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ANALYSIS STABILITY OF THE NONLINEAR EPIDEMIC MODEL WITH TEMPORARY IMMUNITY

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Abstract. The present paper present a nonlinear mathematical model, which analyzes the spread and stability of the model epidemic. A population of size $N(t)$ at time t, is divided into three sub classes, with $N(t) = S(t) + I(t) + Q(t)$; where $S(t)$,$I(t)$, and $Q(t)$ denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary immunity, respectively.This paper deals with the equilibrium and stability, precisely the global asymptotic stability of endemic equilibrium under certain conditions on the parameter. Second stochastic stability and finally the equilibrium and stability of the epidemic model with age.

1. Introduction

Motivated by study the dynamics of the different nonlinear epidemic models in all the references, the present paper addresses a mathematical model, which examine the evolution of the infection, reported in the total population.

The spread of the epidemic might naturally dependent on possible contacts between susceptible and infectious.

In this paper, we discuss the equilibrium and stability of the model with temporary immunity and the different positives parameters. We have made the following contributions:

- The equilibrium and stability of the model, we obtain a disease-free equilibrium in the absence of infection but in the presence of infection, it was a unique positive endemic equilibrium and we define the basic reproduction number of the infection .
- Second the global asymptotic stability of endemic equilibrium.
- Next, we introduce a Brownian motion to system and we transform it into an Itô stochastic differential equation.
- Finally study the equilibrium of epidemic model with age.

Key words and phrases. Basic reproduction number; endemic equilibrium; global asymptotic stability; epidemic model; lyapunov functional; temporary immunity; stochastic stability.

2. Mathematical Model

This paper considers the following epidemic model with temporary immunity:

$$\begin{cases} \dot{S}(t) = \rho + \mu - \nu - (\mu_1 + d) S(t) - \beta S(t) Q(t), \\ \dot{I}(t) = \beta S(t) Q(t) - (\mu_2 + d) I(t) - \gamma e^{-\mu_2 \tau} S(t-\tau) Q(t-\tau), \\ \dot{Q}(t) = \gamma e^{-\mu_2 \tau} S(t-\tau) Q(t-\tau) - (\mu_3 + d) Q(t), \end{cases} \tag{1}$$

Consider a population of size $N(t)$ at time t, this population is divided into three subclasses, with $N(t) = S(t) + I(t) + Q(t)$; where $S(t), I(t)$, and $Q(t)$ denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary immunity, respectively. The positive constants μ_1, μ_2, and μ_3 represent the death rates of susceptible, infectious and quarantine. Biologically, it is natural to assume that $\mu_1 \leq \min\{\mu_2, \mu_3\}$. The positive constant d is natural mortality rate. The positive constants μ represent rate of insidence. The positive constant γ represent the recovery rate of infection. The positive constant β is the average numbers of contacts infective for S and I. ρ the positive constant is the parameter of immigration.ν the positive constant is the parameter of emigration. The term $\gamma e^{-\mu_2 \tau} S(t-\tau) Q(t-\tau)$ reflects the fact that an individual has recovered from infection and still are alive after infectious period τ, where τ is the length of immunity period.

The initial condition of (1) is given as.

$$S(\eta) = \Phi_1(\eta), \; I(\eta) = \Phi_2(\eta), \; Q(\eta) = \Phi_3(\eta), \qquad -\tau \leq \eta \leq 0, \tag{2}$$

Where $\Phi = (\Phi_1, \Phi_2, \Phi_3)^T \in \mathbb{C}$ such that $S(\eta) = \Phi_1(\eta) = \Phi_1(0) \geq 0$, $I(\eta) = \Phi_2(\eta) = \Phi_2(0) \geq 0$, $Q(\eta) = \Phi_3(\eta) = \Phi_3(0) \geq 0$.

Let C denote the Banach space $C([-\tau, 0], \mathbb{R}^3)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^3$. With a biological meaning, we further assume that $\Phi_i(\eta) = \Phi_i(0) \geq 0$ for $i = 1, 2, 3$.

