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DERIVATION OF NINTH STAGE RUNGE-KUTTA METHOD FOR THE SOLUTION OF FIRST ORDER DIFFERENTIAL EQUATIONS

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Abstract: We present hybrid block of eight integrators which are of uniform order nine through interpolation and collocation procedures. The properties of the hybrid block integrators are fully investigated and confirmed to be computationally stable also the block method derived are reconstructed to ninth stage Runge-Kutta method which implemented on stiff, physical and life problems. The results obtained compare favourably with the existing methods when implemented in Runge-Kutta mode.

1.0 Introduction

Among the most important mathematical tools used in producing models in the physical sciences, Biological sciences and Engineering are differential equations. But most of these differential equations do not posses closed form or finite solutions. Even if they posses closed form solutions we do not know the method of getting them.

In many real-life situations, the differential equation that models the problem is too complicated to solve exactly. Hence there is need to develop an accurate algorithm for obtaining an equivalent approximating solution to the original problems. Most recent researchers have developed some block methods to cater for this class of problems

$$y' = f(x,y), \qquad y(a) = \rho \tag{1}$$

Among such researchers are [1], [2],[3], [4] , [5],[6] and [7] to mention a few.

In our method, a block of eight integrators were proposed at step length of four which are of uniform order nine and also all the discrete schemes in our block came from a single continuous formula

Definition 1.0 Zero stable

A linear multi-step method is said to be Zero-stable if the roots $R_j, j = 1(1)k$ of the first characteristics polynomials

$$\rho(R) = det\left[\sum_{i=0}^{k} A_i R^{k-i}\right] = 0, A_0 = -1, satisfies \left|R_j\right| \leq 1$$

If one of the roots is $+1$, we call this the principal root of $\rho(R)$. [7]

Key words and phrases: Block method, Ninth stage, Runge-Kutta type method, Uniform order and Computational stable.

2.0 Development of the method

Our objective is to derive a block of eight integrators of the form

$$y(x_{n+v}) = \propto_0 y_n + h\left[\beta_0 f_n + \beta_{\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3} + \beta_{\frac{7}{2}} f_{n+\frac{7}{2}} + \beta_4 f_{n+4}\right] \quad (2)$$

where $x_{n+v}, v = \frac{3}{4}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$.

We proceed by seeking an approximate solution of the form

$$y(x) = \sum_{j=0}^{m+t-1} \propto_j x^j \quad (3)$$

$$y'(x) = \sum_{j=1}^{m+t-1} j \propto_j x^{j-1} = f(x,y) \quad (4)$$

where m and t are the number of collocation and interpolation points used in the method. Specifically for this method $m = 9$, $t = 1$ and the degree of the polynomial is $m + t - 1$. Equation (3) is interpolated at $x = x_n$ and (4) is collocated at

$$x = x_{n+v}, v = 0, \frac{3}{4}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}\ 3$$

and 4 which leads to the following non linear system of equations of the form

$$P(x) = \sum_{j=0}^{m+t-1} \propto_j x_{\mathrm{n}}^j$$

$$P'(x) = \sum_{j=1}^{m+t-1} j \propto_j x_{n+v}^{j-1} = h \sum_{j=0}^{m+t-1} \beta_j (x) f_{n+v} = f(x,y) \quad (5)$$

When using Maple 17 (Mathematical software) to determine the unknown parameters $\propto_j$ and β_j in (5), we obtain the following.

$\propto_0 = y_n$

$$\beta_0 = \left[l - \frac{18027}{7560h} l^2 + \frac{71277}{22680h^2} l^3 - \frac{2732}{1080h^3} l^4 + \frac{6989}{5400h^4} l^5 - \frac{1369}{3240h^5} l^6 + \frac{161}{1890h^6} l^7 - \frac{73}{7560h^7} l^8 + \frac{4}{8505h^8} l^9\right]$$

$$\beta_{\frac{3}{4}} = \left[\frac{4096(2520)}{405405h} l^2 - \frac{8192(8658)}{1216215h^2} l^3 + \frac{2048(1745)}{57915h^3} l^4 - \frac{8192(1316)}{289575h^4} l^5 + \frac{4096(575)}{173745h^5} l^6 - \frac{16384(73)}{405405h^6} l^7 + \frac{4096(35)}{405405h^6} l^8 - \frac{65536}{3648645h^8} l^9\right]$$

