QUADRUPLE BEST PROXIMITY POINTS OF Q-CYCLIC CONTRACTION PAIR IN ABSTRACT METRIC SPACES

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ABSTRACT. We introduce the concepts of quadruple best proximity point and Q-cyclic contraction pair and prove the existence and convergence of quadruple best proximity points in abstract metric spaces. The results are extension of known results in coupled and tripled best proximity points in literature. Examples are given to illustrate our main result.

1. INTRODUCTION

Let (X, d) be a metric space and T a self-mapping on X, the fixed point theory, which is the theory on the existence of solution to the equation of the form $Tx = x$, is a very interesting area of research in Functional Analysis which has fascinated many researchers since 1922 with the celebrated Banach fixed point theorem. This theorem provides a technique for solving a variety of applied problems in economics, mathematical sciences and engineering. Banach contraction principle has been generalised in different directions in different spaces by researchers over the years. (See, for example, [11,20,21,22,25,26,27,28] and all the references therein).

If the contractive operator T is a non-self mapping in one dimension, i.e. $T : A \to B$, T may not have a fixed point. It is therefore of interest to determine an element x called the best proximity point that is in some sense closest to Tx which trivially becomes a fixed point if T is a self mapping where A, B are nonempty subsets of X. Naturally, $d(x, Tx) \geq d(A, B)$ for all x, where $d(A, B) = inf\{d(x, y) : x \in A, \ y \in B\}$. A best proximity point theorem offers sufficient conditions for the existence of an element x, satisfying the condition that

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$d(x, Tx) = d(A, B)$ which is the optimal solution.

One of the most interesting results in this direction is due to Fan [6], where he introduced and established a classical best approximation theorem, that is, if A is a nonempty compact subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Fan's results was improved upon due to their shortcomings. Best approximation theorem only ensures the existence of approximation solutions, such results need not yield optimal solutions. But the best proximity point theorem provides sufficient conditions that ensure the existence of approximate solutions which are also optimal.

Afterwards many authors such as Eldred and Veeramani [5], Kirk, Reich and Veeramani [16] and Mihaela [18] have derived extensions of Fan's Theorem and the best approximation theorem in many directions. Significant best proximity point results are in [5,6,16,17] and other references therein.

In 1987, the notion of coupled fixed point was introduced by Guo and Laksmikantham [8]. Later, Bhaskar and Laksmikantham [7] introduced the mixed monotone property for contractive operators of the type $F : X \times X \to X$, where X is a partially ordered metric space. They also, as an application, obtained the existence and uniqueness of the solution to periodic boundary valued problems. For more results in this area see [1,7,8,18,19,20,30].

In 2011, Berinde and Borcut [3] introduced the concept of tripled fixed point in partially ordered metric spaces and extended both the results of Gnana-Bhaskar and Laskshmikantham [7] and Sabetghadam et al.[28] to the case of contractive maps of the form $F : X \times X \times X \to X$ using the contractive condition similar to the one used in [28]. For more results in this area see [2].

The notion of fixed point of order $N \geq 2$ using a the concept of F-invariant set was first introduced by Samet and Vetro [29]. Results involving partially ordered metric spaces and cone metric spaces were derived. Also, Olaleru and Olaoluwa [23] gave some results on multiple fixed points theorems in abstract metric spaces and Olaoluwa [24] introduced a definition more general than the ones in [29].

In 2012, Karapinar[13] applied a map in four dimensional spaces to prove some fixed point theorems in nonlinear contractive maps with no mixed monotone property. After this, more results on quadruple fixed point theorems were established in partially ordered metric spaces with mixed monotone property. Karapinar et al.[15] proved the quadruple fixed point theorems under ϕ-contractive conditions for a mapping $F : X^4 \to X$ in ordered metric spaces. For more results, see [12,14,17].

If the contractive map F is a non-self mapping, i.e. $F : A \times A \to B$, then a coupled best proximity point (x, y) is defined by $d(x, F(x, y)) = d(y, F(y, x)) = d(A, B)$. This concept was introduced by Sintunavarat and Kuman [31] who proved the coupled best proximity theorem for cyclic contractions. Their results were extended and generalised in [9,22,23] by using interesting contractive conditions similar to Hardy and Rogers [10] and Sabetghadam et al. [28].

Very recently, Cho et al.[4] introduced the notions of tripled best proximity point and cyclic contraction pair. They established existence and convergence theorems of tripled best proximity points in metric and uniformly convex Banach spaces. They also obtained some results on existence and convergence of tripled fixed point in metric spaces and gave illustrative examples of the theorems.

Motivated by the results of Samet and Vetro [29], Cho et al. [4] and Karapinar et al. [15,] we introduce the notion of quadruple best proximity points of Q-cyclic contraction pairs under the contractive conditions analogous to those used in [28]. The maps are defined on abstract metric spaces having the property of uniformly convex Banach spaces. Our results extend and generalised the results mentioned above.

2. Preliminaries

In this section, we give some definitions and concepts related to the main results of this article. Throughout, we denote by $\mathbf{N}$ the set of all positive integers and by $\mathbf{R}$ the set of all real numbers.

Let E be a Banach space

(1) a subset C of E is called convex if for any pair of point $x, y \in E$ the closed segment with the extremities x, y, that is the set $\{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\}$ is contained in C.

(2) E is called strictly convex if for all $x, y \in E$, $x \neq y$, such that $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1 \ \forall \lambda \in (0, 1)$.

(3) E is called uniformly convex if for each ϵ with $0 < \epsilon \leq 2$, there is a $\delta(\epsilon) > 0$ such that whenever $\|x\| = \|y\| = 1$, and $\|x - y\| \geq \epsilon$, the inequality $\|\frac{x+y}{2}\| < 1 - \delta(\epsilon)$, holds for all $x, y \in E$.

Thus, a Banach space is uniformly convex if for any two distinct points x and y on the unit sphere centred at the origin, the midpoint of the line segment joining x and y is never on the sphere but is close to the sphere only if x and y are sufficiently close to each other.

Examples of uniformly Banach space include: Hilbert spaces and l_p

spaces, $1 < p < \infty$.

Note that a uniformly convex space X is strictly convex but the converse is not true.

(4) E is called reflexive if it coincides with the continuous dual of its continuous dual space. The dual space E^* consists of all continuous linear functional

$F : E \to \mathbf{R}$. Examples of reflexive Banach space include: Hilbert spaces, finite dimensional normed space, uniformly convex Banach space, and l_p spaces, $1 < p < \infty$.

Definition 2.1 ([31]). Let A and B be nonempty subsets of an abstract metric space (X, d). The ordered pair (A, B) is said to satisfy the property UC if the following holds:

if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$, then $d(x_n, z_n) \to 0$.

Example 2.2 ([31]). Here are examples of pairs of nonempty subsets (A, B) satisfying the the property UC:

(1) Every pair of nonempty subsets (A, B) of an abstract metric space (X, d) such that $d(A, B) = 0$.

(2) Every pair of nonempty subsets (A, B) of a uniformly convex Banach space X such that A is convex.

(3) Every pair of nonempty subsets (A, B) of a strictly convex Banach space such that A is convex and relatively compact and the closure of B is weakly compact.

(4)) Every pair of nonempty subsets (A, B) of a reflexive Banach space such that A is convex, every bounded sequence in A has a weakly convergence subsequence, and the closed unit sphere S of B is weakly compact.

