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COMMON FIXED POINTS OF WEAK GENERALIZED (α, ψ)-CONTRACTIVE MAPS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we introduce a pair (f, g) of weak generalized (α, ψ)-contractive maps with rational expressions and prove the existence of common fixed points when (f, g) is a pair of weakly compatible maps and the range of g is complete in partially ordered metric spaces where f is a triangular (α, g)-admissible map. Further, we prove the same conclusion by relaxing the condition 'range of g is complete', but by imposing reciprocally continuity of (f, g) and compatibility of (f, g) in partially ordered complete metric spaces. Our results generalize the results of Arshad, Karapinar and Ahmad [1] and Harjani, Lopez and Sadarangani [5].

Keywords: α-admissible, (α, g)-admissible, triangular α-admissible, triangular (α, g)-admissible map f, (α, ψ)-contractive mapping, a pair (f, g) of weak generalized (α, ψ)-contractive maps with rational expressions.

1. Introduction

Proving existence and uniqueness of common fixed points by using weak commutativity assumptions under more general contraction conditions having rational expressions in partially ordered metric space is the present interest. In 2012, Samet, Vetro and Vetro [12] introduced a new concept namely (α, ψ)-contractive mappings which generalize contractive mappings and proved the existence of fixed points of such mappings in metric space setting.

In the following, Ψ denotes the family of non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that ψ is continuous on $[0, \infty)$ and $\Sigma_{n=1}^{\infty} \psi^n(t) < +\infty$ for each t > 0, where ψ^n is the n^{th} iterate of ψ.

Remark 1.1. Any function $\psi \in \Psi$ satisfies $\lim_{n\to\infty} \psi^n(t) = 0$ and $\psi(t) < t$ for any $t > 0$.

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Definition 1.2. [12] Let (X, d) be a metric space, $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that f is *α-admissible*, if
$x,\ y \in X,\ \alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1.$ (1.2.1)

Definition 1.3. [3] Let $f,\ g$ be two self mappings on X. Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that the map f is *(α, g)-admissible map* if for
$x, y \in X, \alpha(gx, gy) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1.$ (1.3.1)

In 2013, Karapinar, Kumam and Salimi [11] introduced the notion of a triangular α-admissible mapping as follows:

Definition 1.4. [12] Let (X, d) be a metric space, and let $f : X \to X$ and
$\alpha : X \times X \to [0, \infty)$. We say that f is *triangular α-admissible*, if

(i) f is α-admissible; and
(ii) $\alpha(x, y) \geq 1, \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$ for any $x,\ y \in X$. (1.4.1)

Definition 1.5. [12] Let (X, d) be a metric space and $f : X \to X$ be a selfmap of X. If there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that
$\alpha(x, y)d(fx, fy) \leq \psi(d(x, y))$ for all $x,\ y \in X$,
then we say that f is a *(α, ψ)-contractive mapping.*

Remark 1.6. If $f : X \to X$ is a contraction with contractive constant $0 < k < 1$, then f is an (α, ψ)-contraction mapping, where $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$.

Theorem 1.7. [12] *Let (X, d) be a complete metric space and $f : X \to X$ be an (α, ψ)-contractive mapping. Suppose that*

(i) *f is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$; and*
(iii) *f is continuous.*

Then there exists $u \in X$ such that $fu = u$.

In 1977, Jaggi [7] introduced 'rational type contraction mappings' as an extension of 'contraction maps' and proved the existence of fixed points of such mappings.

Theorem 1.8. [7] *Let f be a continuous self-map defined on a complete metric space (X, d). Suppose that f satisfies the following condition: there exist $\alpha,\ \beta \in [0, 1)$ with
$\alpha + \beta < 1$ such that*
$d(fx, fy) \leq \alpha \frac{d(x,fx)d(y,fy)}{d(x,y)} + \beta d(x, y)$ *for all $x, y \in X$, $x \neq y$.* (1.8.1)
Then f has a fixed point in X.

Here we note that a mapping $f : X \to X$, X a metric space, that satisfies (1.8.1) is called a *Jaggi contraction map* on X.
Later Karapinar and Samet [10] introduced generalized (α, ψ)-contractive mappings and proved fixed point results and its extension to partially ordered metric spaces.
Harjani, Lopez and Sadarangani [5] extended Theorem 1.8 to the context of partially ordered complete metric spaces.

Theorem 1.9. [5] *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a non-decreasing mapping such that*
$$d(fx, fy) \leq \alpha\frac{d(x,fx)d(y,fy)}{d(x,y)} + \beta d(x, y) \qquad (1.9.1)$$
for all $x, y \in X$ with $x \succeq y$, $x \neq y$ where $0 \leq \alpha, \beta < 1$ with $\alpha + \beta < 1$. Also, assume either

(i) *f is continuous; (or)*
(ii) *if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \to x$ as $n \to \infty$ then $x = sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

A map f that satisfies the inequality (1.9.1) is called *Jaggi contraction map* in partially ordered metric spaces.

In 2013, Arshad, Karapinar and Ahmad [1] extended Theorem 1.9 to almost Jaggi contraction type mappings in partially ordered metric spaces.

Definition 1.10. [1] Let $(X, d, \preceq)$ be a partially ordered metric space. A selfmapping f on X is called an *almost Jaggi contraction* if it satisfies the following condition: there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $L \geq 0$ such that
$$d(fx, fy) \leq \alpha\frac{d(x,fx)d(y,fy)}{d(x,y)} + \beta.d(x, y) + L \min\{d(x, fx), d(x, fy), d(y, fx)\},$$
(1.10.1)
for any distinct $x,\ y \in X$, with $x \preceq y$.

Theorem 1.11. [1] *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that f is a continuous and non-decreasing selfmap that satisfies the following inequality : there exist $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ and $L \geq 0$ such that*
$$d(fx, fy) \leq \alpha\frac{d(x,fx)d(y,fy)}{d(x,y)} + \beta.d(x, y) + L \min\{d(x, fy), d(y, fx)\}, \qquad (1.11.1)$$
for any distinct $x,\ y \in X$, with $x \preceq y$. Suppose that there exists $x_0 \in X$ with $x_0 \preceq fx_0$. Then f has a unique fixed point.

Remark 1.12. Since every almost Jaggi contraction satisfies the inequality (1.11.1), it follows that the conclusion of Theorem 1.11 is valid under the replacement of condition (1.11.1) by almost Jaggi contraction in Theorem 1.11.

In 1986, Jungck [8] introduced the concept of 'compatible maps' as a generalization of 'commuting maps' and proved the existence of fixed points in metric spaces. In 1998, Jungck and Rhoades [9] introduced the concept of 'weakly compatible' maps as a generalization of 'compatible maps'.

Definition 1.13. [8] Two self mappings f and g of a metric space (X, d) are said to be *compatible* if $\lim\limits_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that
$\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = u$ for some $u \in X$.

