applications. Yosida [19], established some results on differentiability and representation of one-parameter semigroup of linear operators.

Preliminaries

Definition 2.1 (C_0-Semigroup) [18]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (ω-OCP_n) [16]
A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in \text{Dom}\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.
Definition 2.3 (Evolution Equation) [13]
An evolution equation is an equation that can be interpreted as the differential law of the development (evolution) in time of a system. The class of evolution equations includes, first of all, ordinary differential equations and systems of the form

$$u' = f(t, u), u'' = f(t, u, u'),$$

etc., in the case where u(t) can be regarded naturally as the solution of the Cauchy problem; these equations describe the evolution of systems with finitely many degrees of freedom.
Definition 2.4 (Mild Solution) [13]
A continuous solution u of the integral equation.

$$u(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s, u(s))ds \tag{2.4}$$

will be called a mild solution of the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)), \ t > t_0 \\ u(t_0) = u_0 \end{cases} \tag{2.5}$$

if the solution is a Lipschitz continuous function.
Definition 2.5 (Analytic Semigroup) [18]

We say that a C_0-semigroup $\{T(t); t \geq 0\}$ is analytic if there exists $0 < \theta \leq \pi$, and a mapping $S : \bar{\mathbb{C}}_\theta \to L(X)$ such that:
(i) $T(t) = S(t)$ for each $t \geq 0$;
(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \bar{\mathbb{C}}_\theta$;
(iii) $lim_{z_1 \in \bar{\mathbb{C}}_\theta, z_1 \to 0} S(z_1)x = x$ for $x \in X$; and
(iv) the mapping $z_1 \to S(z_1)$ is analytic from $\bar{\mathbb{C}}_\theta$ to $L(X)$. In addition, for each $0 < \delta < \theta$, the mapping $z_1 \to S(z_1)$ is bounded from $\mathbb{C}_\delta$ to $L(X)$, then the C_0-Semigroup $\{T(t); t \geq 0\}$ is called analytic and uniformly bounded.

Definition 2.6 [13] Let $A = A(x, D)$ be strong elliptic operator of order $2m$ on a bounded domain Ω in $\mathbb{R}^n$ and let $1 < p < \infty$. Whenever

$$D(A_p) = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega) \tag{2.6}$$

we have that

$$A_p u = A(x, D)u \quad for \quad x \in D(A_p). \tag{2.7}$$

Definition 2.7 [13] Let $A(x, D)$ be the strong elliptic operator of order $2m$ on the bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$ given by (1.2). Set

$$D(A_1) = \{u : u \in W^{2m-1,1}(\Omega) \cap W_0^{m,1}(\Omega),\ A(x, D)u \in \mathscr{L}'(\Omega)\} \tag{2.8}$$

where $A(x, D)u$ is understood in the sense of distributions. For $u \in D(A_1)$, A_1 is defined by

$$A_1 u = A(x, D)u.$$

Example 1
3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 2
3×3 matrix $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 3
Let $X = C_{ub}(\mathbb{N} \cup \{0\})$ be the space of all bounded and uniformly continuous function from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}$, endowed with the sup-norm $\| \cdot \|_\infty$ and let $\{T(t); t \in \mathbb{R}\} \subseteq L(X)$ be defined by

$$[T(t)f](s) = f(t + s)$$

For each $f \in X$ and each $t, s \in \mathbb{R}$, one may easily verify that $\{T(t); t \in \mathbb{R}\}$ satisfies Examples 1 and 2 above.

Main Results

This section present results of semigroup of linear operator by using ω-OCP_n to generates a parabolic equations $\mathscr{L}^p(\Omega)$ theory:

Theorem 3.1
Let $A : D(A) \subseteq \mathscr{L}^p(\Omega) \to \mathscr{L}^p(\Omega)$ be the infinitesimal generator of C_0-semigroup $T(t);\ t \geq 0$. Let $A(x, D)$ be a strong elliptic operator of order $2m$ on Ω and let A_p, $1 < p < \infty$, be the operator associated with it. The operator A_q^{∞}, $q = \frac{p}{(p-1)}$ associated by Definition 2.6 with the formal adjoint $A^*(x, D)$ of $A(x, D)$ on $\mathscr{L}^q(\Omega)$ is the adjoint operator of A_p for all $A \in \omega - OCP_n$.