Consider the system without the parameter of emigrations. Hence system (1) can be rewritten as

$$\begin{cases} \dot{S}(t) = \rho + \mu - \nu - (\mu_1 + d) S(t) - \beta S(t) Q(t), \\ \dot{I}(t) = \beta S(t) Q(t) - (\mu_2 + d) I(t) - \gamma e^{-\mu_2 \tau} S(t-\tau) Q(t-\tau), \\ \dot{Q}(t) = \gamma e^{-\mu_2 \tau} S(t-\tau) Q(t-\tau) - (\mu_3 + d) Q(t), \end{cases} \tag{3}$$

With the initial conditions.

$$S(\eta) = \Phi_1(\eta), \; I(\eta) = \Phi_2(\eta) \; , Q(\eta) = \Phi_3(\eta), \qquad -\tau \leq \eta \leq 0, \tag{4}$$

Where $\Phi_1(0) \geq 0, \Phi_2(0) \geq 0, \Phi_3(0) \geq 0, \qquad -\tau \leq \eta \prec 0.$

The region $\Omega = \{(S, I, Q) \in \mathbb{R}^3_+, \; S + I + Q \leq N < \frac{\rho + \mu - \nu}{\mu_1 + d}\}$ is positively invariant set of (3).

3. Equilibrium and stability

An equilibrium point of system (3) satisfies

$$\begin{cases} \rho+\mu-\nu-(\mu_1+d)\,S-\beta SQ=0,\\ \beta kSQ-(\mu_2+d)\,I-\gamma e^{-\mu_2\tau}S(t-\tau)Q(t-\tau)=0,\\ \gamma e^{-\mu_2\tau}S(t-\tau)Q(t-\tau)-(\mu_3+d)\,Q=0, \end{cases} \tag{5}$$

We calculate the points of equilibrium in the absence and presence of infection.

In the absence of infection $I=0$, the system (5) has a disease-free equilibrium E_0:

$$E_0=\left(\hat{S},\ \hat{I},\ \hat{Q}\right)^T=\left(\frac{\rho+\mu-\nu}{\mu_1+d},\ 0,\ 0\right)^T.$$

The eigenvalues can be determined by solving the characteristic equation of the linearization of (3) near E_0 is

$$\det\begin{pmatrix} -(\mu_1+d)-A & 0 & -\frac{\beta(\rho+\mu-\nu)}{\mu_1+d}\\ 0 & -(\mu_2+d)-A & \frac{(\rho+\mu-\nu)\left(\beta-\gamma e^{-\mu_2\tau}\right)}{\mu_1+d}\\ 0 & 0 & \frac{(\rho+\mu-\nu)\gamma e^{-\mu_2\tau}}{\mu_1+d}-(\mu_3+d)-A \end{pmatrix}=0 \tag{6}$$

So the eigenvalues are

$$A_1=-(\mu_1+d),\ A_2=-(\mu_2+d).$$

In order for λ_1,λ_2, to be negative, it is required that.

$$\frac{(\rho+\mu-\nu)\,\gamma e^{-\mu_2\tau}}{\mu_1+d}<\mu_3+d \tag{7}$$

Then we define the basic reproduction number of the infection R_0 as follows.

$$R_0=\frac{(\rho+\mu-\nu)\,\gamma e^{-\mu_2\tau}}{(\mu_1+d)\,(\mu_3+d)} \tag{8}$$

In the presence of infection $I\neq 0$, substituting in the system, Ω also contains a unique positive, endemic equilibrium $E_\tau^*=(S_\tau^*,\ I_\tau^*,\ Q_\tau^*)^T$ where

$$\begin{cases} S_\tau^*=\frac{\rho+\mu-\nu}{\mu_1+d}\times\frac{1}{R_0},\\ I_\tau^*=\frac{R_0-1}{\mu_2+d}\left[\frac{\rho+\mu-\nu}{R_0}-\frac{(\mu_1+d)(\mu_3+d)}{\beta}\right],\\ Q_\tau^*=\frac{\mu_1+d}{\beta}(R_0-1) \end{cases} \tag{9}$$

Note that

$$\begin{aligned} N_\tau^* &= \frac{(\rho+\mu-\nu)(\mu_2-\mu_1)}{R_0(\mu_1+d)(\mu_2+d)}+\frac{\rho+\mu-\nu}{\mu_2+d}\\ &\quad+\frac{(R_0-1)(\mu_1+d)(\mu_2-\mu_3)}{\beta(\mu_2+d)} \end{aligned} \tag{10}$$

So $E_\tau^*=(S_\tau^*,\ I_\tau^*,\ Q_\tau^*)^T$ is the unique positive endemic equilibrium point which exists if $R_0>1$.