Definition 1.14. [9] Two self mappings f and g of a metric space (X, d) are said to be *weakly compatible* if they commute at their coincidence points, i.e., if $fu = gu$ for some $u \in X$, then $fgu = gfu$.

Definition 1.15. [13] Let (X, d) be a metric space and f, g be self maps of X. We say that f and g are *reciprocally continuous* if $\lim\limits_{n\to\infty} fgx_n = fz$ and $\lim\limits_{n\to\infty} gfx_n = gz$ whenever $\{x_n\}$ is a sequence in X with $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = z$ for some $z \in X$.

Lemma 1.16. [2] *Suppose* $(X,\ d)$ *is a metric space. Let* $\{x_n\}$ *be a sequence in* X *such that* $d(x_{n+1}, x_n) \to 0$ *as* $n \to \infty$. *If* $\{x_n\}$ *is not a Cauchy sequence then there exist an* $\epsilon > 0$ *and sequences of positive integers* $\{m(k)\}$ *and* $\{n(k)\}$ *with* $m(k) > n(k) > k$ *such that*
$d(x_{m(k)}, x_{n(k)}) \geq \epsilon,\ d(x_{m(k)-1}, x_{n(k)}) < \epsilon$ *and*

(i) $\lim\limits_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon$, *(ii)* $\lim\limits_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$,

(iii) $\lim\limits_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$ *and (iv)* $\lim\limits_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \epsilon$.

Definition 1.17. [4] Suppose $(X, \preceq)$ is a partially ordered set and $f, g : X \to X$ are two self mappings of X. f is said to be g-non-decreasing if for $x, y \in X$,

$$gx \preceq gy \text{ implies } fx \preceq fy. \tag{1.17.1}$$

In Section 2, we introduce a pair (f, g) of weak generalized (α, ψ)-contractive maps with rational expressions and discuss various consequences of these maps. In Section 3, we prove the existence of common fixed points when (f, g) is a pair of weakly compatible maps and the range of g is complete in partially ordered metric spaces where f is a triangular (α, g)-admissible map (Theorem 3.1). Further, we prove the same conclusion by relaxing the condition 'range of g is complete', but by imposing reciprocally continuity of (f, g) and compatibility of (f, g) in partially ordered complete metric spaces (Theorem 3.4). In Section 4, we deduce some corollaries from our main results and construct examples in support of our results.

2. Preliminaries

In the following, we now introduce the concept namely 'f is a triangular (α, g)-admissible map'.

Definition 2.1. Let $f,\ g$ be two self mappings on X. Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that the map f *is triangular* (α, g)-*admissible map* if

(i) f is (α, g)-admissible; and
(ii) $\alpha(gx, gy) \geq 1, \alpha(gy, gz) \geq 1 \Rightarrow \alpha(gx, gz) \geq 1$ for any $x, y \in X$. (2.1.1)

If $g = I_X$, the identity map of X in (2.1.1), then we call f is triangular α-admissible.

Example 2.2. Let $X = [0, 3]$ with the usual metric.
Let A $= \Delta \bigcup \{(0, 2), (3, 0), (0, 3), (2, 3)\}$ and B $= \{(x, y) \in X \times X : x \neq y\} \setminus \{(0, 2), (3, 0), (0, 3), (2, 3)\}$.
We define $f, g : X \to X$ by $fx = \begin{cases} \frac{3x}{2} & if\ x \in [0, 2] \\ 3 & if\ x \in (2, 3] \end{cases}$ and $gx = \begin{cases} x & if\ x \in [0, 2] \\ \frac{x+1}{2} & if\ x \in (2, 3]. \end{cases}$
We define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = \begin{cases} 2e^{|x-y|} & if\ (x, y) \in A \\ 0 & otherwise. \end{cases}$
Then it is easy to see that f is a triangular (α, g)-admissible mapping. But g is not (α, f)-admissible mapping, for, we choose $x = 3$ and $y = 0$. In this case
$fx = 3,\ fy = 0;\ gx = 2$ and $gy = 0$, hence we have
$\alpha(fx, fy) = \alpha(3, 0) = 2e^3 \geq 1$ but $\alpha(gx, gy) = \alpha(2, 0) \ngeq 1$.
Therefore g is not (α, f)-admissible mapping.

Example 2.3. Let $X = [0, 2]$ with the usual metric.
Let A $= \Delta \bigcup \{(0, 1), (1, 2)\}$ and B $= \{(x, y) \in X \times X : x \neq y\} \setminus \{(0, 1), (1, 2)\}$. We define $f, g : X \to X$ by

$fx = \begin{cases} x^2 & if\ x \in [0,1] \\ \frac{x}{2} & if\ x \in (1,2] \end{cases}$ and $gx = \begin{cases} x & if\ x \in [0,1] \\ 2 & if\ x \in (1,2]. \end{cases}$

We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases} 2 & if\ (x,y) \in A \\ 0 & otherwise. \end{cases}$

Then it is easy to see that f is (α, g)-admissible map. But g is not (α, f)-admissible map, for, we choose $x = 0$ and $y = 2$.
In this case $fx = 0,\ fy = 1;\ gx = 0$ and $gy = 2$, hence we have
$\alpha(fx, fy) = \alpha(0,1) = 2 \geq 1$ but $\alpha(gx, gy) = \alpha(0,2) \ngeq 1$.
Here, we observe that f is not triangular (α, g)-admissible map, for, by choosing
$(x,y) = (0,1)$ and $(y,z) = (1,2)$ we have
$\alpha(g0, g1) = (0,1) \geq 1,\ \alpha(g1, g2) = (1,2) \geq 1$ but $\alpha(g0, g2) = \alpha(0,2) \ngeq 1$.
Therefore, condition (ii) of inequality (2.1.1) fails to hold.

Example 2.4. Let $X = \{1,2,3\}$ with the usual metric. Let A $= \{(1,1),(2,2),(3,3),$
$(1,2),(2,3),(3,2)\}$ and B $= \{(2,1),(3,1),(1,3)\}$. We define $f : X \to X$ by
$f1 = f3 = 3,\ f2 = 1;\ g1 = 1,\ g2 = 2$ and $g3 = 3$. We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases} 2 & if\ (x,y) \in A \\ 0 & otherwise. \end{cases}$

Then it is easy to see that neither f is (α, g)-admissible nor g is (α, f)-admissible map, for, we choose $x = 1$ and $y = 2$.
In this case $fx = 3,\ fy = 1;\ gx = 1$ and $gy = 2$, and we have
$\alpha(gx, gy) = \alpha(1,2) = 2 \geq 1$ but $\alpha(fx, fy) = \alpha(3,1) \ngeq 1$.
Therefore f is not (α, g)-admissible map.
Here, we observe that f is not triangular (α, g)-admissible map.
For, by choosing $(x,y) = (1,2)$ and $(y,z) = (2,3)$ we have
$\alpha(g1, g2) = (1,2) = 2 \geq 1$ and $\alpha(g2, g3) = (2,3) = 2 \geq 1$ but $\alpha(g1, g3) = (1,3) \ngeq 1$.
Therefore, condition (ii) of inequality (2.1.1) fails to hold. *i.e.*, f is not triangular (α, g)-admissible. Further, we choose $x = 1$ and $y = 3$.
In this case $fx = fy = 3;\ gx = 1$ and $gy = 3$, hence we have
$\alpha(fx, fy) = \alpha(3,3) = 2 \geq 1$ but $\alpha(gx, gy) = \alpha(1,3) \ngeq 1$.
Therefore g is not (α, f)-admissible map.