$$\beta_1 = \left[-\frac{7560}{180h} l^2 + \frac{28494}{270h^2} l^3 - \frac{42783}{360h^3} l^4 + \frac{33713}{450h^4} l^5 - \frac{7605}{270h^5} l^6 + \frac{2(989)}{315h^6} l^7 - \frac{69}{90h^6} l^8 + \frac{16}{405h^8} l^9\right]$$

$$\beta_{\frac{3}{2}} = \left[\frac{2(252)}{135h}l^2 - \frac{4(10338)}{405h^2}l^3 + \frac{1(16867)}{135h^3}l^4 - \frac{4(1425)}{675h^4}l^5 + \frac{2(6805)}{405h^5}l^6 - \frac{16(463)}{945h^6}l^7 + \frac{2(67)}{135h^6}l^8 - \frac{64}{1215h^8}l^9\right]$$

$$\beta_2 = \left[-\frac{1(1890)}{60h}l^2 + \frac{1(16137)}{180h^2}l^3 - \frac{1(13785)}{120h^3}l^4 + \frac{1(24463)}{300h^4}l^5 - \frac{1(6115)}{180h^5}l^6 + \frac{1(867)}{105h^6}l^7 - \frac{1(65)}{60h^6}l^8 + \frac{8}{135h^8}l^9\right]$$

$$\beta_{\frac{5}{2}} = \left[\frac{4(1512)}{315h}l^2 - \frac{8(6606)}{945h^2}l^3 + \frac{2(1659)}{45h^3}l^4 - \frac{8(1522)}{225h^4}l^5 + \frac{4(789)}{135h^5}l^6 - \frac{32(58)}{315h^6}l^7 + \frac{4(63)}{315h^6}l^8 - \frac{128}{2835h^8}l^9\right]$$

$$\beta_3 = \left[-\frac{1(2520)}{324h}l^2 + \frac{1(11178)}{486h^2}l^3 - \frac{1(20033)}{648h^3}l^4 + \frac{1(18821)}{810h^4}l^5 - \frac{1(5017)}{486h^5}l^6 + \frac{2(761)}{567h^6}l^7 - \frac{1(61)}{162h^6}l^8 + \frac{16}{729h^8}l^9\right]$$

$$\beta_{\frac{7}{2}} = \left[\frac{2(1080)}{1155h}l^2 - \frac{4(4842)}{3465h^2}l^3 + \frac{1(1257)}{165h^3}l^4 - \frac{4(1202)}{825h^4}l^5 + \frac{2(655)}{495h^5}l^6 - \frac{16(51)}{1155h^6}l^7 + \frac{2(59)}{1155h^6}l^8 - \frac{64}{10395h^8}l^9\right]$$

$$\beta_4 = \left[-\frac{1(4215)}{14040h}l^2 + \frac{1(8541)}{14040h^2}l^3 - \frac{1(3921)}{4680h^3}l^4 + \frac{1(15203)}{23400h^4}l^5 - \frac{1(4215)}{14040h^5}l^6 + \frac{1(671)}{8190h^6}l^7 - \frac{1(57)}{4680h^6}l^8 + \frac{4}{5265h^8}l^9\right] \quad (6)$$

where $l = (x - x_n)$

$$y(x) = \propto_0 y_n + h\left[\beta_0 f_n + \beta_{\frac{3}{4}} f_{n+\frac{3}{4}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_3 f_{n+3} + \beta_{\frac{7}{2}} f_{n+\frac{7}{2}} + \beta_4 f_{n+4}\right] \quad (7)$$

Equation (6) is substituted in equation (7) to obtain our continuous formula. Also evaluating (7) at $x = x_{n+v},\ v = \frac{3}{4}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}$ *and* 4 to yield eight integrators to form our hybrid block methods as follows:

$$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{3}{4}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2}\\ y_{n+\frac{5}{2}}\\ y_{n+3}\\ y_{n+\frac{7}{2}}\\ y_{n+4} \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n-\frac{13}{4}}\\ y_{n-3}\\ y_{n-\frac{5}{2}}\\ y_{n-2}\\ y_{n-\frac{3}{2}}\\ y_{n-1}\\ y_{n-\frac{1}{2}}\\ y_n \end{bmatrix}$$