Proof:
We denote by $\langle\cdot\rangle$ the pairing between the dual spaces $\mathscr{L}^p(\Omega)$ and $\mathscr{L}^q(\Omega)$ by A' the adjoint of A_p. A simple integration by parts yields

$$\langle A_p u, v\rangle = \langle u, A_q^* v\rangle \tag{3.9}$$

for every $u \in D(A)$, $u \in D(A_q^*)$ and $A \in \omega - OCP_n$. Therefore, $D(A_q^*) \subset D(A')$ and for $v \in D(A_q^*)$, we have $A^*qv = Av$. Let $v \in D(A')$ and $w = A'v$, then by the definition of the adjoint operator we have

$$\langle A_p u, v\rangle = \langle u, w\rangle \quad for\ all\ u \in D(A)\ and\ A \in \omega - OCP_n. \tag{3.10}$$

Since $D(A_q^*)$ is dense in $\mathscr{L}^q(\Omega)$, then there is a sequence $v_i \in D(A_q^*)$ such that $v_i \to V$ in $\mathscr{L}^q(\Omega)$. From (3.9) and (3.10), it follows that

$$\langle u, A_q^* v_i\rangle \to \langle u, w\rangle \quad for\ all\ u \in D(A),\ A \in \omega - OCP_n$$

and since $D(A_p)$ is dense in $\mathscr{L}^p(\Omega)$ we conclude that $A_q^* v_i$ converges weaky to w. the closeness of A_q^* now implies that $v \in D(A_q^*)$ so that $D(A') \subset D(A_q^*)$ and

$$A' = A_q^*.$$

Hence the proof is completed.
Theorem 3.2
Suppose $A(x, D)$ is a strong elliptic operator of order $2m$ on a bounded domain Ω with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ and let $1 < p < \infty$. If $A_p : D(A_p) \subseteq \mathscr{L}^p(\Omega) \to \mathscr{L}^p(\Omega)$ is the operator associated with A by Definition 2.6, then A_p is the infinitesimal generator of an analytic semigroup on $\mathscr{L}^p(\Omega)$ for all $A \in \omega - OCP_n$.
Proof:
We have already noted that $D(A_p)$ is dense in $\mathscr{L}^p(\Omega)$ and that A_p is a closed operator in $\mathscr{L}^p(\Omega)$. Then we have that since A is a strongly elliptic operator of order $2m$ on a bounded domain Ω with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$ with $1 < p < \infty$. Then there exists a constant $C > 0$, $R \geq 0$ and $0 < u < \pi/2$ such that

$$\|u\|_{0,p} \leq \frac{c}{|\lambda|}\|(\lambda I + A)u\|_{0,p} \tag{3.11}$$

for every $u \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$, $\lambda \in \mathbb{C}$ and $A \in \omega - OCP_n$ satisfying $|\lambda| \geq R$ and $\theta - \pi < \arg\lambda < \pi - \theta$. Then it follows that for every

$$\lambda \in \Sigma_\theta = \{\mu : \theta - \pi < \arg\mu < \pi - \theta,\ |\mu| \geq R\} \tag{3.12}$$

the operator $\lambda I + A_p$ is injective and has closed range. Similarly, it follows that since operator A_q^* on $\mathscr{L}^q(\Omega)$, then there are constants $R' \geq 0$ and $0 < \theta' < \pi/2$ such that for every

$$\lambda \in \Sigma_{\theta'} = \{\mu : \theta' - \pi < \arg\mu < \pi - \theta', \ |\mu| \geq R\}, \textit{then}$$

$\lambda I + A_q^*$ is injective. Let $\theta_1 = \min(\theta, \theta')$ and $R_1 = \max(R, R')$, then for every

$$\lambda \in \Sigma_{\theta} = \{\mu : \theta_1 - \pi < \arg\mu < \pi - \theta_1, \ |\mu| \geq R_1\}, \textit{then}$$

