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A NEW NINTH ORDER HYBRID BLOCK INTEGRATOR FOR SOLUTION OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: We present hybrid block of eight integrators which are of uniform order nine through interpolation and collocation procedures. The block method is implemented on a stiff, physical and life problems in which the results obtained compares favourably with the existing method. The properties of the hybrid block integrators is fully investigated and confirmed to be computationally stable with the numerical experiments tested.

1.0 Introduction

Among the most important mathematical tools used in producing models in the physical sciences, Biological sciences and Engineering are differential equations. But most of these differential equations do not posses closed form or finite solutions. Even if they posses closed form solutions we do not know the method of getting them.

In many real-life situations, the differential equation that models the problem is too complicated to solve exactly. Hence there is need to develop an accurate algorithms for obtaining an equivalent approximating solution to the original problems. Most recent researchers has developed some block methods to cater for this class of problems

$$y' = f(x, y), \qquad y(a) = \rho \tag{1}$$

Among such researchers are [1], [2], [3], [4], [5], [6] and [7] to mention a few.

In our method, a block of eight integrators were proposed at step length of four which are of uniform order nine and also all the Discrete schemes in our block came from a single continuous formula.

2.0 Development of the method

Our objective is to derive a block of eight integrators of the form

$$y(x_{n+v}) = \propto_0 y_n + h\left[\beta_0 f_n + \beta_{\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3} + \beta_{\frac{7}{2}} f_{n+\frac{7}{2}} + \beta_4 f_{n+4}\right] \tag{2}$$

where x_{n+v}, $v = \frac{3}{4}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$.

We proceed by seeking an approximate solution of the form

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$$y(x) = \sum_{j=0}^{m+t-1} \propto_j x^j \quad (3)$$

$$y'(x) = \sum_{j=1}^{m+t-1} j \propto_j x^{j-1} = f(x,y) \quad (4)$$

where m and t are the number of collocation and interpolation points used in the method. Specifically for this method $m = 9$, $t = 1$ and the degree of the polynomial is $m + t - 1$. Equation (3) is interpolated at $x = x_n$ and (4) is collocated at $x = x_{n+v}$, $v = 0, \frac{3}{4}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$ 3 and 4 which leads to the following non linear system of equations of the form

$$P(x) = \sum_{j=0}^{m+t-1} \propto_j x_n^j$$

$$P'(x) = \sum_{j=1}^{m+t-1} j \propto_j x_{n+v}^{j-1} = h \sum_{j=0}^{m+t-1} \beta_j (x) f_{n+v} = f(x,y) \quad (5)$$

When using Maple 17 (Mathematical software) to determine the unknown parameters $\propto_j$ and β_j in (5), we obtain the following.

$\propto_0 = y_n$

$$\beta_0 = \left[l - \frac{18027}{7560h} l^2 + \frac{71277}{22680h^2} l^3 - \frac{2732}{1080h^3} l^4 + \frac{6989}{5400h^4} l^5 - \frac{1369}{3240h^5} l^6 + \frac{161}{1890h^6} l^7 - \frac{73}{7560h^7} l^8 + \frac{4}{8505h^8} l^9\right]$$

$$\beta_{\frac{3}{4}} = \left[\frac{4096(2520)}{405405h} l^2 - \frac{8192(8658)}{1216215h^2} l^3 + \frac{2048(1745)}{57915h^3} l^4 - \frac{8192(1316)}{289575h^4} l^5 + \frac{4096(575)}{173745h^5} l^6 - \frac{16384(73)}{405405h^6} l^7 + \frac{4096(35)}{405405h^6} l^8 - \frac{65536}{3648645h^8} l^9\right]$$

$$\beta_1 = \left[-\frac{7560}{180h} l^2 + \frac{28494}{270h^2} l^3 - \frac{42783}{360h^3} l^4 + \frac{33713}{450h^4} l^5 - \frac{7605}{270h^5} l^6 + \frac{2(989)}{315h^6} l^7 - \frac{69}{90h^6} l^8 + \frac{16}{405h^8} l^9\right]$$

$$\beta_{\frac{3}{2}} = \left[\frac{2(252)}{135h} l^2 - \frac{4(10338)}{405h^2} l^3 + \frac{1(16867)}{135h^3} l^4 - \frac{4(1425)}{675h^4} l^5 + \frac{2(6805)}{405h^5} l^6 - \frac{16(463)}{945h^6} l^7 + \frac{2(67)}{135h^6} l^8 - \frac{64}{1215h^8} l^9\right]$$