$$+h\begin{bmatrix} \frac{985667}{400400} & -\frac{72842607}{22937600} & \frac{14005219}{5734400} & -\frac{89105481}{45875200} & \frac{72842607}{22937600} & -\frac{2085791}{4587520} & \frac{6797493}{63078400} & -\frac{1372587}{1192755200} \\ \frac{46741504}{18243225} & -\frac{340769}{113400} & \frac{102397}{42525} & -\frac{72607}{37800} & \frac{16082}{141750} & -\frac{91943}{204120} & \frac{973}{7425} & -\frac{8413}{737100} \\ \frac{4848}{1925} & -\frac{244719}{89600} & \frac{61343}{22400} & -\frac{360477}{179200} & \frac{1881}{1600} & -\frac{8327}{17920} & \frac{27081}{246400} & -\frac{4203}{358400} \\ \frac{46235648}{18243225} & -\frac{39386}{14175} & \frac{128144}{42525} & -\frac{8098}{4725} & \frac{2272}{2025} & -\frac{11462}{368550} & \frac{5552}{51975} & -\frac{4213}{368550} \\ \frac{1841200}{729729} & -\frac{797975}{290304} & \frac{641525}{217728} & -\frac{271175}{193536} & \frac{50135}{36288} & -\frac{1247875}{2612736} & \frac{29675}{266112} & -\frac{178775}{15095808} \\ \frac{63488}{25025} & -\frac{3897}{1400} & \frac{527}{175} & -\frac{2133}{1400} & \frac{306}{175} & -\frac{71}{280} & \frac{27}{275} & -\frac{99}{9100} \\ \frac{502544}{200475} & -\frac{2809513}{1036800} & \frac{2243563}{777600} & -\frac{916153}{691200} & \frac{194089}{129600} & \frac{368039}{1866240} & \frac{272629}{950400} & -\frac{64141}{4147200} \\ \frac{48234496}{18243225} & -\frac{43072}{14175} & \frac{145408}{42525} & -\frac{9896}{4725} & \frac{32768}{14175} & -\frac{11584}{25515} & \frac{47104}{51975} & -\frac{24242}{184275} \end{bmatrix}\begin{bmatrix} f_{n+\frac{3}{4}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \\ f_{n+\frac{7}{2}} \\ f_{n+4} \end{bmatrix} + h\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{16196113}{91750400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{59977}{340200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{63341}{358400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15011}{85050} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{615535}{3483648} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{247}{1400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2202641}{12441600} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1058}{6075} \end{bmatrix}\begin{bmatrix} f_{n-\frac{13}{4}} \\ f_{n-3} \\ f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_{n} \end{bmatrix} \tag{8}$$

$$\text{Let } A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By multiplying (8) by the inverse of $A^{(0)}$ and rearrange it in Butcher Table as

C	A								
$\frac{3}{16}$	$\frac{16196113}{91750400}$	$\frac{985667}{400400}$	$-\frac{72842607}{22937600}$	$\frac{14005219}{5734400}$	$-\frac{89105481}{45875200}$	$\frac{72842607}{22937600}$	$-\frac{2085791}{4587520}$	$\frac{6797493}{63078400}$	$-\frac{1372587}{1192755200}$
$\frac{1}{4}$	$\frac{59977}{340200}$	$\frac{46741504}{18243225}$	$-\frac{340769}{113400}$	$\frac{102397}{42525}$	$-\frac{72607}{37800}$	$\frac{16082}{141750}$	$-\frac{91943}{204120}$	$\frac{973}{7425}$	$-\frac{8413}{737100}$
$\frac{3}{8}$	$\frac{63341}{358400}$	$\frac{4848}{1925}$	$-\frac{244719}{89600}$	$\frac{61343}{22400}$	$-\frac{360477}{179200}$	$\frac{1881}{1600}$	$-\frac{8327}{17920}$	$\frac{27081}{246400}$	$-\frac{4203}{358400}$
$\frac{1}{2}$	$\frac{15011}{85050}$	$\frac{46235648}{18243225}$	$-\frac{39386}{14175}$	$\frac{128144}{42525}$	$-\frac{\mathbf{8089}}{4725}$	$\frac{2272}{2025}$	$-\frac{11462}{\mathbf{25515}}$	$\frac{5552}{51975}$	$-\frac{4213}{368550}$
$\frac{5}{8}$	$\frac{615535}{3483648}$	$\frac{1841200}{729729}$	$-\frac{797975}{290304}$	$\frac{641525}{217728}$	$-\frac{271175}{193536}$	$\frac{50135}{36288}$	$-\frac{1247875}{2612736}$	$\frac{29675}{266112}$	$-\frac{178775}{15095808}$
$\frac{3}{4}$	$\frac{247}{1400}$	$\frac{63488}{25025}$	$-\frac{3897}{1400}$	$\frac{527}{175}$	$-\frac{2133}{1400}$	$\frac{306}{175}$	$-\frac{71}{280}$	$\frac{27}{275}$	$-\frac{99}{9100}$
$\frac{7}{8}$	$\frac{2202641}{12441600}$	$\frac{502544}{200475}$	$-\frac{2809513}{1036800}$	$\frac{2243563}{777600}$	$-\frac{916153}{691200}$	$\frac{194089}{129600}$	$\frac{368039}{1866240}$	$\frac{272629}{950400}$	$-\frac{64141}{4147200}$
1	$\frac{1058}{6075}$	$\frac{48234496}{18243225}$	$-\frac{43072}{14175}$	$\frac{145408}{42525}$	$-\frac{9896}{4725}$	$\frac{32768}{14175}$	$-\frac{11584}{25515}$	$\frac{47104}{51975}$	$\frac{24242}{184275}$
1	$\frac{1058}{6075}$	$\frac{48234496}{18243225}$	$-\frac{43072}{14175}$	$\frac{145408}{42525}$	$-\frac{9896}{4725}$	$\frac{32768}{14175}$	$-\frac{11584}{25515}$	$\frac{47104}{51975}$	$\frac{24242}{184275}$