Example 2.5. Let $X = \{0,1,2,3\}$ with the usual metric.
Let A $= \{((0,0),(1,1),(2,2),(3,3),(3,1),(0,2),(0,3)\}$ and B $= \{(0,1),(2,0),(1,0),$
$(2,1),(1,2),(3,0),(1,3),(2,3),(3,2)\}$. We define $f : X \to X$ by $f0 =$
$f3 = 0,$
$f1 = f2 = 2;\ g0 = g2 = 0,\ g1 = 1$ and $g3 = 3$. We define

$\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = \begin{cases} 2 & if\ (x, y) \in A \\ 0 & otherwise. \end{cases}$

Then it is easy to see that f is (α, g)-admissible map, but f is not triangular (α, g)-admissible map. For, by choosing $(x, y) = (0, 3)$ and $(y, z) = (3, 1)$ we have

$\alpha(g0, g3) = (0, 3) = 2 \geq 1$ and $\alpha(g3, g1) = (3, 1) = 2 \geq 1$ but $\alpha(g0, g1) = (0, 1) \ngeq 1$.

Here, we observe that g is triangular (α, f)-admissible map, for, by choosing

$(x, y) = (3, 0)$ and $(y, z) = (0, 1)$ we have

$\alpha(f3, f0) = \alpha(0, 0) = 2 \geq 1$ and $\alpha(f0, f1) = \alpha(0, 2) = 2 \geq 1$

implies $\alpha(f3, f1) = \alpha(0, 2) \geq 1$.

Hence g is triangular (α, f)-admissible map, but g is not (α, f)-admissible map.

Remark 2.6. Let f be a triangular (α, g)-admissible mapping and suppose $f(X) \subseteq g(X)$. Assume that there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $gx_{n+1} = fx_n$. Then $\alpha(gx_m, gx_n) \geq 1$ for all $m, n \in N$ with $m < n$.

Definition 2.7. Let $(X, \preceq)$ be a partially ordered metric space and suppose that $f : X \to X$ be a mapping. If there exist two functions $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$ and $L \geq 0$ such that

$\alpha(x, y)d(fx, fy) \leq \psi(M(x, y)) + L.N(x, y)$, where (2.7.1)

$$M(x, y) = \begin{cases} \max\{d(x, y), \frac{d(x,fx)d(y,fy)}{d(x,y)}, \frac{d(x,fy)d(y,fx)}{d(x,y)}, \\ \quad \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{2d(x,y)}\} & if\ x \preceq y,\ x \neq y \\ 0 & if\ x = y \end{cases}$$

and $N(x, y) = \min\{d(x, fx), d(x, fy), d(y, fx)\}$, $x,\ y \in X$ with $x \preceq y$, then we say that f is a *weak generalized* (α, ψ)-contractive map with rational expressions.

Note: Clearly, if f is Jaggi contraction *i.e.*, a map f that satisfies (1.9.1) with $\alpha + \beta < 1$ then it satisfies the inequality (2.7.1) with $\alpha(x, y) = 1\ \forall\ x, y \in X$, L = 0 and $\psi(t) = (\alpha + \beta)t$, $t \geq 0$ so that f is a weak generalized (α, ψ)-contractive map with rational expressions. But, its converse need not be true.

Example 2.8. Let $X = \{0, 1, 2\}$ with the usual metric.

We define a partial order $\preceq$ on X by $\preceq := \{(0, 0), (1, 1), (2, 2), (0, 1)(0, 2), (1, 2)\}$.

Let A $= \{(0, 0), (0, 2), (1, 1), (2, 2), (2, 0), (1, 2)\}$ and B $= \{(0, 1), (1, 0), (2, 1)\}$.

We define $f : X \to X$ by $f0 = 2,\ f1 = 0$ and $f2 = 2$.

We define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = \begin{cases} \frac{3}{2} & if\ (x, y) \in A \\ 0 & otherwise. \end{cases}$

Then it is easy to see that f is a weak generalized (α, ψ)-contractive map with rational expressions. In the following, we mention the importance of L in the inequality (2.7.1).
If $L = 0$ then the inequality (2.7.1) fails to hold.
For, by choosing $x = 1$ and $y = 2$ we have
$d(f1, f2) = 2 \not\leq \psi(2) = \psi(M(x, y))$ for any $\psi \in \Psi$.
Here we observe that the inequality (1.9.1) fails to hold.
For, when $(x, y) = (1, 2)$ we have
$d(fx, fy) = d(f1, f2) = 2 \not\leq \alpha.0 + \beta.1 = \alpha\frac{d(1,f1)d(2,f2)}{d(1,2)} + \beta d(1, 2)$
$= \alpha\frac{d(x,fx)d(y,fy)}{d(x,y)} + \beta d(x, y)$.
Hence f is not a Jaggi contraction map. Also, we observe that the inequality (1.11.1) fails to hold. For, when $(x, y) = (0, 1)$ we have
$d(fx, fy) = d(f0, f1) = 2$
$\not\leq \alpha.0 + \beta.1 = \alpha\frac{d(0,f0)d(1,f1)}{d(0,1)} + \beta d(0, 1) + L.0$
$= \alpha\frac{d(x,fx)d(y,fy)}{d(x,y)} + \beta d(x, y) + L \min\{d(x, fy), d(y, fx)\}$.

In the following, we extend Definition 2.7 two maps f and g.

Definition 2.9. Let $(X, \preceq)$ be a partially ordered metric space and let f and g be two self mappings on X. If there exist two functions $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$ and $L \geq 0$ such that
$\alpha(gx, gy)d(fx, fy) \leq \psi(M(x, y))+L.N(x, y)$, where (2.9.1)

$$M(x,y) = \begin{cases} \max\{d(gx, gy), \frac{d(gx,fx)d(gy,fy)}{d(gx,gy)}, \frac{d(gx,fy)d(gy,fx)}{d(gx,gy)}, \\ \frac{d(gx,fx)d(gx,fy)+d(gy,fy)d(gy,fx)}{2d(gx,gy)}\} & if\ x \preceq y,\ x \neq y \\ 0 & if\ x = y \end{cases}$$

and $N(x, y) = \min\{d(gx, fx), d(gx, fy), d(gy, fx)\}$, $x,\ y \in X$ with $x \preceq y$, then we say that (f, g) *is a pair of weak generalized* (α, ψ)-contractive maps with rational expressions.