$\lambda I + A_p$ is bijective. Indeed, we have already seen that it is injective so it remains only to show that it is surjective. Let $\lambda \in \Sigma_{\theta_1}$ and $A \in \omega - OCP_n$. If $v \in \mathscr{L}^q(\Omega)$ satisfies $\langle(\lambda I + A_p)u, v\rangle = 0$ for all $u \in D(A_p)$, then from Theorem 3.1, we have that $v \in D(A_q^*)$ and that $\langle u, (\lambda I + A_q^*)v\rangle = 0$ for all $u \in D(A_p)$ and $A \in \omega - OCP_n$. Since $D(A_p)$ is dense in $\mathscr{L}^p(\Omega)$, $(\lambda I + A_q^*)v = 0$ and the injectivity of $\overline{\lambda} I + A_q^*$ implies $v = 0$. Thus for $\lambda \in \Sigma_\theta$, $(\lambda I + A_p)$ is invertible and from (3.11) it follows that

$$\|(\lambda I + A_p)^{-1}\| \leq \frac{C}{|\lambda|} \quad \textit{for all } \lambda \in \Sigma_{\theta_1} \textit{ and } A \in \omega - OCP_n$$

which implies that A_p is the infinitesimal generator of an analytic semigroup on $\mathscr{L}^p(\Omega)$, and this achieved the proof.

Theorem 3.3

Let $1 < p < \infty$, then the operator $A_p : D(A_p) \subseteq \mathscr{L}^q(\Omega) \to \mathscr{L}^q(\Omega)$ is the infinitesimal generator of an analytic semigroup of contractions on $\mathscr{L}^q(\Omega)$ for all $A \in \omega - OCP_n$.

Proof:

Let $1 < p < \infty$ be fixed and let $q = p/(p-1)$. We denote the pairing between $\mathscr{L}^p(\Omega)$ and $\mathscr{L}^q(\Omega)$ by $\langle\cdot\rangle$. If $u \in D(A_p)$, then the function $u^* = |u|^{p-2}\overline{u}$ s in $\mathscr{L}^q(\Omega)$ and $\langle u, u^*\rangle = \|u\|_{0,p}^p$. Integration by parts yields

$$\begin{aligned} \langle A_p u, u^*\rangle &= -\int_\Omega \sum_{k,l=1}^n \frac{\partial}{\partial x_k}\left(a_{k,l}\frac{\partial u}{\partial x_1}\right)\overline{u}|u|^{p-2}dx \\ &= \int_\Omega \sum_{k,l=1}^n a_{k,l}\frac{\partial u}{\partial x_1}\frac{\partial}{\partial x_k}(\overline{u}|u|^{p-2})dx \\ &= \int_\Omega \sum_{k,l=1}^n a_{k,l}\left(|u|^{p-2}\frac{\partial u}{\partial x_1}\frac{\partial \overline{u}}{\partial x_k} + \overline{u}\frac{\partial u}{\partial x_k}\frac{\partial |u|^{p-2}}{\partial x_k}\right)dx. \end{aligned}$$

But

$$\frac{\partial}{\partial x_k}|u|^{p-2} = \frac{1}{2}(p-2)|u|^{p-4\left(\overline{u}\frac{\partial u}{\partial x_k} + u\frac{\partial \overline{u}}{\partial x_k}\right)}.$$

Denoting $|u|^{(p-4)/2}\overline{u}(\partial u/\partial x_k) = \alpha_k + i\beta_k$, we find after a simple computation that

$$\langle A_p u, u^*\rangle = \int_\Omega \sum_{l=1}^n a_{k,l}((p-1)\alpha_k\alpha_l + \beta_k\beta_l + i(p-2)\alpha_k\beta_l dx. \tag{3.13}$$

