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NO TOUCH DIGITAL OPTION IN LEVY-DRIVEN MODEL WITH STOCHASTIC INTEREST RATE OF CIR-TYPE

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Abstract. The price of no-touch digital option in a Lévy driven-market is considered. The underlying log-asset price X_t is the double exponential jump-diffusion process, and the risk-less interest rate in the market r_t, is a stochastic function of a Markov process Y_t. Under the assumption that X_t is independent of r_t, we obtain a semi close-form formula for the option price, containing two factors: a bond price with the same maturity date as the option and the expectation of the infirmum process of X_t. Numerical methods are applied to the latter to realize its values, while the former is known analytically.

1. Introduction

In this article, we consider the valuation of the no-touch option in Lévy market. The model we use comprises of a Lévy process X_t, which models the log-asset of the firm and a risk-less interest rate r_t. We assume that the risk-less interest rate is a stochastic function of the Markov process Y_t: $r_t = r(Y_t)$. The Markov process has the state space $D = \mathbb{R}$, and it models the evolution of the risk-free interest rate. Lévy models were introduced in finance as an alternative to Black and Scholes [1] model, which assumes that asset price evolves as a geometric Brownian motion. Lévy processes have jumps in their paths, and therefore models the evolution of asset prices better than Brownian motion. In literature, there exists several articles on the application of Levy processes to different areas of financial modeling e.g, see Boyarnchenko and Levendorskiǐ[2], Innocentis and Levendorskiǐ[3], Madan and Seneta[11], Carr and Wu[5],etc. In these models, one of the major inputs is the constant riskless interest rate. They assume that the riskless interest rate remains constant throughout the option life. In this article, we

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relax this assumption by allowing the interest rate to vary stochastically as the Cox et al [6] process. we assume that the Lévy X_t and the interest rate are stochastically independent. Many methods have been used by researchers to obtain option prices in Lévy-driven models e.g Carr randomization; Carr[4], eigenfunction expansion; Davydov and Linetsky[7], transform method; Jeannin and Pistorius [8], etc. In our case, through the assumption of independence, and with a chosen risk neutral measure Q, we obtain a semi close-form formula for the option price. In particular, the formula comprises two factors: the price of pure discount bond at time t and the expectation of the infirmum process of X_t. Therefore, if we know the price of a pure discount bond that has the same maturity date as the no-touch option, one can easily obtain the option price. The price of the pure discount bond in CIR model is known in close-form, while for the remaining factor, we apply the Laplace and Fourier transforms. The resulting integral is evaluated with simplified trapezoidal rule.

The rest of this paper is organized as follows: In the rest of section 1, we present some general facts about Lévy processes as needed in this article. In section 2, the model used for pricing is constructed. The major theorem that gives the price of the no-touch option is presented here. In section three, a numerical experiment is conducted to validate model. The findings and conclusion are also given here.

1.1. **Lévy Processes.** By simple definition, a Lévy process X_t is a stochastic process with stationary and independent increments. Every Lévy process X_t has a moment generating function which can be represented in the form

$$E[e^{i\xi X_t}] = e^{-t\psi(\xi)}, \xi \in \mathbb{R}, t \geq 0$$

The function ψ is called the characteristic exponent for X, and has the general representation given by the Lévy -Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{\mathbb{R}\backslash 0}(1 + i\xi x 1_{[-1,1]}(x) - e^{ix\xi})F(dx) \tag{1.1}$$

where $\sigma \geq 0, \mu \in \mathbb{R}$, and F is a measure on $\mathbb{R}\backslash 0$ satisfying

$$\int_{\mathbb{R}\backslash 0}\{\mid x \mid^2, 1\}F(dx) < \infty$$

For Brownian motion, there are no jumps. Hence, $F = 0$ and (1) reduces to

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - ib\xi$$

A good example of Lévy processes which I shall use as the underlying log-price in our model is the double exponential jump-diffusion model, hereafter DEJDM, with characteristic exponent:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + c_1\frac{i\xi}{\lambda_+ + i\xi} + c_2\frac{i\xi}{\lambda_- + i\xi} \tag{1.2}$$

with $\sigma > 0, \mu \in \mathbb{R}$ and $c_+, c_- > 0, \lambda_+ > 0, \lambda_- < 0$. The coefficients c_+ and c_- characterizes the intensity of upward and downward jumps respectively. DEJDM is a special case of hyper-exponential jump-diffusion process discovered by Kou[9]. It is a process of finite variation, which means that its sample paths have bounded variation on every compact time, Boyarchenko and Levendorskiĭ[2] .