If $g = I_X$, the identity map of X, in (2.9.1), then the inequality (2.9.1) reduces to (2.7.1) so that f is a weak generalized (α, ψ)-contractive map with rational expressions.
Note: Clearly, a map f that satisfies (1.9.1) with $\alpha + \beta < 1$ also satisfies the inequality (2.9.1) with $\alpha(x, y) = 1\ \forall\ x, y \in X$, L $= 0$, $g = I_X$ and $\psi(t) = (\alpha + \beta)t$, $t \geq 0$ so that f is a weak generalized (α, ψ)-contractive map with rational expressions. But, the following example suggests that its converse need not be true.

Example 2.10. Let $X = \{1, 2, 4, 6\}$ with the usual metric. We define a partial order $\preceq$ on X as follows, $\preceq := \{(1, 1), (2, 2), (4, 4), (6, 6), (1, 2), (1, 4), (2, 6), (1, 6)\}$.
Let A $= \{(1, 1), (2, 2), (4, 4), (6, 6), (1, 2), (1, 4), (2, 6)\}$, B $= \{(2, 1), (4, 1), (1, 6),$

$(6,1),(6,2),(2,4),(4,2),(4,6),(6,4)\}$. We define $f,g : X \to X$ by f1 = f2 = 1,
f4 = f6 = 2; g1 = 1 g2 = g4 = 4 and g6 = 2. We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases} \frac{3}{2} & if\ (x,y) \in A \\ 0 & otherwise. \end{cases}$ and $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{4}{5}t$.
The following three cases arise to verify the inequality (2.9.1).
Case (i): $x = 1$ and $y = 2$.
In this case, the inequality (2.9.1) holds trivially.
Case (ii): $x = 1$ and $y = 4$.
In this case, $d(f1, f4) = 1,\ M(1,4) = 3$ and $N(1,4) = 0$.
$\alpha(gx, gy)d(fx, fy) = \alpha(g1, g4)d(f1, f4)$
$= \frac{3}{2} \leq \psi(3) + L.0 = \psi(M(1,4)) + L.N(1,4)$
$= \psi(M(x,y)) + L.N(x,y)$ holds for any $L \geq 0$.
Case (iii): $x = 2$ and $y = 6$.
In this case, $d(f2, f6) = 1,\ M(2,6) = 3$ and $N(2,6) = 1$.
$\alpha(gx, gy)d(fx, fy) = \alpha(g2, g6)d(f2, f6)$
$= \frac{3}{2} \leq \psi(3) + L.1 = \psi(M(2,6)) + L.N(2,6)$
$= \psi(M(x,y)) + L.N(x,y)$ holds with L = 1.
If $x,\ y \in B$ then the inequality (2.9.1) holds trivially.
Hence, from above cases, we choose $L = 1$, so that the pair (f,g) is of weak generalized (α,ψ)-contractive maps with rational expressions with $L = 1$.
Now, we observe that the inequality (1.11.1) fails to hold. For, by choosing $x = 2$ and $y = 6$ we have
$d(f2, f6) = 1 \nleq \alpha(1) + \beta(4) + L.0 < 1$
$= \alpha\frac{d(2,f2)d(6,f6)}{d(2,6)} + \beta d(2,6) + L.\min\{d(2,f6), d(6,f2)\}$.
$i.e.$, f is not an almost Jaggi contraction map.
Further, we observe that the inequality (1.9.1) also fails to hold.
For, when $(x,y) = (1,4)$ we have
$d(fx, fy) = d(f1, f4) = 1\ \nleq\ \alpha.0 + \beta(3)$
$= \alpha\frac{d(1,f1)d(4,f4)}{d(1,4)} + \beta d(1,4) = \alpha\frac{d(x,fx)d(y,fy)}{d(x,y)} + \beta d(x,y)$.
This shows that the inequality (1.9.1) fails to hold so that f is not a Jaggi contraction map.

Example 2.11. Let $X = [0,\ 2]$ with the usual metric. We define a partial
order $\preceq$ on X by $\preceq := \{(x,y) \in X \times X : x = y\} \bigcup \{(1,0),(2,0),(2,1)\}$.
Let A = $\Delta \bigcup \{(1,0),(2,0),(2,1)\}$ and B = $\{(x,y) \in X \times X : x \neq y\} \setminus \{(1,0),(2,0),(2,1)\}$. We define $f,g : X \to X$ and $\alpha : X \times X :\to [0,\ \infty)$

by $f(x) = \begin{cases} x^2 & if\ x \in [0,1] \\ 1+\frac{x}{2} & if\ x \in (1,2], \end{cases}$

$g(x) = \begin{cases} 2x & if\ x \in [0,1] \\ 1 & if\ x \in (1,2] \end{cases}$ and $\alpha(x,y) = \begin{cases} \frac{3}{2} & if\ (x,y) \in A \\ 0 & otherwise. \end{cases}$

Now, we verify the inequality (2.9.1) by choosing $\psi \in \Psi$ by $\psi(t) = \begin{cases} \frac{1}{2} & if\ t \in [0,1) \\ \frac{t^2}{1+t} & if t \geq 1. \end{cases}$

The following three cases arise to verify the inequality (2.9.1).

Case (i): $x = 2$ and $y = 1$.

In this case, the inequality (2.9.1) holds trivially.

Case (ii): $x = 1$ and $y = 0$.

In this case, $d(f1, f0) = 1,\ M(1,0) = 2$ and $N(1,0) = 1$.

$\alpha(gx, gy)d(fx, fy) = \alpha(g1, g0)d(f1, f0)$
$= \frac{3}{2} \leq \psi(2) + L.1 = \psi(M(1,0)) + L.N(1,0)$
$= \psi(M(x,y)) + L.N(x,y)$ holds with $L = 3$.

Case (iii): $x = 2$ and $y = 0$.

In this case, $d(f2, f0) = 2,\ M(2,0) = 2$ and $N(2,0) = 1$.

$\alpha(gx, gy)d(fx, fy) = \alpha(g2, g0)d(f2, f0)$
$= 3 \leq \psi(2) + L.1 = \psi(M(2,0)) + L.N(2,0)$
$= \psi(M(x,y)) + L.N(x,y)$ holds with L = 3.

If $x,\ y \in B$ then the inequality (2.9.1) holds trivially.

Hence, from above cases, we choose $L = 3$, so that the pair (f, g) is a weak generalized (α, ψ)-contractive maps with rational expressions.

Here we note that if $L = 0$ in the inequality (2.9.1), then for $x = 2$ and $y = 0$ we have
$\alpha(g2, g0)d(f2, f0) = 3 \nleq \psi(2) = \psi(M(2,0))$, for any $\psi \in \Psi$, so that the inequality (2.9.1) fails to hold, which shows the importance of L.

Further, we observe that the inequality (1.9.1) also fails to hold.

For, by choosing $(x,y) = (1,0)$ we have
$d(f1, f0) = 1 \nleq \alpha.0 + \beta.1 = \alpha\frac{d(1,f1)d(0,f0)}{d(1,0)} + \beta d(1,0)$.