Definition 2.3 ([31]). Let A and B be nonempty subsets of an abstract metric space (X, d). The ordered pair (A,B) is said to satisfy the property UC^* if (A, B) has property UC and the following holds:

if $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B satisfying:

(1) $d(z_n, y_n) \to d(A, B)$

(2)For every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that

$$d(x_m, y_n) \to d(A, B) + \epsilon$$

for all $m > n \geq N$,
then, for every $\epsilon > 0$, there exists $N_1 \in \mathbf{N}$ such that

$$d(x_m, z_n) \to d(A, B) + \epsilon$$

for all $m > n \geq N_1$.

Example 2.4 ([31]). Here are examples of pairs of nonempty subsets (A, B) satisfying the property UC^*:
(1) Every pair of nonempty subsets (A, B) of an abstract metric space (X, d) such that $d(A, B) = 0$.
(2) Every pair of nonempty subsets (A, B) of uniformly convex Banach space X such that A is convex.

Sintunavarat and Kuman [31] introduced the notion of cyclic contractions as follows:

Definition 2.5([31]). Let A and B be nonempty subsets of a metric space X and let $F : A \times A \to B$ and $G : B \times B \to A$ be two maps. The ordered pair (F, G) is said to be a cyclic contraction if there exists $\alpha \in [0, \frac{1}{2})$ such that
$d(F(x, x'), G(y, y')) \leq \frac{\alpha}{2}[d(x, y) + d(x', y')] + (1 - \alpha)d(A, B)$ (2.1)
for all $(x, x') \in A \times A$ and $(y, y') \in B \times B$.

As mentioned earlier, Samet and Vetro [29] introduced a fixed point of order $N \geq 3$ using the concept of F-invariant set. In particular, they gave the following definition of fixed point of order $N = 3$.

Definition 2.6([29]). An element $(x, y, z) \in X^3$ is called a tripled fixed point of a given mapping $F : X^3 \to X$ if $x = F(x, y, z)$, $y = F(y, z, x)$ and $z = F(z, x, y)$.

Recently, in an attempt to extend the tripled fixed points to nonself mappings, Cho et al. [4] introduced the tripled best proximity point for cyclic contractions. They adopted the definition of tripled fixed points given by Berinde and Borcut [3], defined for maps with mixed monotone property:

Definition 2.7 ([3]). Let A and B be nonempty subsets of a metric space X and $F : A \times A \times A \to B$. A point $(x, y, z) \in A \times A \times A$ is called a tripled best proximity point of F if
$d(x, F(x, y, z)) = d(y, F(y, x, y)) = d(z, F(z, y, x)) = d(A, B)$.

They proved the existence and convergent of tripled best proximity point for the following cyclic contraction map:

Definition 2.8([3]). Let A and B be nonempty subsets of a metric space X and $F : A^3 \to B$ and $G : B^3 \to A$. The ordered pair (F, G) is said to be a cyclic contraction if there exists $\alpha \in [0, \frac{1}{3})$ such that
$d(F(x, y, z), G(u, v, w)) \leq \frac{\alpha}{3}[d(x, u) + d(y, v) + d(z, w)] + (1 - \alpha)d(A, B)$ (2.2)
for all $(x, y, z) \in A^3$ and $(u, v, w) \in B^3$.

Theorem 2.9[(3)]. Let A and B be nonempty closed subsets of a complete metric space (X, d), such that (A, B) and (B, A) satisfy the property UC^*.
Let the pair (F, G) of maps $F : A^3 \to B$ and $G : B^3 \to A$ be a cyclic contraction in the sence of Definition 2.8. If $(x_0, y_0, z_0) \in A^3$, define the sequence $\{x_n\}, \{y_n\}, \{z_n\}$ in X by
$x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n})$; $y_{2n+1} = F(y_{2n}, x_{2n}, y_{2n})$; $z_{2n+1} = F(z_{2n}, y_{2n}, x_{2n})$
and
$x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1})$; $y_{2n+2} = G(y_{2n+1}, x_{2n+1}, y_{2n+1})$;
$z_{2n+2} = G(z_{2n+1}, y_{2n+1}, x_{2n+1})$ $\forall n \in N \cup \{0\}$. F has a tripled best proximity point $(p, q, r) \in A^3$ and G has a tripled best proximity point $(p', q', r') \in B^3$. Moreover, we have $x_{2n} \to p$, $y_{2n} \to q$, $z_{2n} \to r$, $x_{2n+1} \to p'$, $y_{2n+1} \to q'$ and $z_{2n+1} \to r'$.
Furthermore, if $p = q = r$ and $p' = q' = r'$, then

$$d(p, p') + d(q, q') + d(r, r') = 3d(A, B).$$

As a consequence of the above result, Cho et al. [4] gave the following corollary:

Collollary 2.10[(4)]. Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and $F : A^3 \to B, G : B^3 \to A$ be two mappings such that the ordered pair (F, G) is a cyclic contraction. For any $(x_0.y_0, z_0) \in A^3$, we define the sequence $\{x_n\}, \{y_n\}, \{z_n\}$ in X by
$x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n})$; $y_{2n+1} = F(y_{2n}, x_{2n}, y_{2n})$; $z_{2n+1} = F(z_{2n}, y_{2n}, x_{2n})$
and
$x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1})$; $y_{2n+2} = G(y_{2n+1}, x_{2n+1}, y_{2n+1})$;
$z_{2n+2} = G(z_{2n+1}, y_{2n+1}, x_{2n+1})$ $\forall n \in N \cup \{0\}$. F has a tripled best proximity point $(p, q, r) \in A^3$ and G has a tripled best proximity point $(p', q', r') \in B^3$ Moreover, we have $x_{2n} \to p$, $y_{2n} \to q$, $z_{2n} \to r$, $x_{2n+1} \to p'$, $y_{2n+1} \to q'$ and $z_{2n+1} \to r'$.

Furthermore, if $p = q = r$ and $p' = q' = r'$, then

$$d(p, p') + d(q, q') + d(r, r') = 3d(A, B).$$

We recall the following definition from [14].

Definition 2.11 ([14]). Let X be a nonempty set and let $F : X^4 \to X$ be a given mapping. An element $(x, y, z, w) \in X \times X \times X \times X$ is called a quadruple fixed point of F if
$F(x, y, z, w) = x$, $F(y, z, w, x) = y$, $F(z, y, z, w) = z$, and $F(w, x, y, z) = w$.
We now introduce the definition of quadruple best proximity point as follows:

Definition 2.12. Let A and B be a nonempty subset of an abstract metric space and let $F : A^4 \to B$ be a given mapping. A point $(x, y, z, t) \in A^4$ is called a quadruple best proximity point of F if
$d(x, F(x, y, z, t)) = d(y, F(y, z, t, x)) = d(z, F(z, t, x, y))$
$= d(t, F(t, x, y, z)) = d(A, B)$.
Remark. Definition 2.12 reduces to quadruple fixed point if $d(A, B) = 0$.

We introduce the notion of T-cyclic and Q-cyclic contraction pairs in three and four dimensions respectively:

Definition 2.13. Let A and B be a nonempty subset of an abstract metric space X and $F : A^3 \to B$ and $G : B^3 \to A$. The ordered pair (F, G) is said to be a T-cyclic contraction if
$d(F(x, y, z), G(u, v, w)) \leq [a_1 d(x, u) + a_2 d(y, v) + a_3 d(z, w)]$
$+ (1 - \eta)d(A, B)$ (2.3)
for all $(x, y, z) \in A^3$ and $(u, v, w) \in B^3$, where $a_i, \; i = 1, ..., 3$ are non negative real numbers such that $\sum_{i=1}^{3} a_i = \eta < 1$.