$$\beta_2 = \left[-\frac{1(1890)}{60h} l^2 + \frac{1(16137)}{180h^2} l^3 - \frac{1(13785)}{120h^3} l^4 + \frac{1(24463)}{300h^4} l^5 - \frac{1(6115)}{180h^5} l^6 + \frac{1(867)}{105h^6} l^7 - \frac{1(65)}{60h^6} l^8 + \frac{8}{135h^8} l^9\right]$$

$$\beta_{\frac{5}{2}} = \left[\frac{4(1512)}{315h} l^2 - \frac{8(6606)}{945h^2} l^3 + \frac{2(1659)}{45h^3} l^4 - \frac{8(1522)}{225h^4} l^5 + \frac{4(789)}{135h^5} l^6 - \frac{32(58)}{315h^6} l^7 + \frac{4(63)}{315h^6} l^8 - \frac{128}{2835h^8} l^9\right]$$

$$\beta_3 = \left[-\frac{1(2520)}{324h}l^2 + \frac{1(11178)}{486h^2}l^3 - \frac{1(20033)}{648h^3}l^4 + \frac{1(18821)}{810h^4}l^5 - \frac{1(5017)}{486h^5}l^6 + \frac{2(761)}{567h^6}l^7 - \frac{1(61)}{162h^6}l^8 + \frac{16}{729h^8}l^9\right]$$

$$\beta_{\frac{7}{2}} = \left[\frac{2(1080)}{1155h}l^2 - \frac{4(4842)}{3465h^2}l^3 + \frac{1(1257)}{165h^3}l^4 - \frac{4(1202)}{825h^4}l^5 + \frac{2(655)}{495h^5}l^6 - \frac{16(51)}{1155h^6}l^7 + \frac{2(59)}{1155h^6}l^8 - \frac{64}{10395h^8}l^9\right]$$

$$\beta_4 = \left[-\frac{1(4215)}{14040h}l^2 + \frac{1(8541)}{14040h^2}l^3 - \frac{1(3921)}{4680h^3}l^4 + \frac{1(15203)}{23400h^4}l^5 - \frac{1(4215)}{14040h^5}l^6 + \frac{1(671)}{8190h^6}l^7 - \frac{1(57)}{4680h^6}l^8 + \frac{4}{5265h^8}l^9\right] \qquad (6)$$

where $l = (x - x_n)$

$$y(x) = \alpha_0\, y_n + h\left[\beta_0 f_n + \beta_{\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3} + \beta_{\frac{7}{2}} f_{n+\frac{7}{2}} + \beta_4 f_{n+4}\right] \qquad (7)$$

Equation (6) is substituted in equation (7) to obtain our continuous formula. Also evaluating (7) at $x = x_{n+v},\ v = \frac{3}{4}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$ *and* 4 to yield eight integrators to form our hybrid block methods as follows:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{3}{4}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \\ y_{n+\frac{7}{2}} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{13}{4}} \\ y_{n-3} \\ y_{n-\frac{5}{2}} \\ y_{n-2} \\ y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}$$

$$+h\left[\begin{matrix} \frac{985667}{400400} & -\frac{72842607}{22937600} & \frac{14005219}{5734400} & -\frac{89105481}{45875200} & \frac{72842607}{22937600} & -\frac{2085791}{4587520} \\ \frac{46741504}{18243225} & -\frac{340769}{113400} & \frac{102397}{42525} & -\frac{72607}{37800} & \frac{16082}{141750} & -\frac{91943}{204120} \\ \frac{4848}{1925} & -\frac{244719}{89600} & \frac{61343}{22400} & -\frac{360477}{179200} & \frac{1881}{1600} & -\frac{8327}{17920} \\ \frac{46235648}{18243225} & -\frac{39386}{14175} & \frac{128144}{42525} & -\frac{8098}{4725} & \frac{2272}{2025} & -\frac{11462}{368550} \\ \frac{1841200}{729729} & -\frac{797975}{290304} & \frac{641525}{217728} & -\frac{271175}{193536} & \frac{50135}{36288} & -\frac{1247875}{2612736} \\ \frac{63488}{25025} & -\frac{3897}{1400} & \frac{527}{175} & -\frac{2133}{1400} & \frac{306}{175} & -\frac{71}{280} \\ \frac{502544}{200475} & -\frac{2809513}{1036800} & \frac{2243563}{777600} & -\frac{916153}{691200} & \frac{194089}{129600} & \frac{368039}{1866240} \\ \frac{48234496}{18243225} & -\frac{43072}{14175} & \frac{145408}{42525} & -\frac{9896}{4725} & \frac{32768}{14175} & -\frac{11584}{25515} \end{matrix}\right.$$