(9)

The Table (9) satisfies Runge-Kutta conditions for solution of first order ODEs since

$$(\boldsymbol{i}) \quad \sum_{j=1}^{s} \boldsymbol{a_{ij} = c_i}$$

$$(\boldsymbol{ii}) \quad \sum_{j=1}^{s} \boldsymbol{b_j = 1}$$

The method (8) is formally given as Runge-Kutta type method as

$$y_{n+1} = y_n + h\left(\frac{529}{12150}k_1 + \frac{12058624}{18243225}k_2 - \frac{10768}{14175}k_3 + \frac{36352}{42525}k_4 - \frac{2474}{4725}k_5 + \frac{8192}{14175}k_6 - \frac{2896}{25515}k_7 + \frac{11776}{51975}k_8 + \frac{12121}{368550}k_9\right)$$

$$k_1 = f\left(x_n, y_n\right)$$

$$k_2 = f\left(x_n + \frac{3}{16}h, \quad y_n + h\left(\frac{16196113}{367001600}k_1 + \frac{985667}{1601600}k_2 - \frac{7284260}{91750400}k_3 + \frac{14005219}{22937600}k_4 - \frac{89105481}{183500800}k_5 + \frac{3285531}{11468800}k_6 - \frac{2085791}{18350080}k_7 - \frac{6797493}{252313600}k_8 - \frac{13732587}{4771020800}k_9\right)\right)$$

$$k_3 = f\left(x_n + \frac{1}{4}h, \quad y_n + h\left(\begin{array}{l}\frac{59977}{1360800}k_1 + \frac{11685376}{18243225}k_2 - \frac{340769}{453600}k_3 + \frac{102397}{170100}k_4 - \frac{72607}{151200}k_5 \\ + \frac{8041}{28350}k_6 - \frac{91943}{816480}k_7 + \frac{793}{29700}k_8 - \frac{8413}{2948400}k_9\end{array}\right)\right)$$

$$k_4 = f\left(x_n + \frac{3}{8}h, \quad y_n + h\left(\begin{array}{l}\frac{63341}{1433600}k_1 + \frac{1212}{1925}k_2 - \frac{244719}{358400}k_3 + \frac{61343}{89600}k_4 - \frac{360477}{716800}k_5 + \\ \frac{1881}{6400}k_6 - \frac{8327}{71600}k_7 + \frac{27081}{985600}k_8 - \frac{4203}{1433600}k_9\end{array}\right)\right)$$

$$k_5 = f\left(x_n + \frac{1}{2}h, \quad y_n + h\left(\begin{array}{l}\frac{15011}{340200}k_1 + \frac{11558912}{18243225}k_2 - \frac{19693}{28350}k_3 + \frac{32036}{42525}k_4 - \frac{4049}{9450}k_5 \\ + \frac{568}{2025}k_6 - \frac{5731}{51030}k_7 + \frac{1388}{51975}k_8 - \frac{4213}{1474200}k_9\end{array}\right)\right)$$