Note that if (F, G) is a T-cyclic contraction pair then (G, F) is also a T-cyclic contraction pair. If we take $a_1 = a_2 = a_3 = \frac{\alpha}{3}$ in Definition 2.13 then we get Definition 2.8.
Definition 2.14. Let A and B be a nonempty subset of an abstract metric space X and $F : A^4 \to B$ and $G : B^4 \to A$. The ordered pair (F, G) is said to be a Q-cyclic contraction if
$d(F(x, y, z, t), G(u, v, w, a)) \leq [a_1 d(x, u) + a_2 d(y, v) + a_3 d(z, w)$
$+ a_4 d(t, a)] + (1 - \eta)d(A, B)$ (2.4)

for all $(x, y, z, t) \in A^4$ and $(u, v, w, a) \in B^4$, a_i, $i = 1, ..., 4$ are non negative real numbers such that $\sum_{i=1}^{4} a_i = \eta < 1$.

Note that if (F, G) is a Q-cyclic contraction pair then (G, F) is also a Q-cyclic contraction pair.
The following examples show that T-cyclic and Q-cyclic contractions are generalisations of cyclic contractions.

Example 2.15. Let $X = \mathbf{R}$ with the usual metric space $d(x, y) = |x - y|$ and let A = [2,6] and B = [-6, -2]. We see that d(A, B) = 4. Define $F : A^3 \to B$ and $G : B^3 \to A$ by $F(x, y, z) = \frac{-2x-3y-z-6}{12}$ and $G(x, y, z) = \frac{-2x-3y-4z+6}{12}$.
For arbitrary $(x, y, z) \in A^3$ and $(u, v, w) \in B^3$, and fixed $\eta = \frac{3}{4}$, we get

$$
\begin{aligned}
&d(F(x, y, z), G(u, v, w)) \\
&= \left| \frac{-2x-3y-4z-6}{12} - \frac{-2u-v-4w+6}{12} \right| \\
&\leq 2\left| \frac{x-u}{12} \right| + 3\left| \frac{y-v}{12} \right| + 4\left| \frac{z-w}{12} \right| + 1 \\
&= \frac{1}{6}d(x, u) + \frac{1}{4}d(y, v) + \frac{1}{3}d(z, w) + (1 - \frac{3}{4}) \times 4 \\
&= a_1 d(x, u) + a_2 d(y, v) + a_3 d(z, w)] + (1 - \eta)d(A, B).
\end{aligned}
$$

Observe that $a_1 + a_2 + a_3 = \eta = \frac{3}{4}$.
This implies that (F, G) is a T-cyclic contraction with $\eta = \frac{3}{4}$. See that (2.3) above holds with the constants $a_1 = \frac{1}{6}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{3}$.
It is obvious that (F, G) is not a cyclic contraction and does not satisfy 2.2.
Suppose F satisfies (2.2), let $x = y \neq z, v = v \neq w$ in (2.2). Precisely, taking $x = y = 0, z = 6$ and $u = v, w = 6$ in (2.2), we get $\frac{11}{12} \leq \frac{1}{2}$ a contradiction. This proves that F does not satisfy (2.2). Hence T-cyclic contractions are generalisations of cyclic contractions.

Example 2.16. Let $X = \mathbf{R}$ with the usual metric space $d(x, y) = |x - y|$ and let A = [3,6] and B = [-6, -3]. We see that d(A,B) = 6. Define $F : A^4 \to B$ and $G : B^4 \to A$ by $F(x, y, z, t) = \frac{-x-3y-4z-2t-6}{12}$ and $G(x, y, z, t) = \frac{-x-3y-4z-2t+6}{12}$.
For arbitrary $(x, y, z, t) \in B^4$ and $(u, v, w, a) \in A^4$ and fixed $\eta = \frac{5}{6}$, we get

$$
\begin{aligned}
&d(F(x, y, z, t), G(u, v, w, a)) \\
&= \left| \frac{-x-3y-4z-2t-6}{12} - \frac{-u-3v-4w-2a+6}{12} \right| \\
&\leq \left| \frac{x-u}{12} \right| + 3\left| \frac{y-v}{12} \right| + 4\left| \frac{z-w}{12} \right| + 2\left| \frac{t-a}{12} \right| + 1 \\
&= \frac{1}{12}d(x, u) + \frac{1}{4}d(y, v) + \frac{1}{3}d(z, w) + \frac{1}{6}d(t, a) + (1 - \frac{5}{6}) \times 6 \\
&= a_1 d(x, u) + a_2 d(y, v) + a_3 d(z, w) + a_4 d(t, a)] + (1 - \eta)d(A, B).
\end{aligned}
$$

Observe that $a_1 + a_2 + a_3 + a_4 = \eta = \frac{5}{6}$.

This implies that (F, G) is a Q-cyclic contraction with $\eta = \frac{5}{6}$.

It is also obvious that (F, G) is not a cyclic contraction. Hence, Q-cyclic contraction is more general than cyclic contraction.

The following Lemmas which are analogous to tripled best proximity point in [4], play an important role in our main result.

Lemma 2.17. Let A and B be nonempty subsets of a metric space (X, d) and let $F : A^4 \to B$, $G : B^4 \to A$ be two maps such that (F, G) be a Q-cyclic contraction. If for

$(x_0, y_0, z_0, t_0) \in A^4$, we define

$$\begin{cases} x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}, t_{2n}) \ ; \\ y_{2n+1} = F(y_{2n}, z_{2n}, t_{2n}, x_{2n}) \ ; \\ z_{2n+1} = F(y_{2n}, t_{2n}, x_{2n}, y_{2n}) \ ; \\ t_{2n+1} = F(t_{2n}, x_{2n}, y_{2n}, z_{2n}) \end{cases} \quad \text{and}$$

$$\begin{cases} x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}, t_{2n+1}) \ ; \\ y_{2n+2} = G(y_{2n+1}, z_{2n+1}, t_{2n+1}, x_{2n+1}) \ ; \\ z_{2n+2} = G(z_{2n+1}, t_{2n+1}, x_{2n+1}, y_{2n+1}) \ ; \\ t_{2n+2} = G(t_{2n+1}, x_{2n+1}, y_{2n+1}, z_{2n+1}) \end{cases}$$

$\forall n \in \mathbf{N} \cup \{0\}$, then

$d(x_{2n}, x_{2n+1}) \to d(A, B), d(y_{2n}, y_{2n+1}) \to d(A, B), d(z_{2n}, z_{2n+1}) \to d(A, B)$ and $d(t_{2n}, t_{2n+1}) \to d(A, B)$.

Proof. For each $n \in \mathbf{N}$ we have

$d(x_{2n}, x_{2n+1})$
$= d(F(x_{2n}, y_{2n}, z_{2n}, t_{2n}), G(x_{2n-1}, y_{2n-1}, z_{2n-1}, t_{2n-1})$
$\leq a_1 d(x_{2n}, x_{2n-1}) + a_2 d(y_{2n}, y_{2n-1}) + a_3 d(z_{2n}, z_{2n-1}) + a_4 d(t_{2n}, t_{2n-1})]$
$+ (1 - \eta) d(A, B)$.

$d(y_{2n}, y_{2n+1})$
$= d(F(y_{2n}, z_{2n}, t_{2n}, x_{2n}), G(y_{2n-1}, z_{2n-1}, t_{2n-1}, x_{2n-1}))$
$\leq a_1 d(y_{2n}, y_{2n-1}) + a_2 d(z_{2n}, z_{2n-1}) + a_3 d(t_{2n}, t_{2n-1}) + a_4 d(x_{2n}, x_{2n-1})$
$+ (1 - \eta) d(A, B)$.