This shows that the inequality (1.9.1) fails to hold so that f is not a Jaggi contraction map.

Thus we conclude that the class of (f, g) weak generalized (α, ψ)-contractive maps
with rational expressions is more general than the class of almost Jaggi contraction maps which in turn, it is more general than the class of all Jaggi contraction maps.

3. Main Results

In the following, first we prove the existence of coincidence points of a pair (f, g) of weak generalized (α, ψ)-contractive maps with rational expressions.

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a metric space. Let $f, g : X \to X$ be two selfmaps on X. Suppose that f is a triangular (α, g)-admissible and g-non-decreasing mapping. Suppose that there exist two functions $\alpha : X \times X \to [0,\ \infty)$, $\psi \in \Psi$ and $L \geq 0$ such that (f, g) is a pair of weak generalized (α, ψ)-contractive maps with rational expressions. Also, assume that*

(i) *$fX \subseteq gX$;*
(ii) *there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$ with $gx_0 \preceq fx_0$;*
(iii) *$g(X)$ is a complete subset of X*
(iv) *if $\{gx_n\}$ is a non-decreasing sequence in X such that $gx_n \to gx$ as $n \to \infty$ then*
$gx = Sup\{gx_n\}$; and $gx_n \preceq ggx$.

Then f and g have a coincidence point.

Proof. Let $x_0 \in X$ be as in (ii), $\alpha(gx_0, fx_0) \geq 1$ with $gx_0 \preceq fx_0$. Since $fX \subseteq gX$, we choose $x_1 \in X$ such that $gx_1 = fx_0$. Since $gx_0 \preceq fx_0 = gx_1$, and f is g-non-decreasing, we have $fx_0 \preceq fx_1$, so that $gx_1 \preceq gx_2$. By using the similar argument we choose a sequence $\{x_n\}$ in X with
$fx_n = gx_{n+1}$ for $n = 1, 2, \ldots$. (3.1.1)
Further, since $gx_1 \preceq gx_2$ and f is g-non-decreasing, we have
$fx_1 \preceq fx_2$ so that $gx_2 \preceq gx_3$.
Inductively, it follows that $gx_n \preceq gx_{n+1}$ for all $n = 0, 1, 2, \ldots$. (3.1.2)
Now, $\alpha(gx_0, gx_1) = \alpha(gx_0, fx_0) \geq 1$, and by using the property that f is an
(α, g)-admissible map, we have
$\alpha(fx_0, fx_1) \geq 1$, *i.e.*, $\alpha(gx_1, gx_2) \geq 1$. By a repeated application of this property,
$\alpha(fx_1, fx_2) \geq 1$, *i.e.*, $\alpha(gx_2, gx_3) \geq 1$, and inductively, it follows that
$\alpha(fx_n, fx_{n+1}) \geq 1$ *i.e.*, $\alpha(gx_n, gx_{n+1}) \geq 1$ for all $n = 0, 1, 2, \ldots$. (3.1.3)
If $gx_{n+1} = gx_{n+2}$, for some n, then $gx_{n+1} = fx_{n+1}$ so that x_{n+1} is a coincidence point of f and g .
If $gx_{n+1} \neq gx_{n+2}$ for all n then we have $d(gx_{n+2}, gx_{n+1}) > 0$.
Now, from (2.9.1), (3.1.2) and (3.1.3), we have
$d(gx_{n+2}, gx_{n+1}) = d(fx_{n+1}, fx_n)$

$\leq \alpha(gx_{n+1}, gx_n)d(fx_{n+1}, fx_n)$
$\leq \psi(M(x_{n+1}, x_n)) + L.N(x_{n+1}, x_n)$, where (3.1.4)
$M(x_{n+1}, x_n) = \max\{d(gx_{n+1}, gx_n), \frac{d(gx_{n+1},fx_{n+1})d(gx_n,fx_n)}{d(gx_{n+1},gx_n)}, \frac{d(gx_{n+1},fx_n)d(gx_n,fx_{n+1})}{d(gx_{n+1},gx_n)},$
$\frac{d(gx_{n+1},fx_{n+1})d(gx_{n+1},fx_n)+d(gx_n,fx_n)d(gx_n,fx_{n+1})}{2d(gx_{n+1},gx_n)}\}$
$= \max\{d(gx_{n+1}, gx_n), \frac{d(gx_{n+1},gx_{n+2})d(gx_n,gx_{n+1})}{d(gx_{n+1},gx_n)}, \frac{d(gx_{n+1},gx_{n+1})d(gx_n,gx_{n+2})}{d(gx_{n+1},gx_n)},$
$\frac{d(gx_{n+1},gx_{n+2})d(gx_{n+1},gx_{n+1})+d(gx_n,gx_{n+1})d(gx_n,gx_{n+2})}{2d(gx_{n+1},gx_n)}\}$
$= \max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}, 0, \frac{d(gx_n,gx_{n+2})}{2}\}$
$\leq \max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}, \frac{d(gx_n,gx_{n+1})+d(gx_{n+1},gx_{n+2})}{2}\}$
$= \max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}\}$, and
$N(x_{n+1}, x_n) = \min\{d(gx_{n+1}, fx_{n+1}), d(gx_{n+1}, fx_n), d(gx_n, fx_{n+1})\}$
$= \min\{d(gx_{n+1}, gx_{n+2}), d(gx_{n+1}, gx_{n+1}), d(gx_n, gx_{n+2})\} = 0$
Now, from (3.1.4), we have
$d(gx_{n+2}, gx_{n+1}) \leq \psi(\max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}\}).$ (3.1.5)
If $\max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}\} = d(gx_{n+1}, gx_{n+2})$ then from (3.1.5) we have
$d(gx_{n+2}, gx_{n+1}) \leq \psi(d(gx_{n+2}, gx_{n+1})) < d(gx_{n+2}, gx_{n+1}),$
a contradiction.
Hence from (3.1.5) we have
$\max\{d(gx_{n+1}, gx_n), d(gx_{n+1}, gx_{n+2}\} = d(gx_{n+1}, gx_n)$ so that
$d(gx_{n+2}, gx_{n+1}) \leq \psi(d(gx_{n+1}, gx_n))$ (3.1.6)
which implies that $d(gx_{n+2}, gx_{n+1}) < d(gx_{n+1}, gx_n)$ for all n.
Thus it follows that $\{(d(gx_{n+1}, gx_n)\}$ is a strictly decreasing sequence of reals and hence $\lim_{n\to\infty} d(gx_{n+1}, gx_n)$ exists and it is r (say).
i.e., $\lim_{n\to\infty} d(gx_{n+1}, gx_n) = r \geq 0$.
We now show that $r = 0$.
Suppose that $r > 0$. Then from (3.1.6), we have
$d(gx_{n+2}, gx_{n+1}) \leq \psi(d(gx_{n+1}, gx_n))$.
On letting $n \to \infty$, we have
$r \leq \psi(r) < r$,
a contradiction.
Hence $\lim_{n\to\infty} d(gx_{n+1}, gx_n) = 0$, *i.e.*, $r = 0$.
Step (i): $\{gx_n\}$ is a Cauchy sequence in X.
Suppose that $\{gx_n\}$ is not a Cauchy sequence. Then, there exist $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that
$d(gx_{m(k)}, gx_{n(k)}) \geq \epsilon.$ (3.1.7)
We choose $m(k)$, the least positive integer satisfying (3.1.7). Then, we have
$m(k) > n(k) > k$ with