$$\begin{bmatrix} \frac{6797493}{63078400} & -\frac{1372587}{1192755200} \\ \frac{973}{7425} & -\frac{8413}{737100} \\ \frac{27081}{246400} & -\frac{4203}{358400} \\ \frac{5552}{51975} & -\frac{4213}{368550} \\ \frac{29675}{266112} & -\frac{178775}{15095808} \\ \frac{27}{275} & -\frac{99}{9100} \\ \frac{272629}{950400} & -\frac{64141}{4147200} \\ \frac{47104}{51975} & \frac{24242}{184275} \end{bmatrix} \begin{bmatrix} f_{n+\frac{3}{4}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+4} \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{16196113}{91750400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{59977}{340200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{63341}{358400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15011}{85050} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{615535}{3483648} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{247}{1400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2202641}{12441600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1058}{6075} \end{bmatrix} \begin{bmatrix} f_{n-\frac{13}{4}} \\ f_{n-3} \\ f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{bmatrix} \tag{8}$$

3.0 Basic Analysis of the proposed hybrid block method

3.1 Order and Error constants of the newly hybrid block integrators according to [8], given a linear multi-step method of the form

$$\sum_{j=0}^{k} \propto_j (x) y_{n+j} = h \sum_{j=0}^{k} \beta_j (x) f_{n+j} \tag{9}$$

We associate the linear difference operator

$$\mathcal{L}[y(x), h] = \sum_{j=0}^{k} \left[\propto_j \mathrm{y}(x + ih) - h\beta_j \mathrm{y}'(\mathrm{x} + \mathrm{j}h) \right] \tag{10}$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a, b]$. the test function Expanding $\mathrm{y}(x + ih)$ and $\mathrm{y}'(\mathrm{x} + \mathrm{j}h)$ as Taylor series about x and collecting like terms in (10) gives

$$\mathcal{L}[y(x), h] = c_0 y(x) + c_1 h y'^{(x)} + \cdots + c_q h^q y^q(x) \tag{11}$$

Definition 3.1.1

A linear multistep method of the form

$$\sum_{j=0}^{m+t-1} \propto_j (x) y_{n+j} = h \sum_{j=0}^{m+t-1} \beta_j (x) f_{n+j}$$

is said to be of order p, if from (11), we have $c_0 = c_1 = c_2 = \cdots c_p = 0$ but $C_{p+1} \neq 0$ and C_{p+1} is called error constant. [8]

Definition 3.1.2

A LMM of (9) is said to satisfy the root conditions if all of the roots of the first characteristics polynomial have modulus less than or equal to unity and those of modulus unity are simple. The method (9) is said to be zero-stable if it satisfies the root condition. [8]

Table 1 Order and error constants of integrators (8)

Schemes	Order	Error Constants
$y_{n+\frac{3}{4}}$	9	$\frac{1627529}{93952409600}$
y_{n+1}	9	$\frac{1027}{59719680}$
$y_{n+\frac{3}{2}}$	9	$\frac{6439}{367001600}$
y_{n+2}	9	$\frac{2257}{130636800}$
$y_{n+\frac{5}{2}}$	9	$\frac{187975}{10701766656}$
y_{n+3}	9	$\frac{7}{409600}$
$y_{n+\frac{7}{2}}$	9	$\frac{140581}{7644119040}$
y_{n+4}	9	$\frac{23}{2041200}$

4.0 Consistency and Stability of the block schemes

The hybrid block integrators (8) is said to be zero stable, if the roots $Z_s\ , S = 1,2,..k$ of the first characteristic polynomial $\rho(z) = \det \lambda\left(A^{(0)} - A^{(1)}\right)$ satisfies $|z_s| \leq 1$ has multiplicity not exceeding the order of the differential equation.

$$\boldsymbol{\rho(z)} = \left|\boldsymbol{\lambda}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix} - \begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}\right| = 0$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 4, \lambda_5 = 0, \lambda_6 = 0, \lambda_7 = 0, \lambda_8 = 1$$

From definition 3.1.2, the newly hybrid block method (8) is zero stable and also consistent since the order of all the integrators in (8) are $9 > 1$

5.0 Numerical experiments

The hybrid block method are demonstrated on stiff, life and physical problems

Example 5.1

$y' = 1 - y^2$

$y(0) = 0, h = 0.1$

Analytic solution: $\frac{e^{2x}-1}{e^{2x}+1}$ or $Tanhx$

Example 5.2 (SIR Model)