$d(z_{2n}, z_{2n+1})$
$= d(F(z_{2n}, t_{2n}, x_{2n}, y_{2n}), G(z_{2n-1}, t_{2n-1}, x_{2n-1}, y_{2n-1}))$
$\leq a_1 d(z_{2n}, z_{2n-1}) + a_2 d(t_{2n}, t_{2n-1}) + a_3 d(x_{2n}, x_{2n-1}) + a_4 d(y_{2n}, y_{2n-1})$
$+ (1 - \eta) d(A, B)$.

And,

$d(z_{2n}, z_{2n+1})$
$= d(F(z_{2n}, x_{2n}, y_{2n}, z_{2n}), G(t_{2n-1}, x_{2n-1}, y_{2n-1}, z_{2n-1}))$
$\leq a_1 d(t_{2n}, t_{2n-1}) + a_2 d(x_{2n}, x_{2n-1}) + a_3 d(y_{2n}, y_{2n-1}) + a_4 d(z_{2n}, z_{2n-1})$
$+ (1 - \eta) d(A, B).$

Now, let $d_{2n} = d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}) + d(z_{2n}, z_{2n+1}) + d(t_{2n}, t_{2n+1})$, by taking the sum of the inequalities above, we get
$d_{2n} \leq (a_1 + a_2 + a_3 + a_4) d_{2n-1} + 4(1 - \eta) d(A, B).$
$= \eta d_{2n-1} + 4(1 - \eta) d(A, B).$
Similarly, we have
$d_{2n-1} \leq (a_1 + a_2 + a_3 + a_4) d_{2n-2} + 4(1 - \eta) d(A, B).$
$= \eta d_{2n-2} + 4(1 - \eta) d(A, B).$ Hence
$d_{2n} \leq \eta d_{2n-1} + 4(1 - \eta) d(A, B)$
$\leq \eta[\eta d_{2n-2} + 4(1 - \eta) d(A, B) + 4(1 - \eta) d(A, B)]$
$\leq \eta^2 d_{2n-2} + 4(1 - \eta)(1 + \eta) d(A, B)$
$\leq \eta^2 d_{2n-2} + 4(1 - \eta^2) d(A, B)$
$\leq \ldots$
$d_{2n} \leq \eta^n d_0 + 4(1 - \eta^n) d(A, B).$
$\begin{cases} x_{2n}, y_{2n}, z_{2n}, t_{2n} \in A \\ x_{2n+1}, y_{2n+1}, z_{2n+1}, t_{2n+1} \in B \end{cases}$ Thus

$d(A, B) \leq d(x_{2n}, x_{n+1}), \ d(A, B) \leq d(y_{2n}, y_{n+1}), \ d(A, B) \leq d(z_{2n}, z_{n+1})$
and
$d(A, B) \leq d(t_{2n}, t_{n+1}) \ \forall n.$
Hence, $4d(A, B) \leq d_{2n}.$
Consequently, $\eta^n d_0 + 4(1 - \eta^n) d(A, B) \geq d_0 \geq 4d(A, B).$
Letting $n \to \infty$, $\lim_{n \to \infty} d_{2n} = 4d(A, B)$, i.e.
$\lim_{n \to \infty}[d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}) + d(z_{2n}, z_{2n+1}) + d(t_{2n}, t_{2n+1})] = 4d(A, B).$
Since each of the 4 terms $d(x_{2n}, x_{2n+1}), d(y_{2n}, y_{2n+1}), d(z_{2n}, z_{2n+1})$ and $d(t_{2n}, t_{2n+1})$

are greater or equal to $d(A, B)$, it follows that $\begin{cases} d(x_{2n}, x_{2n+1}) \to d(A, B) \\ d(y_{2n}, y_{2n+1}) \to d(A, B) \\ d(z_{2n}, z_{2n+1}) \to d(A, B) \\ d(t_{2n}, t_{2n+1}) \to d(A, B). \end{cases}$

Lemma 2.18. Let A and B be nonempty subsets of a metric space X such that (A, B) and (B, A) have a property UC and $F : A^4 \to B$ and $G : B^4 \to A$ be two maps such that (F, G) be a Q-cyclic contraction. If for $(x_0, y_0, z_0, t_0) \in A^4$, we define

$$\begin{cases} x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}, t_{2n}), \\ y_{2n+1} = F(y_{2n}, z_{2n}, t_{2n}, x_{2n}), \\ z_{2n+1} = F(z_{2n}, t_{2n}, x_{2n}, y_{2n}), \\ t_{2n+1} = F(t_{2n}, x_{2n}, y_{2n}, z_{2n}) \end{cases} \text{, and}$$

$$\begin{cases} lx_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}, t_{2n+1}), \\ y_{2n+2} = G(y_{2n+1}, z_{2n+1}, t_{2n+1}, x_{2n+1}), \\ z_{2n+2} = G(z_{2n+1}, t_{2n+1}, x_{2n+1}, y_{2n+1}), \\ t_{2n+2} = G(t_{2n+1}, x_{2n+1}, y_{2n+1}, z_{2n+1}) \end{cases}$$

$\forall n \in \mathbf{N} \cup \{0\}$, then for $\epsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$

$a_1 d(x_{2m}, x_{2n+1}) + a_2 d(y_{2m}, y_{2n+1}) + a_3 d(z_{2m}, z_{2n+1}) + a_4 d(t_{2m}, t_{2n+1})$
$< d(A, B) + \epsilon.$ (2.4)

Proof. By Lemma 2.17, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$,
$d(x_{2n+1}, x_{2n+2}) \to d(A, B), d(y_{2n}, y_{2n+1}) \to d(A, B)$,
$d(y_{2n+1}, y_{2n+2}) \to d(A, B), d(z_{2n}, z_{2n+1}) \to d(A, B)$,
$d(z_{2n+1}, z_{2n+2}) \to d(A, B), \ d(t_{2n}, t_{2n+1}) \to d(A, B)$ and
$d(t_{2n+1}, t_{2n+2}) \to d(A, B)$.
Since (A, B) has a property UC, we also have $d(x_{2n}, x_{2n+2}) \to 0$,
$d(y_{2n}, y_{2n+2}) \to 0, d(z_{2n}, z_{2n+2}) \to 0$ and $d(t_{2n}, t_{2n+2}) \to 0$.
As (B, A) has a property UC, we also have,
$d(x_{2n+1}, x_{2n+3}) \to 0 \ d(y_{2n+1}, y_{2n+3}) \to 0$,
$d(z_{2n+1}, z_{2n+3}) \to 0$ and $d(t_{2n+1}, t_{2n+3}) \to 0$.
Suppose that (2.4) does not hold. Then there exists $\epsilon' > 0$ such that for all $k \in \mathbf{N}$, there is $m_k > n_k \geq k$ satisfying
$a_1 d(x_{2m_k}, x_{2n_k+1}) + a_2 d(x'_{2m_k}, x'_{2n_k+1}) + a_3 d(x''_{2m_k}, x''_{2n_k+1}) + a_4 d(x'''_{2m_k}, x'''_{2n_k+1}) \geq d(A, B) + \epsilon'$.
We can choose m_k in such a way that it is the smallest integer with $m_k > n_k$ and satisfying the relation above.
Then, $a_1 d(x_{2m_k-2}, x_{2n_k+1}) + a_2 d(y_{2m_k-2}, y_{2n_k+1}) + a_3 d(z_{2m_k-2}, z_{2n_k+1})$
$+ a_4 d(t_{2m_k-2}, t_{2n_k+1}) < d(A, B) + \epsilon'$.
Therefore we get,
$d(A, B) + \epsilon \leq a_1 d(x_{2m_k}, x_{2n_k+1}) + a_2 d(y_{2m_k}, x_{2n_k+1})$
$+ a_3 d(z_{2m_k}, z_{2n_k+1}) + a_4 d(t_{2m_k}, t_{2n_k+1})$
$\leq a_1 [d(x_{2m_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2n_k+1})]$
$+ a_2 [d(y_{2m_k}, y_{2m_k-2}) + d(y_{2m_k-2}, y_{2n_k+1})]$
$+ a_3 [d(z_{2m_k}, z_{2m_k-2}) + d(z_{2m_k-2}, z_{2n_k+1})]$
$+ a_4 [d(t_{2m_k}, t_{2m_k-2}) + d(t_{2mk-2}, t_{2n_k+1})]$
$< a_1 d(x_{2m_k}, x_{2m_k-2}) + a_2 d(y_{2m_k}, y_{2m_k-2})$
$+ a_3 d(z_{2m_k}, z_{2m_k-2}) + a_4 d(t_{2m_k}, t_{2m_k-2}) + d(A, B) + \epsilon'$.
Letting $k \to \infty$, we have