1.2. **Wiener-Hopf factors.** For $q > 0$, let T_q be an exponentially distributed random variable independent of X_t, with mean $1/q$. If we define the infirmum and supremum processes of a Lévy process X_t as $\underline{X_t} = inf_{0\leq s\leq t}X_s$ and $\overline{X_t} = sup_{0\leq s\leq t}X_s$ respectively, then

$$E[e^{i\xi X_{T_q}}] = E[e^{i\xi\overline{X}_{T_q}}]E[e^{i\xi\underline{X}_{T_q}}]$$

$\forall\xi \in \mathbb{R}$ is called the Wiener-Hopf factorization of $E[e^{i\xi X_{T_q}}]$, with $E[e^{i\xi\overline{X}_{T_q}}]]$ and $E[e^{i\xi\underline{X}_{T_q}}]$ as the Wiener-Hopf factors.

2. The Model

Our model assumes that X_t and Y_t satisfy the following stochastic differential equations

$$dX_t = \mu(X_t)dt + dZ_t \tag{2.1}$$

$$dY_t = \alpha(\beta - Y_t)dt + \sigma_y\sqrt{y}d\omega_t \tag{2.2}$$

where $\alpha, \beta \geq 0$, $\sigma_y \geq 0$ is the volatility of the interest rate and Z is a Lévy process . For (3) and (4) to have a unique strong solution, we assume that Z_t and ω_t are independent, and $\mu(x)$ are sufficiently regular on $\mathbb{R}_t$.

In general, consider a contingent claim with fixed expiry date T and payoff $G(X_T, Y_T)$. Assume that the asset pays no dividend, the risk-neutral price of this contingent claim at time $t \leq T$ is

$$V(t, x, y) = E^{Q,x,y}[exp(-\int_t^T r(Y_s)ds)G(X_T, Y_T)] \tag{2.3}$$

where Q is the chosen risk-neutral measure for pricing. Applying Feynman-Kac theorem, we see that the option price satisfies the partial-integro differential equation

$$(\partial_t + \mu(x)\partial_x + \alpha(\beta - y)\partial_y + \frac{\sigma^2}{2}y\partial_y^2 + L - r(y))V(t,x,y) = 0 \tag{2.4}$$

subject to

$$V(T,x,y) = G(X_T, Y_T)$$

where L is the infinitesimal generator of Z, which acts as follows $Lu(x) = \int_{\mathbb{R}\setminus 0}(u(x+x') - u(x))F(dx')$

2.1. **No-Touch Digital Option.**

A no-touch digital option is characterized by its maturity date T, a barrier h, and the first entrance date T_h into the interval $(-\infty, h]$. The owner of the option receives one unit of currency if the price of the underlying asset does not touch or enter the interval $(-\infty, h]$ until expiry date T. If the price X_t touches or enters the interval prior to maturity, the option expires worthless. Below, we state the theorem which gives the price of no-touch digital option that matures at T.

Theorem:
Let h be the agreed barrier, and $V(t,T,x,y)$ be the time t price of a no-touch digital option with maturity date T and first entrance date T_h. Then we have

$$V(t,T,x,y) = B(t,T,y)\frac{1}{2\pi i}\int_{Req=\alpha}\frac{e^{q(T-t)}}{q}E^{Q,x}[1_{\underline{X_t}>h}]dq \tag{2.5}$$

where $B(t,T,y)$ is the time-zero price of pure discount bond with maturity date T.