Let $|\alpha_{k,l}\infty| \leqq M$ for $1 \leqq k, l \leqq n$, $x \in \overline{\Omega}$ and $A \in \omega - OCP_n$ and set

$$|\alpha|^2 = \sum_{k=1}^n \int_\Omega \alpha_k^2 dx \quad |\beta|^2 = \sum_{k=1}^n \int_\Omega \beta_k^2 dx,$$

then if follows that if we assume coefficients $a_{k,l}(x) = a_{l,k}(x)$ are real valued and continuously differentiable in $\overline{\Omega}$ and that $A(x, D)$ is strongly elliptic, that is there is a constant $C_0 > 0$ such that

$$\sum_{k,l=1} a_{k,l}(x)\xi_k\xi_l \geqq C_0 \sum_{k=1}^n \xi_k^2 = C_0|\xi|^2 \tag{3.14}$$

for all real ξ_k, $1 \leq k \leq n$. Then it follow easily from (3.13) and (3.14) that

$$Re\langle A_p u, u^*\rangle \geqq C_0((p-1)|\alpha|^2 + |\beta|^2) \geqq 0 \tag{3.15}$$

and

$$\frac{|Im\langle A_p u, u^*\rangle|}{|Re\langle A_p u, u^*\rangle|} \leqq \frac{|p-2|M(\frac{p}{2}|\alpha|^2 + \frac{1}{2p}|\beta|^2)}{C_0((p-1)|\alpha|^2 + |\beta|^2)} \tag{3.16}$$

for every $p > 0$. Choosing $p = \sqrt{p-1}$ in (3.16) yields

$$\frac{|Im\langle A_p u, u^*|}{|Re\langle A_p u, u^*\rangle|} \leq \frac{M|p-2|}{2C_0\sqrt{p-1}}. \tag{3.17}$$

From (3.15), it follows readily that for every $\lambda > 0$ and $u \in D(A_p)$ we have

$$\lambda\|u\|_{o,p} \leqq \|(\lambda I + A_p)u\|_{0,p} \tag{3.18}$$

and therefore, $\lambda I + A_p$ is injective and has closed range for every $\lambda > 0$. Since (3.18) holds for every $1 < p < \infty$, it follows that for $\lambda > 0$, $\lambda I + A_p$ is also surjective. Indeed, if $v \in \mathscr{L}^p(\Omega)$ satisfies

$$\langle(\lambda I + A_p)u, v\rangle = 0$$

for all $u \in D(A_p)$ and $A \in \omega - OCP_n$. Since $A(x, D)$ is formally self adjoint, then it follows from Theorem 3.1 that $v \in D(A_p)$, $q = p/(p-1)$, and that

$$\langle u, (\overline{\lambda} I + A_q)v\rangle = 0$$

for every $u \in D(A_p)$. Since $D(A_p)$ is dense in $\mathscr{L}^p(\Omega)$, $(\overline{\lambda} I + A_q)v = 0$ and (3.18) with p replaced by q, implies $v = 0$. Thus, $\lambda I + A_p$ is bijective for $\lambda > 0$ and as a consequence of (3.18) we have

$$\|(\lambda I + A_p)^{-1}\|_{0,p} \leqq \frac{1}{\lambda} \quad for\ \lambda > 0. \tag{3.19}$$

The Hille-Yosida theorem now implies that A_p is the infinitesimal generator of a contraction semigroup on $\mathscr{L}^p(\Omega)$ for every $1 < p < \infty$. Finally, to prove that the semigroup generated by A_p is analytic we observe by (3.15) and (3.17) the numerical range $S(A_p)$ of A_p is contained in the set

$$S_{\theta_1} = (\lambda : |\arg\lambda| > \pi - \theta_1)$$

where

$$\theta = \arctan(M|p-2|/2C_0\sqrt{p-1}), \quad 0 < \theta_1 < \pi/2.$$

Choosing $\theta_1 < \theta < \pi/2$ and denoting

$$\Sigma_\theta = (\lambda : |\arg\lambda| < \pi - \theta) \tag{3.20}$$

it follows that there is a constant $C_0 > 0$ for which

$$d(\lambda : \overline{S(A_p)}) \geq C_0|\lambda| \quad for\ \lambda \in \Sigma_\theta.$$