$a_1 d(x_{2m_k}, x_{2n_k+1}) + a_2 d(y_{2m_k}, x_{2n_k+1}) + a_3 d(z_{2m_k}, z_{2n_k+1})$
$+ a_4 d(t_{2m_k}, t_{2n_k+1}) \to d(A, B) + \epsilon'.$

By triangle inequality, we have

$a_1 d(x_{2m_k}, x_{2n_k+1}) + a_2 d(y_{2m_k}, y_{2n_k+1}) + a_3 d(z_{2m_k}, z_{2n_k+1}) + a_4 d(t_{2m_k}, t_{2n_k+1})$
$\leq a_1[d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1})]$
$+ a_2[d(y_{2m_k}, y_{2m_k+2}) + d(y_{2m_k+2}, y_{2n_k+3}) + d(y_{2n_k+3}, y_{2n_k+1})]$
$+ a_3[d(z_{2m_k}, z_{2m_k+2}) + d(z_{2m_k+2}, z_{2n_k+3}) + d(z_{2n_k+3}, z_{2n_k+1})]$
$+ a_4[d(t_{2m_k}, t_{2m_k+2}) + d(t_{2m_k+2}, t_{2n_k+3}) + d(t_{2n_k+3}, t_{2n_k+1})]$
$= a_1[d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, y_{2m_k+1}, z_{2m_k+1}, t_{2m_k+1}),$
$F(x_{2n_k+2}, y_{2n_k+2}, z_{2n_k+2}, t_{2n_k+2})) + d(x_{2nk+3}, x_{2n_k+1})]$
$+ a_2[d(y_{2m_k}, y_{2m_k+2}) + d(G(y_{2m_k+1}, z_{2m_k+1}, t_{2m_k+1}, x_{2m_k+1}),$
$F(y_{2n_k+2}, z_{2n_k+2}, t_{2n_k+2}, x_{2n_k+2})) + d(y_{2n_k+3}, y_{2n_k+1})]$
$+ a_3[d(z_{2m_k}, z_{2m_k+2}) + d(G(z_{2m_k+1}, t_{2m_k+1}, x_{2m_k+1}, y_{2m_k+1}),$
$F(z_{2n_k+2}, t_{2n_k+2}, x_{2n_k+2}, y_{2n_k+2})) + d(z_{2n_k+3}, z_{2n_k+1})]$
$+ a_4[d(t_{2m_k}, t_{2m_k+2}) + d(G(t_{2m_k+1}, x_{2m_k+1}, y_{2m_k+1}, z_{2m_k+1}),$
$F(t_{2n_k+2}, x_{2n_k+2}, y_{2n_k+2}, z_{2n_k+2})) + d(t_{2n_k+3}, t_{2n_k+1})]$

$\leq a_1[d(x_{2m_k}, x_{2m_k+2}) + a_1 d(x_{2m_k+1}, x_{2n_k+2}) + a_2 d(y_{2m_k+1}, y_{2n_k+2})$
$+ a_3 d(z_{2m_k+1}, z_{2n_k+2}) + a_4 d(t_{2m_k+1}, t_{2n_k+2}) + (1 - \eta) d(A, B)$
$+ d(x_{2n_k+3}, x_{2n_k+1})] + a_2[d(y_{2m_k}, y_{2m_k+2}) + a_1 d(y_{2m_k+1}, y_{2n_k+2})$
$+ a_2 d(z_{2m_k+1}, z_{2m_k+2}) + a_3 d(t_{2m_k+1}, t_{2n_k+2}) + a_4 d(x_{2m_k+1}, x_{2n_k+2})$
$+ (1 - \eta) d(A, B) + d(y_{2n_k+3}, y_{2n_k+1})] + a_3[d(z_{2n_k}, z_{2m_k+2})$
$+ a_1 d(z_{2m_k+1}, z_{2n_k+2}) + a_2 d(t_{2_m k+1}, t_{2n_k+2}) + a_3 d(x_{2m_k+1}, x_{2n_k+2})$
$+ a_4 d(y_{2m_k+1}, y_{2n_k+2}) + (1 - \eta) d(A, B) + d(z_{2n_k+3}, z_{2n_k+1})]$
$+ a_4[d(t_{2m_k}, t_{2m_k+2}) + a_1 d(t_{2m_k+1}, t_{2n_k+2}) + a_2 d(x_{2m_k+1}, x_{2n_k+2})$
$+ a_3 d(y_{2m_k+1}, y_{2n_k+2}) + a_4 d(z_{2m_k+1}, z_{2n_k+2}) + (1 - \eta) d(A, B)$
$+ d(x'''_{2nk+3}, t_{2n_k+1})]$
$\leq \eta[d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}) + d(y_{2m_k}, y_{2m_k+2}) + d(y_{2n_k+3}, y_{2n_k+1})$
$+ d(z_{2m_k}, z_{2m_k+2}) + d(z_{2n_k+3}, z_{2n_k+1}) + d(t_{2m_k}, t_{2m_k+2}) + d(x'''_{2mk+3}, t_{2n_k+1})]$
$+ \eta^2[d(x_{2m_k+1}, x_{2n_k+2}) + d(y_{2m_k+1}, y_{2n_k+2}) + d(z_{2m_k+1}, z_{2n_k+2}) + d(t_{2m_k+1}, t_{2n_k+2})$
$+ (1 - \eta^2) d(A, B).$

As $k \to \infty$, we have
$d(A, B) + \epsilon' \leq \eta^2[d(A, B) + \epsilon'] + (1 - \eta^2) d(A, B) = d(A, B) + \eta^2 \epsilon',$
which is a contradiction. Hence,
$a_1 d(x_{2m}, x_{2n+1}) + a_2 d(y_{2m}, y_{2n+1}) + a_3 d(z_{2m}, z_{2n+1}) + a_4 d(t_{2m}, t_{2n+1})$
$< d(A, B) + \epsilon.$

3. Main Results

Here, we state the existence and convergence of quadruple best proximity points for Q-cyclic contraction pairs on nonempty subsets of metric spaces satisfying the property of UC^*.