$$k_6 = f\left(x_n + \frac{5}{8}h, \quad y_n + h\left(\begin{array}{l}\frac{615535}{13934592}k_1 + \frac{460300}{729729}k_2 - \frac{797975}{1161216}k_3 + \frac{641525}{870912}k_4 - \frac{271175}{774144}k_5 + \\ \frac{50135}{145152}k_6 - \frac{1247875}{10450944}k_7 + \frac{29675}{1064448}k_8 - \frac{178775}{60383232}k_9\end{array}\right)\right)$$

$$k_7 = f\left(x_n + \frac{3}{4}h, \quad y_n + h\left(\begin{array}{l}\frac{247}{5600}k_1 + \frac{15872}{25025}k_2 - \frac{3897}{5600}k_3 + \frac{527}{700}k_4 - \frac{2133}{5600}k_5 + \\ \frac{153}{350}k_6 - \frac{71}{1120}k_7 + \frac{27}{1100}k_8 - \frac{99}{36400}k_9\end{array}\right)\right)$$

$$k_8 = f\left(x_n + \frac{7}{8}h, \quad y_n + h\left(\begin{array}{l}\frac{2202641}{49766400}k_1 + \frac{125636}{200475}k_2 - \frac{2809513}{4147200}k_3 + \frac{2243563}{3110400}k_4 - \frac{916153}{2764800}k_5 \\ + \frac{194089}{518400}k_6 + \frac{368039}{7464960}k_7 + \frac{272629}{3801600}k_8 - \frac{64141}{16588800}k_9\end{array}\right)\right)$$

$$k_9 = f\left(x_n + h, \quad y_n + h\left(\begin{array}{l}\frac{529}{12150}k_1 + \frac{12058624}{18243225}k_2 - \frac{10768}{14175}k_3 + \frac{36352}{42525}k_4 - \frac{2474}{4725}k_5 \\ + \frac{8192}{14175}k_6 - \frac{2896}{25515}k_7 + \frac{11776}{51975}k_8 + \frac{12121}{368550}k_9\end{array}\right)\right) \quad (10)$$

3.0 Consistency and Stability of the block method

Thus obtain the normalized form of (8),the first characteristics polynomial of the normalized matrix will be expressed as

$$\rho(R) = det\left[RA^{(0)} - A^{(1)}\right]$$

$$\rho(z) = \left| \lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right| = \mathbf{0}$$

$\lambda^7(\lambda - 1) = 0$

$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 4, \lambda_5 = 0, \lambda_6 = 0, \lambda_7 = 0, \lambda_8 = 1$

From definition 1.0, the newly hybrid block method (8) is zero stable and also consistent since the order of all the integrators are uniform order $9 > 1$

4.0 Numerical Experiments

The following examples are used to confirm the efficiency of our method

Example 4.1

$y' = 20x^2 - 20y + 2x, \qquad y(0) = \frac{1}{3} \qquad h = 0.05 \qquad 0 \le x \le 1.0$

Exact Solution: $y(x) = x^2 + \frac{1}{3}e^{-20x}$

Example 4.2

$y' = -y, \qquad y(0) = 1 \qquad h = 0.05 \qquad 0 \le x \le 1.0$

Exact Solution: $y(x) = e^{-x}$

Example 4.3 (SIR Model)

The Susceptible Infected Recovery (SIR) model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations relating the number of susceptible people $S(t)$, number of people infected $I(t)$, and the number of people who have recovered $R(t)$,. This is a good and simple model for many infectious diseases including measles, mumps and rubella. It is given by the following three coupled equations.

$$\frac{dS}{dt} = \mu(I - S) - \beta IS$$

$$\frac{dI}{dt} = \mu I - \gamma I + \beta IS \qquad \text{(i)}$$

$$\frac{dR}{dt} = \mu R + \gamma I$$

where μ, γ and β are positive parameters. Define y to be,

$$y = S + I + R \qquad (ii)$$

when solutions in (i) are substituted in (ii) we have

$$y' = \mu(1 - y)t \text{ then}$$

$$y' = \mu(1 - y), \qquad y(a) = p \qquad (iii)$$

Taking $\mu = 0.5$ and attaching an initial condition $y(0) = 0.5$ (for a particular closed population), we obtain

$$y' = 0.5(1 - y), \qquad y(0) = 0.5 \qquad (iv)$$

Exact solution: $y(t) = 1 - 0.5e^{-0.5t}$

Example 4.4

$y' = \lambda(sinx - y), \qquad y(0) = 0 \qquad h = 0.05 \qquad 0 \le x \le 1.0$

Exact Solution: $y(x) = \frac{\lambda^2}{\lambda^2+1} sinx - \frac{\lambda}{\lambda^2+1} cosx + \frac{\lambda}{\lambda^2+1} e^{-\lambda x}$