$d(gx_{m(k)}, gx_{n(k)}) \geq \epsilon,\ d(gx_{m(k)-1}, gx_{n(k)}) < \epsilon$
and by Lemma 1.16, it follows that $\lim_{k\to\infty} d(gx_{m(k)+1}, gx_{n(k)+1}) = \epsilon$.
Now from (2.9.1), we have
$d(gx_{m(k)+1}, gx_{n(k)+1}) = d(fx_{m(k)}, fx_{n(k)})$
$\leq \alpha(gx_{m(k)}, gx_{n(k)})d(fx_{m(k)}, fx_{n(k)})$
$\leq \psi(M(x_{m(k)}, x_{n(k)})) + L.N(x_{m(k)}, x_{n(k)})$, where
$M(x_{m(k)}, x_{n(k)}) = \max\{d(gx_{m(k)}, gx_{n(k)}), \frac{d(gx_{m(k)},fx_{m(k)})d(gx_{n(k)},fx_{n(k)})}{d(gx_{m(k)},gx_{n(k)})},$
$\frac{d(gx_{m(k)},fx_{n(k)})d(gx_{n(k)},fx_{m(k)})}{d(gx_{m(k)},gx_{n(k)})},$
$\frac{(d(gx_{m(k)},fx_{m(k)})d(gx_{m(k)},fx_{n(k)})+(d(gx_{n(k)},fx_{n(k)})d(gx_{n(k)},fx_{m(k)})}{2d(gx_{m(k)},gx_{n(k)})}\}$
$= \max\{d(gx_{m(k)}, gx_{n(k)}), \frac{d(gx_{m(k)},gx_{m(k)+1})d(gx_{n(k)},gx_{n(k)+1})}{d(gx_{m(k)},gx_{n(k)})},$
$\frac{d(gx_{m(k)},gx_{n(k)+1})d(gx_{n(k)},gx_{m(k)+1})}{d(gx_{m(k)},gx_{n(k)})},$
$\frac{(d(gx_{m(k)},gx_{m(k)+1})d(gx_{m(k)},gx_{n(k)+1})+(d(gx_{n(k)},gx_{n(k)+1})d(gx_{n(k)},gx_{m(k)+1})}{2d(gx_{m(k)},gx_{n(k)})}\}$
and
$N(x_{m(k)}, x_{n(k)}) = \min\{d(gx_{m(k)}, fx_{m(k)}), d(gx_{m(k)}, fx_{n(k)}), d(gx_{n(k)}, fx_{m(k)})\}$
$= \min\{d(gx_{m(k)}, gx_{m(k)+1}), d(gx_{m(k)}, gx_{n(k)+1}), d(gx_{n(k)}, gx_{m(k)+1})\}$
On letting $k \to \infty$, and using the conclusion of Lemma 1.16, it follows that
$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = \max\{\epsilon, 0, \epsilon, 0\} = \epsilon$, $\lim_{k\to\infty} N(x_{m(k)}, x_{n(k)}) = 0$ and
$\epsilon \leq \lim_{k\to\infty} \psi(M(x_{m(k)}, x_{n(k)}) + L.0.$
Hence $\epsilon \leq \psi(\epsilon) < \epsilon$,
a contradiction .
Hence $\lim_{k\to\infty} d(gx_{m(k)+1}, gx_{n(k)+1}) = 0$.
Therefore $\{gx_n\}$ is a Cauchy sequence in (X, d). Since $g(X)$ is complete, there exists $z \in g(X)$ such that

$$\lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} fx_n = gx = z \text{ for some } x \in X. \tag{3.1.8}$$

<u>*Step*</u> *(ii)*: $gx = fx$.
Suppose that $gx \neq fx$.
Now, suppose that the condition *(iv)* holds. Since $\{gx_n\}$ is a non-decreasing sequence and $gx_n \to gx$ we have $gx = Sup\{gx_n\}$ and $gx_n \preceq ggx$.
Particularly, $gx_n \preceq ggx$ for all n. Since f is g-non-decreasing, we have $fx_n \preceq fgx$ for all n. *i.e.*, $gx_{n+1} \preceq fgx$ for all n.
Moreover, as $gx_n \preceq gx_{n+1} \preceq fgx$ for all n and $gx_n \preceq ggx$, we get $gx \preceq fgx$.
Let us now consider the sequence $\{gy_n\}$ that is constructed as follows:
$gy_0 = gx,\ gy_{n+1} = f(gy_n),\ n = 0, 1, 2, ...$.
Then $gy_0 \preceq f(gy_0)$ and since f is triangular (α, g)-admissible.

i.e., $\alpha(gx_0, gz_0) \geq 1$, $\alpha(gz_0, gy_0) \geq 1$ implies $\alpha(gx_0, gy_0) \geq 1$. Since f is g non-decreasing, we obtain that $\{gy_n\}$ is a non-decreasing sequence and $\{gy_n\}$ is cauchy (similar to the argument to show $\{gx_n\}$ is cauchy) $gy_n \to gy$ (say), $y \in X$. Again, by the first part of the condition (iv), we have $gy = Sup\{gy_n\}$. Since $gx_n \preceq gx = gy_0 \preceq f(gx) = f(gy_0) \preceq gy_n \preceq gy$ for all n. By using Remark 2.6, we have $\alpha(gx_{n+1}, gy_{n+1}) \geq 1$, for $n = 0, 1, 2, \ldots$.
Suppose that $gx \neq gy$. Now from (2.9.1), we have
$d(gx_{n+1}, gy_{n+1}) = d(fx_n, f(gy_n))$
$\leq \alpha(gx_n, ggy_n)d(fx_n, f(gy_n))$
$\leq \psi(M(x_n, gy_n)) + L.N(x_n, gy_n)$, where
$M(x_n, gy_n) = \max\{d(gx_n, gy_n), \frac{d(gx_n,fx_n)d(ggy_n,fgy_n)}{d(gx_n,ggy_n)}, \frac{d(gx_n,fgy_n)d(ggy_n,fx_n)}{d(gx_n,ggy_n)}, \frac{d(gx_n,fx_n)d(gx_n,fgy_n)+d(ggy_n,fgy_n)d(ggy_n,fx_n)}{2d(gx_n,ggy_n)}\}$ and
$N(x_n, gy_n) = \min\{d(gx_n, fx_n), d(gx_n, fgy_n), d(ggy_n, fx_n)\}$
$= \min\{d(gx_n, gx_{n+1}), d(gx_n, gy_{n+1}), d(ggy_n, gx_{n+1})\}$.
On letting $n \to \infty$, we have
$d(gx, gy) \leq \psi(\max\{d(gx, gy), 0, d(gx, gy)\}) + L \min\{d(gx, gx), d(gx, gy), d(gy, gx)\}$
so that $d(gx, gy) \leq \psi(\max\{d(gx, gy), 0, d(gx, gy)\}) + L.0$
$= \psi(d(gx, gy)) < d(gx, gy)$,
a contradiction.
Hence $gx = gy$, and we have
$gy_0 = gx \preceq fgx = fgy_0 = gy_1 \preceq gy_2 \preceq .... \preceq gy_n \preceq gy = gx$ so that $gx = fgx$.
By (iv), it follows that $gx \preceq ggx$. Now, since f is g non-decreasing we have