Theorem 3.1. Let A and B be nonempty closed subsets of a metric space X such that (A, B) and (B, A) have a property UC^*, and $F : A^4 \to B$ and $G : B^4 \to A$ be two mappings such that the ordered pair (F, G) is a Q-cyclic contraction. If $(x_0, y_0, z_0, t_0) \in A^4$, define,

$$\begin{cases} x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}, t_{2n}), \\ y_{2n+1} = F(y_{2n}, z_{2n}, t_{2n}, x_{2n}), \\ z_{2n+1} = F(z_{2n}, t_{2n}, x_{2n}, y_{2n}), \\ t_{2n+1} = F(t_{2n}, x_{2n}, y_{2n}, z_{2n}) \end{cases} \quad \text{and}$$

$$\begin{cases} x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}, t_{2n+1}), \\ y_{2n+2} = G(y_{2n+1}, z_{2n+1}, t_{2n+1}, x_{2n+1}), \\ z_{2n+2} = G(z_{2n+1}, t_{2n+1}, x_{2n+1}, y_{2n+1}), \\ t_{2n+2} = G(t_{2n+1}, x_{2n+1}, y_{2n+1}, z_{2n+1}) \end{cases}$$

$\forall n \in \mathbf{N} \cup \{0\}$. Then, F has a quadruple best proximity point $(l, j, k, r) \in A^4$ and G has a quadruple best proximity point $(l', j', k', r') \in B^4$. Moreover, $x_{2n} \to l$, $y_{2n} \to j$, $z_{2n} \to k$, $t_{2n} \to r$, $x_{2n+1} \to l'$, $y_{2n+1} \to j'$, $z_{2n+1} \to k'$, $t_{2n+1} \to r'$. In addition, if $l = j = k = r$ and $l' = j' = k' = r'$ then
$$d(l, l') + d(j, j') + d(k, k') + d(r, r') = 4d(A, B).$$

Proof. From Lemma 2.17, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$, and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$. Since (A, B) satisfies property UC, we get $d(x_{2n}, x_{2n+2}) \to 0$. Similarly, we have $d(x_{2n+1}, x_{2n+3}) \to 0$ because (B, A) satisfies property UC.

We now show that for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that

$$d(x_{2m}, x_{2n+1}) \le d(A, B) + \epsilon \ (3.1)$$

for all $m > n \ge N$.

Suppose (3.1) does not hold, then there exists $\epsilon > 0$ such that for all $k \in \mathbf{N}$ there exists $m_k > n_k \ge k$ such that

$$d(x_{2m_k}, x_{2n_k+1}) > d(A, B) + \epsilon. \ (3.2)$$

Now we choose m_k in such a way that it is the smallest integer with $m_k > n_k$ and satisfying the inequality above. We get

$$\begin{aligned} d(A, B) + \epsilon \ &< \ d(x_{2m_k}, x_{2n_k+1}) \\ &\le \ d(x_{2m_k}, x_{2m_k-2}) + d(x_{2m_k-2}, x_{2n_k+1}) \\ &\le \ d(x_{2m_k}, x_{2m_k-2}) + d(A, B) + \epsilon. \end{aligned}$$

Taking $k \to \infty$, we have $d(x_{2m_k}, x_{2n_k+1}) \to d(A, B) + \epsilon$. By Lemma 2.18, there exists $N \in \mathbf{N}$ such that

$a_1 d(x_{2m_k}, x_{2n_k+1}) + a_2 d(y_{2m_k}, y_{2n_k+1}) + a_3 d(z_{2m_k}, z_{2n_k+1}) + a_4 d(t_{2m_k}, t_{2n_k+1})$
$< d(A, B) + \epsilon$ (3.3)

for all $m_k > n_k \geq \mathbf{N}$. By triangle inequality, we have

$d(A, B) + \epsilon < d(x_{2m_k}, x_{2n_k+1})$
$\leq d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, , x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1})$
$= d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, y_{2m_k+1}, z_{2m_k+1}, t_{2m_k+1}),$
$F(x_{2n_k+2}, y_{2n_k+2}, z_{2n_k+2}, t_{2n_k+2})) + d(x_{2n_k+3}, x_{2n_k+1})$
$\leq d(x_{2m_k}, x_{2m_k+2}) + a_1 d(x_{2m_k+1}, x_{2n_k+2}) + a_2 d(y_{2m_k+1}, y_{2n_k+2})$
$+ a_3 d(z_{2m_k+1}, z_{2n_k+2}) + a_4 d(t_{2m_k+1}, t_{2n_k+2}) + (1 - \eta) d(A, B)$
$+ d(x_{2n_k+3}, x_{2n_k+1})$
$= a_1 [d(F(x_{2m_k}, y_{2m_k}, z_{2m_k}, t_{2m_k}), G(x_{2n_k+1}, y_{2n_k+1}, z_{2n_k+1}, t_{2n_k+1}))]$
$+ a_2 [d(F(y_{2m_k}, z_{2m_k}, t_{2m_k}, x_{2m_k}), G(y_{2n_k+1}, z_{2n_k+1}, t_{2n_k+1}, x_{2n_k+1}))]$
$+ a_3 [d(F(z_{2m_k}, t_{2m_k}, x_{2m_k}, y_{2m_k}), G(z_{2n_k+1}, t_{2n_k+1}, x_{2n_k+1}, y_{2n_k+1}))]$
$+ a_4 [d(F(t_{2m_k}, x_{2m_k}, y_{2m_k}, z_{2m_k})), G(t_{2n_k+1}, x_{2n_k+1}, y_{2n_k+1}, z_{2n_k+1}))]$
$+ (1 - \eta) d(A, B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})$
$\leq a_1 [a_1 d(x_{2m_k}, x_{2n_k+1}) + a_2 d(y_{2m_k}, y_{2n_k+1}) + a_3 d(z_{2m_k}, z_{2n_k+1})$
$+ a_4 d(t_{2m_k}, t_{2n_k+1}) + (1 - \eta) d(A, B)]$
$+ a_2 [a_1 d(y_{2m_k}, y_{2n_k+1}) + a_2 d(z_{2m_k}, z_{2n_k+1}) + a_3 d(t_{2m_k}, t_{2n_k+1})$
$+ a_4 d(x_{2m_k}, x_{2n_k+1}) + (1 - \eta) d(A, B)$
$+ a_3 [a_1 d(z_{2m_k}, z_{2n_k+1}) + a_2 d(t_{2m_k}, t_{2n_k+1}) + a_3 d(x_{2m_k}, x_{2n_k+1})$
$+ a_4 d(y_{2m_k}, y_{2n_k+1}) + (1 - \eta) d(A, B)$
$+ a_4 [a_1 d(t_{2m_k}, t_{2n_k+1}) + a_2 d(x_{2m_k}, x_{2n_k+1}) + a_3 d(y_{2m_k}, y_{2n_k+1})$

$+ \quad a_4 d(z_{2mk}, z_{2nk+1}) + (1 - \eta) d(A, B)] + (1 - \eta) d(A, B)$
$+ d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})$
$= \quad \eta^2 [d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1})$
$+ d(z_{2m_k}, z_{2n_k+1}) + d(t_{2m_k}, x'''_{2n_k+1}] + (1 - \eta^2) d(A, B)$
$+ d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})$
$< \quad \eta^2 (d(A, B) + \epsilon) + (1 - \eta^2) d(A, B) + d(x_{2m_k}, x_{2m_k+2})$
$+ d(x_{2n_k+3}, x_{2n_k+1})$
$= \quad \eta^2 \epsilon + d(A, B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1}).$

As $k \to \infty$, we have
$d(A, B) + \epsilon < d(A, B) + \eta^2 \epsilon$. A contradiction, hence the condition (3.1) holds. Since (3.1) holds, and $d(x_{2n}, x_{2n+1}) \to d(A, B)$, by using property UC^* of (A, B) we obtain $d(x_{2m}, x_{2n+1}) \to d(A, B)$ then $d(x_{2n}, x_{2m}) \to 0$. Hence, $\{x_{2n}\}$ is a Cauchy sequence. Similarly, we can show that $\{y_{2n}\}, \{z_{2n}\}, \{t_{2n}\},$
$\{x_{2n+1}\}, \{y_{2n+1}\}, \{z_{2n+1}\}$ and $\{t_{2n+1}\}$ are Cauchy sequences.