Absolute errors, with stiffness ratio $\lambda = 100$ for fixed step size $h = 0.01, 0 \le x \le 1.0$

Table 1: Approximate solution of Example 4.1 at k=4

Mesh values	Exact solution	Present method k = 4	Absolute error
0.05	0.125126480400000	0.125126480300000	1.0×10^{-10}
0.10	0.055111761070000	0.055111761050000	2.0×10^{-11}
0.15	0.039095689460000	0.039095689450000	1.0×10^{-11}
0.20	0.046105212960000	0.046105212980000	2.0×10^{-11}
0.25	0.064745982330000	0.064745982360000	3.0×10^{-11}
0.30	0.090826250730000	0.090826250760000	1.0×10^{-10}
0.35	0.122803960700000	0.122803960700000	……………...
0.40	0.160111820900000	0.160111820900000	……………...
0.45	0.202541136600000	0.202541136300000	3.0×10^{-10}
0.50	0.250015133300000	0.250015133200000	1.0×10^{-10}
0.55	0.302505567200000	0.302505567200000	………………
0.60	0.360002048100000	0.36000204800000	1.0×10^{-10}
0.65	0.422500753400000	0.422500753400000	……………...
0.70	0.490000277200000	0.490000277200000	……………...
0.75	0.562500102000000	0.562500102000000	……………...
0.80	0.640000037500000	0.640000037400000	1.0×10^{-10}
0.85	0.722500013800000	0.722500013800000	……………...
0.90	0.810000005100000	0.810000005000000	1.0×10^{-10}
0.95	0.902500001900000	0.902500001800000	1.0×10^{-10}
1.00	1.0000000001000000	1.0000000000000000	1.0×10^{-9}

Table 2: Approximate solution of Example 4.2 at k=4

Mesh values	Method [9] at k=4	Method [8] at k=4	Present method at k=4
0.05		1.7509×10^{-14}	--------
0.10	1.456×10^{-8}	1.3173×10^{-14}	1.0×10^{-15}
0.15		1.3607×10^{-14}	1.0×10^{-15}
0.2	7.2078×10^{-9}	1.2558×10^{-14}	1.0×10^{-15}
0.25		1.3668×10^{-14}	1.0×10^{-15}
0.3	1.2499×10^{-8}	7.5660×10^{-14}	1.0×10^{-15}
0.35		8.1734×10^{-14}	1.0×10^{-15}
0.4	3.4300×10^{-9}	3.1670×10^{-13}	1.0×10^{-15}
0.45		3.1299×10^{-13}	1.0×10^{-15}
0.5	6.6468×10^{-9}	2.9542×10^{-13}	1.0×10^{-15}
0.55		2.8179×10^{-13}	2.0×10^{-15}
0.60	2.0233×10^{-9}	2.6783×10^{-13}	1.0×10^{-15}
0.65		2.5574×10^{-13}	1.0×10^{-15}
0.7	5.8373×10^{-9}	2.3961×10^{-13}	1.0×10^{-15}
0.75		2.7780×10^{-13}	1.0×10^{-15}
0.8	4.5984×10^{-9}	4.2459×10^{-13}	1.0×10^{-15}
0.85		4.1166×10^{-13}	1.0×10^{-15}
0.90	2.3751×10^{-9}	3.9011×10^{-13}	1.0×10^{-15}
0.95		3.77160×10^{-13}	1.0×10^{-15}
1.00	5.2617×10^{-9}	3.5322×10^{-13}	1.0×10^{-15}