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations relating the number of susceptible people $S(t)$, number of people infected $I(t)$, and the number of people who have recovered $R(t)$,. This is a good and simple model for many infectious diseases including measles, mumps and rubella (Rosser 1967). It is given by the following three coupled equations.

$$\frac{dS}{dt} = \mu(I - S) - \beta IS$$

$$\frac{dI}{dt} = \mu I - \gamma I + \beta IS \qquad (12)$$

$$\frac{dR}{dt} = \mu R + \gamma I$$

where μ, γ and β are positive parameters. Define y to be,

$$y = S + I + R \qquad (13)$$

When solutions in (12) are substituted in (13) we have

$y = \mu(1 - y)t$ then

$$y' = \mu(1 - y), \qquad y(a) = p \qquad (14)$$

Taking $\mu = 0.5$ and attaching an initial condition $y(0) = 0.5$ (for a particular closed population), we obtain

$$y' = 0.5(1 - y), \qquad y(0) = 0.5 \qquad (15)$$

Exact solution: $y(t) = 1 - 0.5e^{-0.5t}$

Table 2 Performance of Hybrid block method (8) on problem 5.1

x	Exact solution	Computed solution	Error of [9]	New Error
0.1	0.099667994624956	0.099667994698116	1.2312 E(-09)	7.31 E(-11)
0.2	0.197375320224904	0.197375320296238	1.2343 E(-09)	7.1334 E(-11)
0.3	0.291312612451591	0.291312612518669	1.2814 E(-09)	6.7078 E(-11)
0.4	0.379948962255225	0.379948962292507	2.2283336 E(-09)	3.7282 E (-11)
0.5	0.462117157260010	0.462117157294015	3.3949 E(-09)	3.4005 E (-11)
0.6	0.537049566998035	0.537049567020542	3.5428 E(-09)	2.2507 E (-11)
0.7	0.604367777117164	0.604367777137633	3.8244 E(-09)	2.0469 E (-11)
0.8	0.664036770267849	0.664036770297224	3.8293 E(-09)	2.9375 E (-11)
0.9	0.716297870199024	0.716297870224885	1.3275 E(-09)	2.5861 E (-11)
1.0	0.761594155955765	0.761594155986542	1.4427 E(-09)	3.0777 E (-11)

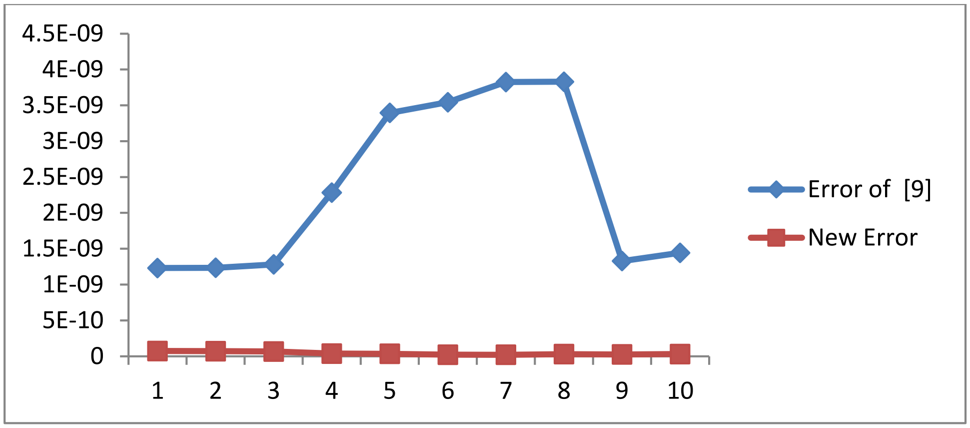

Figure 1: Error graph of solution of example 5.1

Table 3 Performance of Hybrid block method (8) on problem 5.2

t	Exact solution	Computed solution	Error of [3]	New Err
0.1	0.524385287749643	0.524385287749643	5.57443 E (-12)	0.00
0.2	0.547581290982020	0.547581290982018	3.946177 E(-12)	2.00 E(-15)
0.3	0.569646011787471	0.569646011787471	8.183232 E(-11)	0.00
0.4	0.590634623461009	0.590634623461001	3.436118 E(-11)	8.00 E (-15)
0.5	0.610599608464298	0.610599608464286	1.92974 E(-10)	1.20 E (-14)
0.6	0.629590889659141	0.629590889659125	1.87904 E(-10)	1.60 E (-14)
0.7	0.647655955140644	0.647655955140626	1.776835 E(-10)	1.80 E (-14)
0.8	0.664839976982180	0.664839976982157	1.724676 E(-10)	2.30 E (-14)
0.9	0.681185924189114	0.681185924189090	1.847545 E(-10)	2.40 E (-14)
1.0	0.696734670143684	0.696734670143655	3.00577 E(-10)	2.90 E (-14)