Thus there exists $l, j, k, r \in A$ such that
$\{x_{2n}\} \to l, \ \{y_{2n}\} \to j, \{z_{2n}\} \to k, \ \{t_{2n}\} \to r$. We have
$d(A, B) \le d(l, x_{2n-1}) \le d(l, x_{2n}) + d(x_{2n}, x_{2n-1})$. (3.4)
Letting $n \to \infty$ in(3.4), we have $d(l, x_{2n-1}) \to d(A, B)$. Similarly,
$d(j, x_{2n-1}) \to d(A, B), d(k, x_{2n-1}) \to d(A, B), d(r, x_{2n-1}) \to d(A, B)$.
Then

$$
\begin{aligned}
d(x_{2n}, F(l, j, k, r)) &= d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}, t_{2n-1}), F(l, j, k, r)) \\
&\le a_1 d(x_{2n-1}, l) + a_2 d(y_{2n-1}, j) + a_3 d(z_{2n-1}, k) \\
&\quad + a_4 d(t_{2n-1}, r) + (1 - \eta) d(A, B).
\end{aligned}
$$

Taking $n \to \infty$, we have $d(l, F(l, j, k, r)) = d(A, B)$. Similarly,
$d(j, F(j, k, r, l)) = d(A, B), \ d(k, F(k, r, l, j)) = d(A, B)$, and
$d(r, F(r, l, j, k)) = d(A, B)$.
Therefore, we have, (l, j, k, r) is a quadruple best proximity point of
F. By similar argument we can prove that there exists $l', j', k', r' \in B$
such that $x_{2n+1} \to l', \ y_{2n+1} \to j', \ z_{2n+1} \to k', \ t_{2n+1} \to r'$.
Moreover, we have,
$d(l', F(l', j', k', r')) = d(A, B), \ d(j', G(j', k', r', l')) = d(A, B)$,
$d(k', G(k', r', l', j')) = d(A, B)$ and $d(r', G(r', l', j', k')) = d(A, B)$.
So (l', j', k', r') is a quadruple best proximity point of G.
Finally, let $l = j = k = r$ and $l' = j' = k' = r'$, we want to show that

$$d(l, l') + d(j, j') + d(k, k') + d(r, r') = 4d(A, B).$$

For all $n \in \mathbf{N}$, we get

$$
\begin{aligned}
d(x_{2n}, x_{2n+1}) &= d(G(x_{2n-1}, y_{2n-1}, z_{2n-1}, t_{2n-1}), F(x_{2n}, y_{2n}, z_{2n}, t_{2n})) \\
&\le a_1 d(x_{2n-1}, x_{2n}) + a_2 d(y_{2n-1}, y_{2n}) + a_3 d(z_{2n-1}, z_{2n}) \\
&\quad + a_4 d(t_{2n-1}, t_{2n}) + (1 - \eta) d(A, B).
\end{aligned}
$$

Letting $n \to \infty$ we have

$$d(l, l') \le a_1 d(l, l') + a_2 d(j, j') + a_3 d(k, k') + a_4 d(r, r') + (1 - \eta) d(A, B). \ (3.5)$$

For all $n \in \mathbf{N}$, we have

$$
\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(G(y_{2n-1}, z_{2n-1}, t_{2n-1}, x_{2n-1}), F(y_{2n}, z_{2n}, t_{2n}, x_{2n})) \\
&\le a_1 d(y_{2n-1}, y_{2n}) + a_2 d(z_{2n-1}, z_{2n}) + a_3 d(t_{2n-1}, t_{2n}) \\
&\quad + a_4 d(x_{2n-1}, x_{2n}) + (1 - \eta) d(A, B).
\end{aligned}
$$

Letting $n \to \infty$ we have

$$d(j, j') \le a_1 d(j, j') + a_2 d(k, k') + a_3 d(r, r') + a_4 d(l, l') + (1 - \eta) d(A, B). \ (3.6)$$

Also, For all $n \in \mathbf{N}$

$$
\begin{aligned}
d(z_{2n}, z_{2n+1}) &= d(G(z_{2n-1}, t_{2n-1}, x_{2n-1}, y_{2n-1}), F z_{2n}, t_{2n}, x_{2n}, y_{2n})) \\
&\le a_1 d(z_{2n-1}, z_{2n}) + a_2 d(t_{2n-1}, t_{2n}) + a_3 d(x_{2n-1}, x_{2n}) \\
&\quad + a_4 d(y_{2n-1}, y_{2n}) + (1 - \eta) d(A, B).
\end{aligned}
$$

Letting $n \to \infty$ we also have

$$d(k, k') \le a_1 d(k, k') + a_2 d(r, r') + a_3 d(l, l') + a_4 d(j, j') + (1-\eta)d(A, B). \quad (3.7)$$

Finally, For all $n \in \mathbf{N}$

$$\begin{aligned} d(t_{2n}, t_{2n+1}) &= d(G(t_{2n-1}, x_{2n-1}, y_{2n-1}, z_{2n-1}), F(t_{2n}, x_{2n}, y_{2n}, z_{2n})) \\ &\le a_1 d(t_{2n-1}, t_{2n}) + a_2 d(x_{2n-1}, x_{2n}) + a_3 d(y_{2n-1}, y_{2n}) \\ &\quad + a_4 d(z_{2n-1}, z_{2n}) + (1-\eta)d(A, B). \end{aligned}$$

Letting $n \to \infty$ we also have

$$d(r, r') \le a_1 d(r, r') + a_2 d(l, l') + a_3 d(j, j') + a_4 d(k, k') + (1-\eta)d(A, B). \quad (3.8)$$

Summing $(3.5), (3.6), (3.7)$ and (3.8) yields

$$d(l, l') + d(j, j') + d(k, k') + d(r, r')$$
$$\le \eta d(l, l') + \eta d(j, j') + \eta d(k, k') + \eta d(r, r')$$
$$+ 4(1-\eta)d(A, B).$$

Which implies that

$$d(l, l') + d(j, j') + d(k, k') + d(r, r') \le 4d(A, B).$$

Since $d(A, B) \le d(l, l')$, $d(A, B) \le d(j, j')$, $d(A, B) \le d(k, k')$ and $d(A, B) \le d(r, r')$, we have

$$d(l, l') + d(j, j') + d(k, k') + d(r, r') \ge 4d(A, B).$$

Thus from $(3.5), (3.6), (3.7)$ and (3.8),

$$d(l, l') + d(j, j') + d(k, k') + d(r, r') = 4d(A, B).$$

This ends the proof.