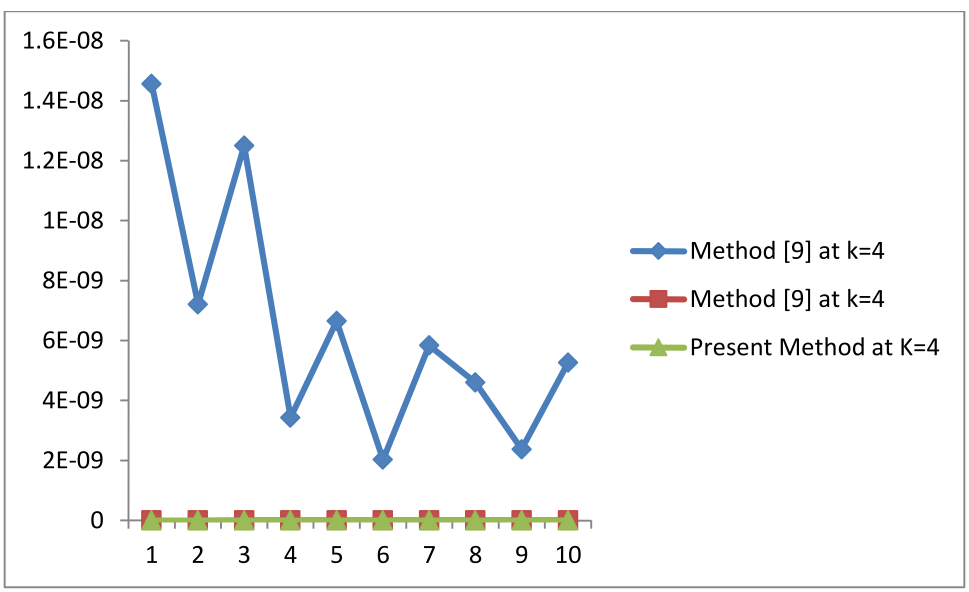

Figure 1: Error graph of Example 4.2 at k= 4

Table 3: Approximate solution of Example 4.3 at k=4

$\boldsymbol{Mesh\ values}$	**Method [5] at k=4**	**Method [2] at k=4**	**Present Method at k=4**
0.1	5.57443 E (-12)	-----------	--------------
0.2	3.946177 E(-12)	2.00 E(-15)	-------------
0.3	8.183232 E(-11)	----------	-----------
0.4	3.436118 E(-11)	8.00 E (-15)	----------------
0.5	1.92974 E(-10)	1.20 E (-14)	1.0 E (-15)
0.6	1.87904 E(-10)	1.60 E (-14)	1.0 E (-15)
0.7	1.776835 E(-10)	1.80 E (-14)	2.0 E (-15)
0.8	1.724676 E(-10)	2.30 E (-14)	1.0 E (-15)
0.9	1.847545 E(-10)	2.40 E (-14)	2.0 E (-15)
1.0	3.00577 E(-10)	2.90 E (-14)	2.0 E (-15)