$$fx \preceq fgx = gx, i.e., fx \preceq gx. \qquad (3.1.9)$$

Since $gx_n \preceq gx$ and f is g non-decreasing we have $fx_n \preceq fx$ for all n; *i.e.*, $gx_{n+1} \preceq gx$ for all n so that $gx = \lim_{n\to\infty} gx_{n+1} = sup\{gx_{n+1}\} \preceq fx$. (3.1.10)
From (3.1.9) and (3.1.10), we have $fx = gx$.
So that x is a coincidence point of f and g. □

Corollary 3.2. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that $(X,\ d)$ is a complete metric space. Let $f : X \to X$ be a non-decreasing mapping. Suppose that there exists a function $\alpha : X \times X \to [0,\ \infty)$ and a constant $k \in (0, 1)$ such that*

$$\alpha(x, y)d(fx, fy) \leq k \max\{d(x, y), \frac{d(x,fx)d(y,fy)}{d(x,y)}\} \qquad (3.2.1)$$

for all $x, y \in X$ with $x \preceq y$, $x \neq y$. Also, assume that

(i) *f is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$ with $x_0 \preceq fx_0$; either*

(iii) *f is continuous;* (*or*)
(iv) *$\{x_n\}$ is non-decreasing in X such that $x_n \to x$ as $n \to \infty$ then $x = sup\{x_n\}$; and also $\alpha(x_0, x) \geq 1$ and $\alpha(x, fx) \geq 1$.*

Then f has a fixed point.

Proof. The conclusion of this corollary follows by taking $g = I_X,\ \psi(t) = kt,\ t \geq 0$ and L $= 0$, in Theorem 3.1. □

Theorem 3.3. *In addition to the hypotheses of Theorem 3.1, if f and g are weakly compatible then f and g have a unique common fixed point in X.*

Proof. From the proof of Theorem 3.1 we have $\{gx_n\}$ is non-decreasing sequence that converges to gx and $fx = gx$. Since f and g are weakly compatible, we have $fgx = gfx$.
$i.e.,\ gx = fgx = ggx$.
Hence $fz = gz = z$, so that f and g have a common fixed point z.
$\underline{Uniqueness}$: Let z and z' be two common fixed points of f and g
$i.e.,\ fz = gz = z$ and $fz' = gz' = z'$.
Now, we show that $z = z'$
Suppose that $z \neq z'$.
Now from (2.9.1), we have
$d(z, z') = d(fz, fz')$
$\leq \alpha(d(gz, gz'))d(fz, fz')$
$\leq \psi(M(z, z')) + L.N(z, z'), where$
$M(z, z') = \max\{d(gz, gz'), \frac{d(gz,fz)d(gz',fz')}{d(gz,gz')}, \frac{(d(gz,fz')d(gz',fz))}{d(gz,gz')},$
$\frac{d(gz,fz)d(gz,fz')+d(gz',fz')d(gz',fz)}{2d(gz,gz')}\}$
$= \max\{d(gz, gz'), 0, d(gz', gz), 0\}$
and $N(z, z') = \min\{d(gz, fz), d(gz, gz'), d(gz', gz))\} = 0$
$d(z, z') \leq \psi(d(z, z')) < d(z, z')$,
a contradiction.
Hence $z = z'$.
Therefore f and g have a unique common fixed point.
This completes the proof of the Theorem. □

In the following we prove Theorem 3.1 with different hypotheses, particularly by replacing the condition (iii) and weakly compatible by 'reciprocal continuity' and 'compatibility' when the metric d is complete.

Theorem 3.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric s-pace. Let $f, g : X \to X$ be two selfmaps on X. Suppose that f is a triangular (α, g)-admissible and g-non-decreasing mapping. Suppose that there exist two functions $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$ and $L \geq 0$ such that (f, g) is a pair of weak generalized (α, ψ)-contractive map-s with rational expressions. Also, assume that*

(i) *$fX \subseteq gX$;*
(ii) *f and g are compatible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(gx_0, fx_0) \geq 1$ with $gx_0 \preceq fx_0$;*
(iv) *f and g are reciprocally continuous.*

Then f and g have a coincidence point. Moreover, f and g have a unique common fixed point in X.

Proof. As in the proof of Theorem 3.1, for $x_0 \in X$ of (iii), we choose $\{x_n\}$ in X that satisfies $fx_n = gx_{n+1}$ for $n = 1, 2, ...$. and that $\{gx_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, there exists $z \in X$ such that
$\lim_{n\to\infty} gx_n = z$. Hence $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = z$.
Since f and g are reciprocally continuous, we have
$\lim_{n\to\infty} fgx_n = fz$ and $\lim_{n\to\infty} gfx_n = gz$.
Since f and g are compatible, we have
$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ so that $d(fz, gz) = 0$.
Hence $fz = gz$ so that z is a coincidence point of f and g.
Now, since every compatible pair is weakly compatible, by applying Theorem 3.3 it follows that f and g have a unique common fixed point in X. □

4. Corollaries and Examples

By choosing $g = I_X$ in Theorem 3.1, we have the following corollary.

Corollary 4.1. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space and let $f : X \to X$ be a weak generalized (α, ψ)-contractive map with rational expressions. If there exists x_0 in X such that $x_0 \preceq fx_0$ with $\alpha(x_0, fx_0) \geq 1$ and f is non-decreasing. Further, assume that for any non-decreasing sequence $\{x_n\}$, where $x_n = fx_{n-1},\ n = 1, 2, 3, ...$ in X converges to u, then $x_n \preceq u$ for all $n \geq 0$.
Then f has a fixed point in X.

By choosing $g = I_X$ in Theorem 3.4, we have the following corollary.

Corollary 4.2. Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space and let $f : X \to X$ be a weak generalized (α, ψ)-contractive map with rational expressions. If there exists x_0 in X such that $x_0 \preceq fx_0$, if f is non-decreasing and continuous. Then f has a fixed point.

Remark 4.3. (i) Theorem 1.9 follows as a corollary to Corollary 3.2, since the inequality (1.9.1) implies the inequality (3.2.1) with $k = \alpha + \beta < 1$; and $\alpha(x, y) = 1$ for all $x, y \in X$.
(ii) Theorem 1.11 follows as a corollary to Corollary 3.2, since the inequality (1.11.1) implies the inequality (3.2.1) with $k = \alpha + \beta < 1$; and $\alpha(x, y) = 1$ for all $x, y \in X$.