Since $\lambda > 0$ is in the resolvent set $\rho(A_p)$ of A_p by the first part of the proof, it follows that since $\rho(A_p) \supset \Sigma_\theta$, we have

$$\|(\lambda I + A_p)^{-1}\|_{0,p} \leq \frac{1}{C_0|\lambda|} \quad for\ all\ \lambda \in \Sigma_\theta. \tag{3.21}$$

Hence, A_p is the infinitesimal generator of an analytic semigroup on $\mathscr{L}^p(\Omega)$ and this achieved the proof.

Theorem 3.4
The operator $A : D(A_1) \subseteq \mathscr{L}^1(\Omega) \to \mathscr{L}^1(\Omega)$ is the infinitesimal generator of an analytic semigroup on $\mathscr{L}^1(\Omega)$.

Proof:
Let

$$A(x, D)u = \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha u$$

and

$$\overline{A}(x, D) = \sum_{|\alpha| \le 2m} (-1)^{|\alpha|} D^\alpha (a_n(x)u).$$

Let $\overline{A}_c$ be the operator associated with $\overline{A}(x, D)$ on the space C given by

$$C = \{u : u \in C(\overline{\Omega}), u = 0 \; on \; \partial\Omega\}. \tag{3.22}$$

Since $\overline{A}(x, D)$ is strongly elliptic together with $A(x, D)$ it follows from the infinitesimal generator of analytic semigroup and we have that there are constants $M > 0, R \ge 0$ and $0 < \theta < \pi/2$ such that

$$\|(\lambda I + \overline{A}_c)^{-1}\|_{0,\infty} \le M|\lambda|^{-1} \tag{3.23}$$

for every

$$\lambda \in \Sigma_\theta = \{\mu : |\arg \mu| > \theta, \; |\mu| \ge R\}.$$

Now, let $u \in D(A_1)$, and let Ω be a bounded domain in $\mathbb{R}^n$ for all $u \in \mathscr{L}^1(\Omega)$, then we have

$$\|u\|_{0,1} = \sup\left\{\int_\Omega u(x)\varphi(x)dx : \varphi \in C_0^\infty(\Omega), \; \|\varphi\|_{0,\infty} \le 1\right\}. \tag{3.24}$$

Since $C_0^\infty(\Omega)$ is contained in the range of $\lambda I + \overline{A}_c$ for every $\lambda \in \Sigma_\theta$ and $A \in \omega - OCP_n$, it follows from (3.23) and (3.24) that

$$\|u\|_{0,1} = \sup\left\{\int_\Omega u(\lambda I + \overline{A}_c)v dx : v \in D(\overline{A}_c), \; \|v\|_{0,\infty} \le M|\lambda|^{-1}\right\}$$

which implies that for every $v \in D(\overline{A}_c)$, $\|v\|_{0,\infty} \le M|\lambda|^{-1}$ we have

$$\|u\|_{0,1} \le \left|\int_\Omega u(\lambda I + \overline{A}_c)v dx\right| = \left|\int_\Omega (\lambda I + A_1)uv dx\right|$$
$$\le \|(\lambda I + A_1)u\|_{0,1}\|v\|_{0,\infty} \le M|\lambda|^{-1}\|(\lambda I + A_1)u\|_{0,1}.$$

Thus for every $\lambda \in \Sigma_\theta$, $\lambda I + A_1$ is injective and has closed range. Moreover, since $D(A_2) \subset D(A_1)$ the range of $\lambda I + A$ contains $\mathscr{L}^2(\Omega)$, which is dense in $\mathscr{L}^1(\Omega)$, and therefore the range of $\lambda I + A_1$ is all of $\mathscr{L}^1(\Omega)$ and

$$\|(\lambda I + A_1)\|_{0,1} \le M|\lambda|^{-1}$$

for every $\lambda \in \Sigma_\theta$, it follows therefore that A_1 is the infinitesimal generator of an analytic semigroup on $\mathscr{L}^1(\Omega)$. Hence the proof is completed.