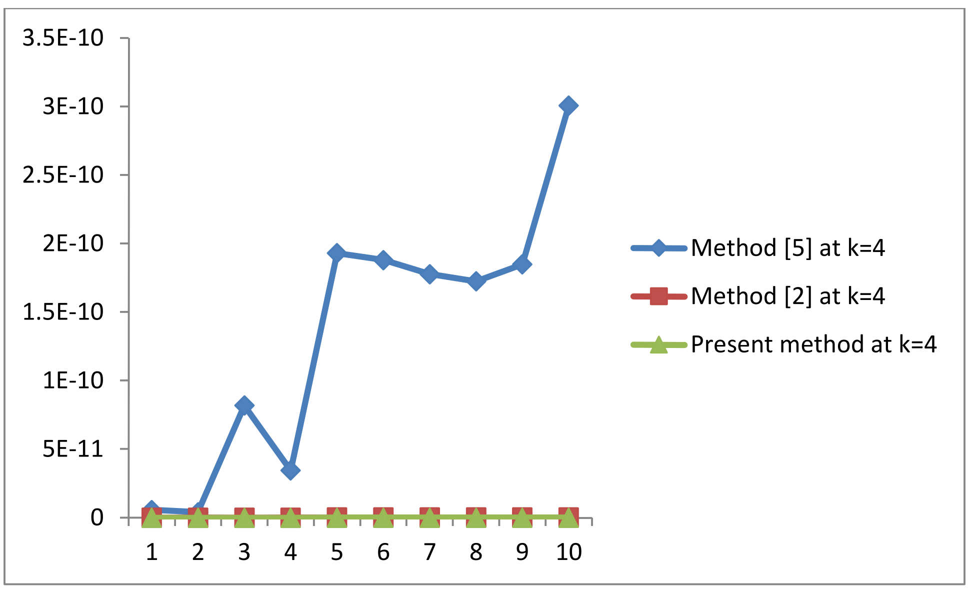

Figure 2: Error graph of Example 4.3

Table 4: Approximate solution of Example 4.4 at k=4

Mesh values	Method [9] at k=4	Method [8] at k=4	Present method k = 4
0.05		5.9679×10^{-4}	4.2826125×10^{-8}
0.10	1.8070×10^{-2}	1.5485×10^{-4}	$8.27482906 \times 10^{-8}$
0.15		8.1938×10^{-5}	5.581025×10^{-9}
0.20	1.0210×10^{-3}	7.5021×10^{-5}	5.543383×10^{-9}
0.25		1.1954×10^{-6}	3.7376×10^{-11}
0.30	1.3225×10^{-3}	3.8056×10^{-6}	2.53×10^{-13}
0.35		1.0512×10^{-6}	2.00×10^{-15}
0.40	4.3646×10^{-3}	1.2499×10^{-6}	1.00×10^{-15}
0.45		6.6184×10^{-7}	---------
0.50	7.8860×10^{-4}	1.9415×10^{-8}	1.00×10^{-15}
0.55		1.0247×10^{-8}	---------------
0.60	4.4568×10^{-4}	9.3785×10^{-8}	-----------------
0.65		1.4945×10^{-8}	----------------
0.70	5.7728×10^{-4}	4.7574×10^{-9}	----------------
0.75		1.3142×10^{-9}	1.0000×10^{-15}
0.80	1.9057×10^{-3}	1.5626×10^{-9}	----------------
0.85		8.2737×10^{-10}	2.0000×10^{-15}
0.90	3.4431×10^{-4}	2.4271×10^{-10}	-------------
0.95		1.2804×10^{-11}	---------------
1.0	1.9459×10^{-4}	1.1724×10^{-11}	1.0000×10^{-15}

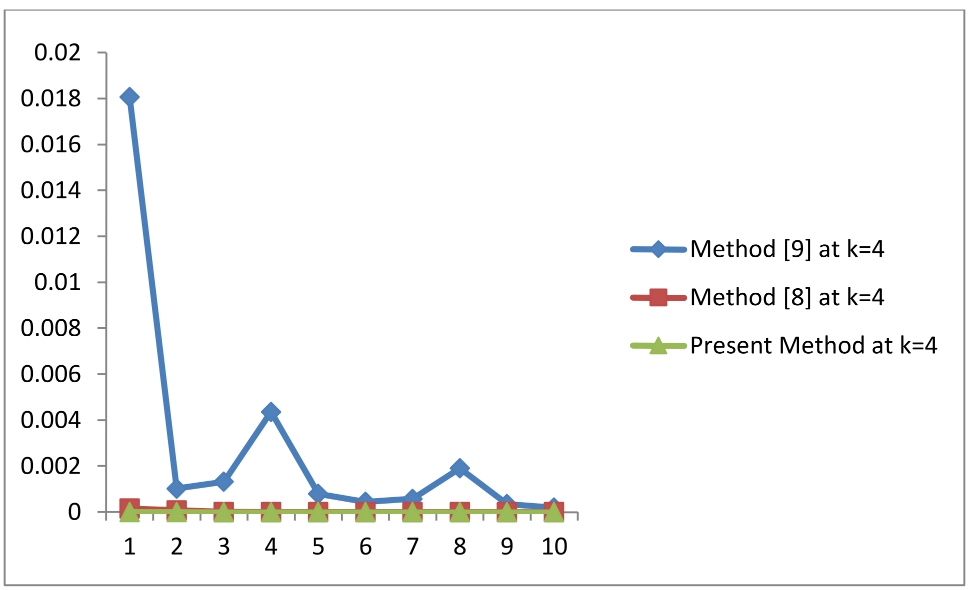

Figure 3: Error graph of Example 4.4

5.0 Discussion of Results

We observed that the Ninth stage Runge-Kutta method performed excellently well with the four problems tested with the method. This shows that our method is good and can be used to solve accurately any model of the form $y' = f(x, y),\ y(a) = \rho$. (see Tables 1, 2,3,4 and figures 1,2 and 3)

6.0 Conclusion

We want to conclude that the newly block integrator (8) is of uniform order 9, zero stable, consistent and self starting and after reformulating into Runge-Kutta type method, the results obtained from the method converges more excellently than the existing method. (see figures 1, 2 and 3). Although the cost of implementation is high when compare with method [2].

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