Hence, we conclude that Theorem 1.9 and Theorem 1.11 follow as corollaries to Corollary 3.2, which in turn these two Theorems follow as corollaries to Theorem 3.1 when g is the identity map in Theorem 3.1.

In the following, we provide examples in support of the results obtained in Section 3.
The following is an example in support of Theorem 3.1.

Example 4.1. Let $X = [0,\ 2]$ with the usual metric. We define a partial
order $\preceq$ on X by $\preceq := \{(x, y) \in X \times X : x = y\} \bigcup \{(0, 1), (0, 2), (1, 2)\}$.
Let A $= \Delta \bigcup \{(0, 1), (0, 2), (1, 2), (2, 1)\}$ and B $= \{(x, y)/x, y \in X$ and $x \neq y\}/\{(0, 1),$
$(0, 2), (1, 2), (2, 1)\}$. We define $f, g : X \to X$ and $\alpha : X \times X \to [0,\ \infty)$ by

$$f(x) = \begin{cases} 1 + x^2 & if\ x \in [0, 1] \\ 1 + \frac{x}{2} & if\ x \in (1, 2], \end{cases} \quad g(x) = \begin{cases} 2x & if\ x \in [0, 1] \\ 2 & if\ x \in (1, 2] \end{cases}$$

and $\alpha(x, y) = \begin{cases} \frac{3}{2} & if\ (x, y) \in A \\ 0 & otherwise. \end{cases}$

Clearly $fX \subseteq gX$. We choose $x_0 = 0$ then $gx_0 \preceq fx_0$. Also f and g are weakly compatible, f is triangular (α, g)-admissible and f is g-non-decreasing. Moreover, we choose $x_0 = 0 \in X$, then $\alpha(gx_0, fx_0) = \alpha(0, 1) \geq 1$ and $0 \preceq 1$. Now, we verify the inequality (2.9.1) by choosing $\psi \in \Psi$ given by $\psi(t) = \frac{2}{5}t$ for $t \geq 0$ and $L = 3$.

Case (1): $(x, y) = (1, 2)$ and $(x, y) = (2, 1)$.
In this case, the inequality (2.9.1) holds trivially.

Case (ii): $(x, y) = (0, 1)$.
In this case, $d(f0, f1) = 1,\ M(0, 1) = 2$ and $N(0, 1) = 1$.
$\alpha(gx, gy)d(fx, fy) = \alpha(g0, g1)d(f0, f1)$

$= \frac{3}{2} \leq \psi(2) + L.1 = \psi(M(0,1)) + L.N(0,1)$
$= \psi(M(x,y)) + L.N(x,y)$ holds with $L = 1$.
Case (iii): $(x,y) = (0,2)$.
In this case, $d(f0,f2) = 1,\ M(0,2) = 2$ and $N(0,2) = 1$.
$\alpha(gx,gy)d(fx,fy) = \alpha(g0,g2)d(f0,f2)$
$= 3 \leq \psi(2) + L.1 = \psi(M(0,2)) + L.N(0,2)$
$= \psi(M(x,y)) + L.N(x,y)$ holds with L = 1.
If $x,\ y \in B$ then the inequality (2.9.1) holds trivially.
Hence, from consider above cases, we choose $L = 1$, so that a pair (f,g) of weak
generalized (α,ψ)-contractive maps with rational expressions with $L = 1$. Further, f and g and ψ satisfy the inequality (2.9.1). Hence f and g satisfy all the hypotheses of Theorem 3.1 and it has fixed point 2.
Here we note that the inequality (1.9.1) also fails to hold.
For, by choosing $(x,y) = (0,2)$ we have
$d(f0,f2) = 1 \nleq \alpha.0 + \beta.2 < 1 = \alpha\frac{d(0,f0)d(2,f2)}{d(0,2)} + \beta d(0,2)$.
This shows that the inequality (1.9.1) fails to hold so that f is not a *Jaggi contration map*. Hence Theorem 1.9 is also not applicable.
Further, we observe that the inequality (1.11.1) also fails to hold.
For, by choosing $(x,y) = (0,1)$ we have
$d(f0,f1) = 1 \nleq \alpha.1 + \beta.1 + L.0 < 1 = \alpha\frac{d(0,f0)d(1,f1)}{d(0,1)} + \beta d(0,1) + L\min\{d(0,f1), d(1,f0)\}$
so that f is not an almost Jaggi contraction map.

Here we note that Remark 4.3 and Example 4.1 suggest that Theorem 3.1 is a generalization of Theorem 1.9 and Theorem 1.11.

Example 4.2. Let $X = \{1,2,4,7\}$ with the usual metric. We define partial order
$\preceq$ on X as follows, $\preceq := \{(1,1),(2,2),(4,4),(7,7),(2,1),(4,1),(7,1),(4,2),(7,2)\}$.
Let A $= \Delta \bigcup \{(1,2),(2,1),(4,1),(7,1),(4,2),(7,2)\}$ and B $= \{(x,y)/x,y \in X$ and
$x \neq y\}/\{(1,2),(2,1),(4,1),(7,1),(4,2),(7,2)\}$. We define $f,g : X \to X$ by
$f1 = f2 = 2,\ f4 = 4,\ f7 = 7;\ g1 = 1,\ g2 = 2,\ g4 = 7$ and $g7 = 4$.
We define $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \begin{cases} \frac{3}{2} & if\ (x,y) \in A \\ 0 & otherwise \end{cases}$ and
$\psi : [0,\infty) \to [0,1)$ by $\psi(t) = \frac{t}{2}$.
Clearly $fX \subseteq gX$. We choose $x_0 = 1$ then $gx_0 \preceq fx_0$. Also f and g are compatible and reciprocally continuous, f is triangular (α,g)-admissible and f is g-non-decreasing. Hence f and g satisfy all the

hypotheses of Theorem 3.3 and 2 is the unique common fixed point of f and g.
In the following, we mention the importance of L of Theorem 3.3. If $L = 0$ then the inequality (2.9.1) fails to hold. For, by choosing $x = 7$ and $y = 2$ we have
$\alpha(g7, g2)d(f7, f2) = 5 \nleq \psi(5) = \psi(M(7, 2))$ for any $\psi \in \Psi$.
Here we note that the inequality (1.9.1) also fails to hold.
For, by choosing $(x, y) = (7, 2)$ we have
$d(f7, f2) = 5 \nleq \alpha.0 + \beta.5 < 1 = \alpha\frac{d(7,f7)d(2,f2)}{d(7,2)} + \beta d(7, 2)$.
This shows that the inequality (1.9.1) fails to hold so that f is not a Jaggi contraction map.

Here we note that Remark 4.3 (i) and Example 4.2 suggest that Theorem 3.3 is a generalization of Theorem 1.9.

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