Proof
Since Lévy market is incomplete, we choose a risk neutral measure Q, for our pricing. Under this measure and risk-less rate $r = r(y)$, we have

$$V(t,T,x;y) = E^{Q,x,y}[exp(-\int_t^T r(y_s)ds)1_{T_h>T}] \tag{2.6}$$

Using the assumption that X and Y are independent, (8) can be written in the form

$$V(t,T,x;y) = B(t,T,y)E^{Q,x}[1_{T_h>T}] \tag{2.7}$$

where

$$B(t,T,y) = E^{Q,y}[exp(-\int_t^T r(y_s)ds)]$$

The Laplace transform of $E^{Q,x}[1_{T_h>T}]$ with respect to time to maturity $\tau = T - t$ of the option is $\frac{1}{q}E^{Q,x}[1_{T_h>T}]$. Using the fact that $T_h > T$ if and only if the infirmum process $\underline{X_t} > h$, we have

$$E^{Q,x}[1_{T_h>T}] = \frac{1}{2\pi i}\int_{Req=\alpha} \frac{e^{q(T-t)}}{q} E^{Q,x}[1_{\underline{X_t}>h}]dq \tag{2.8}$$

Substituting (10) into (9) concludes the proof.
For computational efficiency, instead of numerical realization of (7), we use

$$V(t,T,x,y) = B(t,T,y)(1 - V(t,T,x)) \tag{2.9}$$

where

$$V(t,T,x) = \frac{1}{2\pi i}\int_{Req=\alpha} \frac{e^{q(T-t)}}{q} E^{Q,x}[1_{\underline{X_t}\leq h}]dq \tag{2.10}$$

is the price of first touch digital option when the bond price B(t,T,y) is 1. Using Fourier transform, then for $x > h$, we have

$$E^{Q,x}[1_{\underline{X_t}\leq h}] = \frac{1}{2\pi}\int_{Im\xi=\omega_+} e^{i(x-h)\xi} E^{Q}[e^{i\xi \underline{X_T}}]d\xi \tag{2.11}$$

In Levendorskiǐ[10], the simplest formula for $E^{Q}[e^{i\xi \underline{X_T}}]$ is given as

$$E^{Q}[e^{i\xi \underline{X_T}}] = \exp\{-\frac{1}{2\pi i}\int_{im\eta=\omega_t} \frac{\xi ln(1+\psi(\eta)/q)}{\eta(\xi-\eta)}d\eta\} \tag{2.12}$$

where $\omega_t \in (0, \lambda_t)$ and ψ is as defined in (2). To evaluate (14), we truncate the integral, and use the simplified trapezoidal rule to obtain, for each ξ,

$$E^{Q}[e^{i\xi \underline{X_T}}] = \exp\{-\frac{\zeta}{2\pi i}\sum_{k\in\mathbb{Z}} \frac{\xi ln(1+\psi(\eta_k)/q)}{\eta_k(\xi-\eta_k)}\} \tag{2.13}$$

where ζ is the chosen η grid step

Applying the Feynman-Kac theorem to $B(t,T,y)$, it is easy to show that the bond price satisfies the Kolmogorov backward equation

$$\begin{aligned} (\partial_t + \frac{1}{2}\sigma^2 y\partial_y^2 + \alpha(\beta - y)\partial_y - r(y))B(t,T,y) &= 0 \\ B(T,T,y) &= 1 \end{aligned} \tag{2.14}$$