Corollary 3.2. Let A and B be nonempty closed subsets of a complete metric space X, such that (A, B) and (B, A) satisfy the property UC^*. Let $F : A \times A \times A \times A \to B, G : B \times B \times B \times B \to A$ and (F, G) be a cyclic contraction in the sense of Definition 2.5, there exists, $\alpha \in [0, 1)$ such that
$d(F(x, y, z, t), G(u, v, w, a)) \le \frac{\alpha}{4}[d(x, u) + d(y, v) + d(z, w) + d(t, a)] + (1-\alpha)d(A, B)$.
If $(x_0, y_0, z_0), t_0) \in A \times A \times A \times A$, define

$$\begin{cases} x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}, t_{2n}), \\ y_{2n+1} = F(y_{2n}, z_{2n}, t_{2n}, x_{2n}), \\ z_{2n+1} = F(z_{2n}, t_{2n}, x_{2n}, y_{2n}), \\ t_{2n+1} = F(t_{2n}, x_{2n}, y_{2n}, z_{2n}) \text{ and} \end{cases}$$

$$\begin{cases} x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}, t_{2n+1}), \\ y_{2n+2} = G(y_{2n+1}, z_{2n+1}, t_{2n+1}, x_{2n+1}), \\ z_{2n+2} = G(z_{2n+1}, t_{2n+1}, x_{2n+1}, y_{2n+1}), \\ t_{2n+2} = G(t_{2n+1}, x_{2n+1}, y_{2n+1}, z_{2n+1}) \; \forall n \in N \cup \{0\}. \end{cases}$$

F has a quadruple best proximity point $(x, y, z, t) \in A \times A \times A \times A$ and G has a quadruple best proximity point $(u, v, w, a) \in B \times B \times B \times B$

such that

$$d(x, u) + d(y, v) + d(z, w) + d(t, a) = 4d(A, B).$$

Proof. Taking $a_1 = a_2 = a_3 = a_4 = \frac{\alpha}{4}$, the result follows from Theorem (3.1).

Note that every closed subset A, B of a uniformly convex Banach space X such that A is convex and satisfies the property UC. Therefore, we obtain the following corollary.

Corollary 3.3. Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X, $F : A^4 \to B$ and $G : B^4 \to A$ and the ordered pair (F, G) be a Q-cyclic contraction. If $(x_0, y_0, z_0, t_0) \in A^4$ and define,

$$\begin{cases} x_{2n+1} = F(x_{2n}, y_{2n}, z_{2n}, t_{2n}), \\ y_{2n+1} = F(y_{2n}, z_{2n}, t_{2n}, x_{2n}), \\ z_{2n+1} = F(z_{2n}, t_{2n}, x_{2n}, y_{2n}), \\ t_{2n+1} = F(t_{2n}, x_{2n}, y_{2n}, z_{2n}) \text{ and} \end{cases}$$

$$\begin{cases} x_{2n+2} = G(x_{2n+1}, y_{2n+1}, z_{2n+1}, t_{2n+1}), \\ y_{2n+2} = G(y_{2n+1}, z_{2n+1}, t_{2n+1}, x_{2n+1}), \\ z_{2n+2} = G(z_{2n+1}, t_{2n+1}, x_{2n+1}, y_{2n+1}), \\ t_{2n+2} = G(t_{2n+1}, x_{2n+1}, y_{2n+1}, z_{2n+1}) \ \forall n \in \mathbf{N} \cup \{0\}. \end{cases}$$

Then, F has a quadruple best proximity point $(l, j, k, r) \in A^4$ and G has a quadruple best proximity point $(l', j', k', r') \in B^4$. Moreover, $x_{2n} \to l$, $y_{2n} \to j$, $z_{2n} \to k$, $t_{2n} \to r$, $x_{2n+1} \to l'$, $z_{2n+1} \to j'$, $z_{2n+1} \to k'$, $t_{2n+1} \to r'$.

Furthermore, if $l = j = k = r$ and $l' = j' = k' = r'$, then $d(l, l') + d(j, j') + d(k, k') + d(r, r') = 4d(A, B)$.

We give an illustrative example of Corollary 3.3

Example 3.4. Consider the uniformly convex Banach space $X = \mathbf{R}$ with the usual norm. Let A = [1,2] and B = [-3, -2]. We see that d(A, B) = 3. Define $F : A^4 \to B$ and $G : B^4 \to A$ by $F(x, y, z, t) = \frac{-2x-y-z-3t-3}{9}$ and $G(x, y, z, t) = \frac{-2x-y-z-3t+3}{9}$.

For arbitrary $(x, y, z, t) \in B^4$ and $(u, v, w, a) \in A^4$ and fixed $\eta = \frac{7}{9}$, we get

$d(F(x, y, z, t), G(u, v, w, a))$
$= \mid \frac{-2x - y - z - 3t - 3}{9} - \frac{-2x - y - z - 3t + 3}{9} \mid$
$\leq 2 \mid \frac{x - y}{9} \mid + \mid \frac{y - v}{9} \mid + \mid \frac{z - w}{9} \mid + 3 \mid \frac{t - a}{9} \mid + \frac{2}{3}$
$= \frac{2}{9} d(x, u) + \frac{1}{9} d(y, v + \frac{1}{9} d(z, w) + \frac{1}{3} d(t, a) + (1 - \frac{7}{9}) \times 3$
$= a_1 d(x, u) + a_2 d(y, v) + a_3 d(z, w) + a_4 d(t, a)] + (1 - \eta) d(A, B).$

This implies that (F, G) is a Q-cyclic contraction with $\eta = \frac{7}{9}$.

Since A and B are closed convex, we get (A, B) and (B, A) satisfying the property UC^*. Therefore, Corollary 3.4 holds. Thus, F has a quadruple best proximity point and G has a quadruple best proximity point. The point $(1, 1, 1, 1) \in A^4$ is a unique quadruple best proximity point of F and the point $(-2, -2, -2, -2) \in B^4$ is a unique quadruple best proximity point of G. Moreover, we get

$$d(1, -2) + d(1, -2) + d(1, -2) + d(1, -2) = 12 = 4d(A, B).$$

Example 3.5 Let $X = \mathbf{E}^2$ with the metric
$d(x, y, z, t), (u, v, w, a) = \max\{\mid x - u \mid, \mid y - v \mid, \mid z - w \mid, \mid t - a \mid\}$ and
let $A = \{(x, 1) : 1 \leq x \leq 3\}$ and $A = \{(x, 3) : 1 \leq x \leq 3\}$. We see
that $d(A, B) = 2$. Define $F : A^4 \to B$ and $G : B^4 \to A$ by
$d(F(x, 1), (y, 1), (z, 1), (t, 1)) = (\frac{x + y + z + t}{4}, 3)$ and
$d(G(x, 3), (y, 3), (z, 3, (t, 3)) = (\frac{x + y + z + t}{4}, 1)$.
$d(F(x, 1), (y, 1), (z, 1), (t, 1)), G(u, 3), (v, 3), (w, 3, (a, 3))$
$= d((\frac{x + y + z + t}{4}, 3), \frac{u + v + w + a}{4}, 1)) = 2$.
For all $\eta \in [0, 1)$, and $\sum_{i=1}^{4} a_i = \eta < 1$, we get
$a_1 d(x, 1), (u, 3) + a_2 d(y, 1), (v, 3) + a_3 d(z, 1), (w, 3) + a_4 d(t, 1), (a, 3)$

$+ (1 - \eta) d(A, B).$
$= a_1 \max[\{\mid x - u \mid, 2\}] + a_2 \max[\{\mid y - v \mid, 2\}] + a_3 \max[\{\mid z - w \mid, 2\}]$
$+ a_4 \max[\{\mid t - a \mid, 2\}] + (1 - \eta) d(A, B)$
$= 2(a_1 + a_2 + a_3 + a_4) + (1 - \eta)(2)$
$= 2\eta + (1 - \eta)(2)$
$= 2.$

Remark. It has been shown that most of the coupled fixed point theorems are immediate consequences of the known fixed point results (see[27,31]). But to the best of the authors' knowledge, it is not in literature that the coupled best proximity point result could be derived from their corresponding best proximity point results. Also, to the best of authors' knowledge, non of the few results of coupled best proximity point in literature can be deduced from any of the best proximity point results in literature.

Competing Interest.

The authors declare that they have no competing interest.

Authors' Contributions. All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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