Conclusion
In this paper, it has been established that ω-order preserving partial contraction mapping generates some results of parabolic equations $\mathscr{L}^p(\Omega)$ theory.

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References

[1] S. Agmon, A. Douglis, and L. Nirenberg, Estimates Near the Boundary Problems for Solutions of Elliptic Partial Differential Equation, Comm. Pure Appl. Math. 12 (1959), 623-727.

[2] A. Y. Akinyele, O. E. Jimoh, J. B. Omosowon, and K. A. Bello, Results of Semigroup of Linear Operator Generating a Continuous Time Markov Semigroup, Earthline J. Math. Sci. 10 (2022), 97-108.

[3] A. Y. Akinyele, O. Y. Saka-Balogun, and O. A. Adeyemo, Perturbation of Infinitesimal Generator in Semigroup of Linear Operator, South East Asian J. Math. Math. Sci. 15 (2019), 53-64.

[4] A. V. Balakrishnan, An Operator Calculus for Infinitesimal generators of Semigroup, Trans. Amer. Math. Soc. 91 (1959), 330-353.

[5] S. Banach, Surles Operation Dam Les Eusembles Abstracts et lear Application Aus Equation integrals, Fund. Math. 3 (1922), 133-181.

[6] H. Brezis, and T. Gallouet, Nonlinear Schrodinger evolution equation, Nonlinear Anal. TMA, 4 (1980), 677-682.

[7] R. Chill, and Y. Tomilov, Stability Operator Semigroup, Banach Center Publication 75, Polish Academy of Sciences, Warsaw, (2007), 71-73.

[8] E. B. Davies, Linear Operator and their Spectra, Cambridge Studies in Advanced Mathematics, Cambridge University Press, New York. (2007).

[9] K. Engel, and R. Nagel, One-parameter Semigroups for Linear Equations, Graduate Texts in Mathematics, 194, Springer, New York, (2000).

[10] J. B. Omosowon, A. Y. Akinyele, O. Y. Saka-Balogun, and M.A. Ganiyu, Analytic Results of Semigroup of Linear Operator with Dynamic Boundary Conditions, Asian J. Math. Appl. 2020 (2020), ama0561.

[11] J. B. Omosowon, A. Y. Akinyele, F. M. Jimoh, Dual Properties of ω-order Reversing Partial Contraction Mapping in Semigroup of Linear Operator, Asian J. Math. Appl. 2021 (2021), ama0566.

[12] J. B. Omosowon, A. Y. Akinyele, K. A. Bello, and B. M. Ahmed, Results of Semigroup of Linear Operators Generating a Regular Weak*-continuous Semigroup, Earthline J. Math. Sci. 10 (2022), 289-304.

[13] A. Pazy, asymptotic behavior of the solution of an abstract evolution and some applications, J. Diff. Eqs. 4 (1968), 493-509.

[14] A. Pazy, A class of semi-linear equations of evolution, Isreal J. Math. 20 (1975), 23-36.

[15] J. Pr$\ddot{u}$ss, On semilinear evolution equations in Banach spaces, J. Reine Ang. Mat. 1978 (1978), 144-158.

[16] K. Rauf, and A. Y. Akinyele, Properties of ω-Order-Preserving Partial Contraction Mapping and its Relation to C_0-semigroup, Int. J. Math. Computer Sci. 14 (2019), 61-68.

[17] K. Rauf, A. Y. Akinyele, M. O. Etuk, R. O. Zubair, and M. A. Aasa, Some Result of Stability and Spectra Properties on Semigroup of Linear Operator, Adv. Pure Math. 9 (2019), 43-51.

[18] I. I. Vrabie, C_0-semigroup and application, Mathematics Studies, 191, Elsevier, North-Holland, (2003).

[19] K. Yosida, On The Differentiability and Representation of One-Parameter Semigroups of Linear Operators, J. Math. Soc. Japan, 1 (1948), 15-21.