The solution to (16) is given by

$$B(t,T,y) = \exp(A(\tau)y + C(\tau))$$

where A and B satisfy the system of ordinary differential equations

$$-A_\tau(\tau) - \alpha A(\tau) + \frac{\sigma^2}{2} A^2(\tau) - 1 = 0 \tag{2.15}$$

$$A(0) = 0$$

$$-B_\tau(\tau) + \alpha\beta A(\tau) = 0 \tag{2.16}$$

$$B(0) = 0$$

3. Numerical example and conclusion

: We consider two cases with $h = 100$ and $h = 200$. The parameters of the CIR interest rate process are $\sigma_y = 1.6428$, $\alpha = 1.2$, $\beta = 1$. For the Lévy process, we have $\mu = .55$, $\sigma = .5$, $\lambda_+ = 10$, $\lambda_- = -20$, $c_1 = c_2 = 1$. The results of our calculations are shown in figures 1 and 2. To show the validity and accuracy of our model, we see from the figures, the obvious gaps at the boundaries h=100 and 200. This is because the spot price is not allowed to touch or cross the boundaries. If it does, the option expires with nothing. In conclusion, we have shown that with the aid of independence of stochastic processes, it is possible to use stochastic interest rate in the pricing of no-touch options. Moreover, with price of a pure discount bond in the market with the same expiry date, we efficiently calculated the price of the no-touch option.

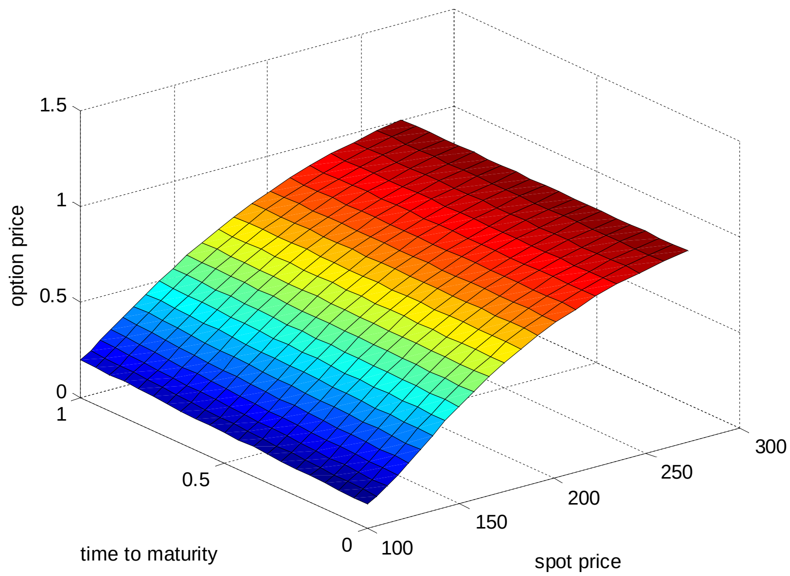

Figure 1: Option price versus time to maturity and spot prices with h=100

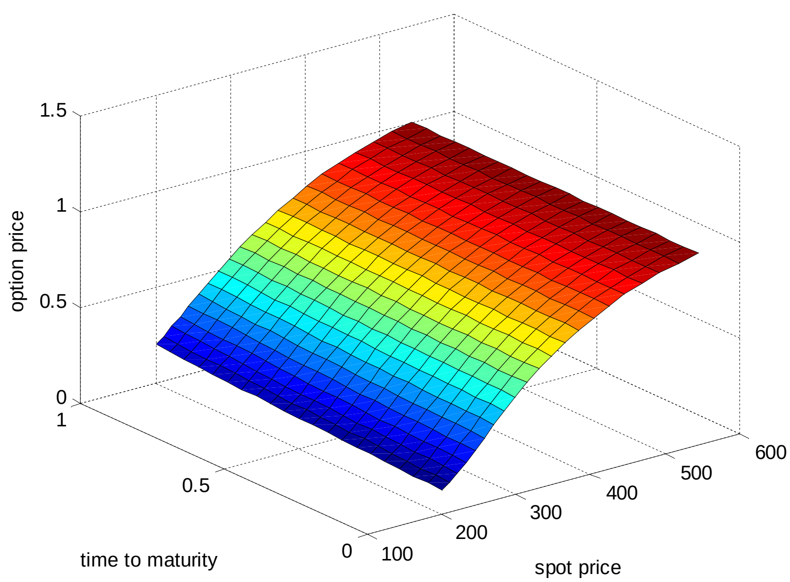

Figure 2: Option price versus time to maturity and spot prices with h=200

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