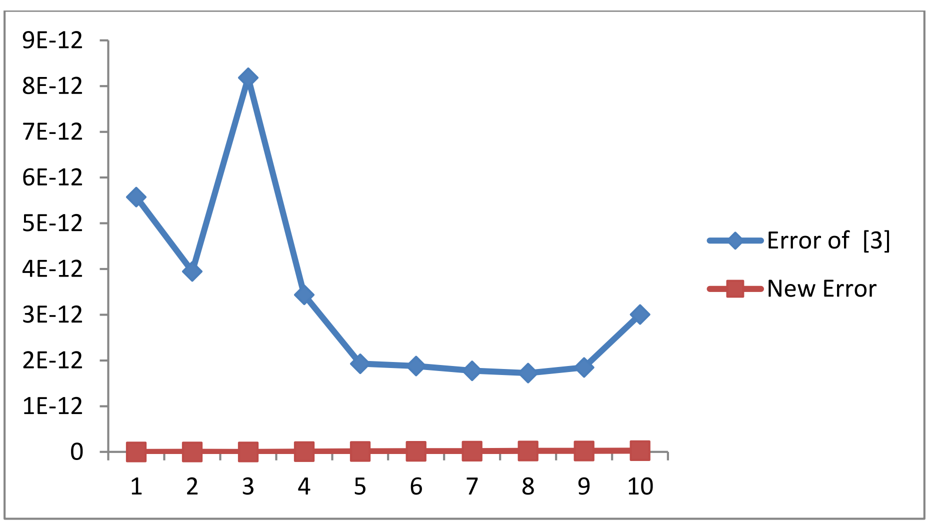

Figure 2: Error graph of solution of example 5.2

6.0 Discussion of Results

We observed that the newly block integrator (8) performed excellently well with the two problems tested with the schemes. This shows that our method is good and can be used to solve accurately any model of the form $y' = f(x, y),\ y(a) = \rho$. (see tables 2,3 and figures 1,2)

7.0 Conclusion

We want to conclude that the newly block integrator (8) is of uniform order 9, zero stable, consistent and self starting. Also result obtained from the method converges more than the existing method. (see figures 1 and 2)

REFERENCES

[1] Badmus A M and Mshelia W D: "Some uniform order block methods for the solution of first order ordinary differential equations" Journal of Nigerian Association of Mathematical Physics, volume 19 pages 149-154. (2011)

[2] Badmus AM, Badmos TA and Ekpenyong FE: "Hybrid Implicit Block algorithms for improved performances in the solution of first order initial value problems". Journal of Nigerian Association of Mathematical Physics volume 23 pages 95-102, (2013)

[3] Sunday J, Odekunle MR and Adesanya AO: Order six block integrator for the solution of first-order ordinary differential equations. International Journal of mathematics and Soft Computing, volume 3, No1, pages 87-96. (2013)

[4] Odekunle MR, Adesanya AO, Sunday J: "A new block integrator for the solution of initial value problems of first order ordinary differential equations" .International Journal of Pure and Applied Science and Technology. (2012);1

[5] E. A. Areo, R. A. Ademiluyi and P.O. Babatola, "Three steps hybrid linear multistep method for the solution of first-order initial value problems in ordinary differential equations", Journal of Nigerian Association of Mathematical Physics, volume 23 pages 95-102, 19, 261-266, (2011)

[6] Onumanyi P and Awoyemi D.O, Jator S.N. and Siriseria U.W "New Linear Multistep Methods with continuous coefficients for first order ivps" Journal of Nigeria mathematics society 13: 37- 51.(1994)

[7] Chollom JP, Olatunbojun IO, Omagu SA. "A class of A-stable block methods for the solution of ordinary differential equation". Research Journal of Mathematics and Statistics.;4(2):52-56. (2012)

[8] Lambert J.D : Computational Methods in Ordinary Differential Equations. John Wiley and Sons, New York. 278 (1973)

[9] Badmus AM, Yahaya YA and Yakubu PC: "Adam type hybrid Block methods associated with chebyshev polynomial for the solution of ordinary differential equations". British Journal of Mathematics & Computer Science (1) 6(6): 464-474. (2015)

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