Theorem 1. *The disease-free equilibrium E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

Theorem 2. *With $R_0 > 1$, system (3) has a unique non-trivial equilibrium $E_\tau^*$is locally asymptotically stable.*

4. Global asymptotic stability of endemic equilibrium

Consider system (3), with introducing the variables,

$$x(t) = S(t) - S_\tau^*, \qquad y(t) = I(t) - I_\tau^*, \qquad z(t) = Q(t) - Q_\tau^*,$$

System (3) is centered at the endemic equilibrium $E_\tau^* = \left(S_\tau^*,\ I_\tau^*,\ Q_\tau^*\right)^T$, then

$$\begin{cases} \dot{x}(t) = \left[-\left(\mu_1 + d\right) - \beta Q_\tau^*\right] x + \left[-\beta S_\tau^*\right] z, \\ \dot{y}(t) = \left[\left(\beta - \gamma e^{-\mu_2 \tau}\right) Q_\tau^*\right] x + \left[-\left(\mu_2 + d\right)\right] y + \left[\left(\beta - \gamma e^{-\mu_2 \tau}\right) S_\tau^*\right] z, \\ \dot{z}(t) = \left[\beta \gamma e^{-\mu_2 \tau} Q_\tau^*\right] x + \left[\gamma e^{-\mu_2 \tau} S_\tau^* - \left(\mu_3 + d\right)\right] z \end{cases} \tag{11}$$

Lemma 3. *Let*

$$S\left(s\right) = S\left(0\right) > 0,\ I\left(s\right) = I\left(0\right) \geq 0 \text{ for all } s \in [-\tau, 0] \text{ and } Q\left(0\right) > 0.$$

$S(t)$, $I(t)$ and $Q(t)$ solutions of system (3) are positive for all $t > 0$.

Proof. For contraduction there exists the first time t_0, such that $S\left(t_0\right) Q\left(t_0\right) = 0$.

- Assume that $S\left(t_0\right) = 0$, then $Q\left(t\right) \geq 0$ for all $t \in [0, t_0]$.With Eq 1 in the system (1) we have
$$\dot{S}\left(t_0\right) = \rho + \mu - \nu > 0.$$
For $S\left(t_0\right) = 0$, $S_0 > 0$, we must have $\dot{S}\left(t_0\right) < 0$ which is contradiction.
- Assume that $I\left(t_0\right) = 0$, then with Eq 2 in the system (1) we have
$$\dot{I}\left(t_0\right) = -\gamma e^{-\mu_2 \tau} S(t-\tau) Q(t-\tau)$$
$\dot{I}\left(t_0\right)$ is positive because $S(t)$ and $Q(t)$ solutions of system (1) are positive for all $t > 0$.
- For $I\left(t_0\right) = 0$, $I > 0$, we must have $\dot{I}\left(t_0\right) < 0$ which is contradiction.
- Assume that $Q\left(t_0\right) = 0$, then $S\left(t\right) \geq 0$ for all $t \in [0, t_0]$. with Eq 3 in the system (3) we have
$$\begin{aligned} \dot{Q}\left(t_0\right) &= \gamma e^{-\mu_2 \tau} S(t-\tau) Q(t-\tau), \\ Q\left(t_0\right) &= \gamma \int_{t_0-\tau}^{t_0} e^{-\mu_2 (t_0 - s)} S(s) Q(s) ds. \end{aligned}$$
$S\left(s\right) > 0, S\left(s\right) > 0$ for all $t \in [0, t_0)$. We have $\gamma \int_{t_0-\tau}^{t_0} e^{-\mu_2 (t_0 - s)} S(s) Q(s) ds > 0$, and $Q\left(t_0\right) = 0$, which is contradiction.

□

Lemma 4. *Let*

$$S(s) = S_0 > 0,\ Q(s) = Q_0 > 0 \text{ for all } s \in [-\tau, 0] \text{ and } Q_0 > 0.$$

Then

$$S(t) \leq \max\left\{\frac{\rho+\mu-\nu}{\mu_1+d}, S_0+I_0+Q_0\right\} = M$$

Proof. We have

$$N(t) = S(t) + I(t) + Q(t),$$

For $R_0 < 1$ the solutions $S(t), I(t)$ and $Q(t)$ approach the disease free equilibrium as $t \to \infty$.

With Eq 2 in the system (3) we have $\dot{I} \leq -(\mu_2+d)I$, hence if $\mu_2 + d < 0$,

$$\lim_{t\to\infty} I(t) = 0,$$

With Eq 3 in the system (3) we have.

$$\lim_{t\to\infty} Q(t) = 0,$$

With Eq 1 in the system (3) we obtain $\dot{S} = \rho + \mu - \nu - (\mu_1 + d)S$.

$$\lim_{t\to\infty} S(t) = \frac{\rho+\mu-\nu}{\mu_1+d},$$

Hence.

$$\lim_{t\to\infty} N(t) = \frac{\rho+\mu-\nu}{\mu_1+d}.$$

From lemma1, $S(t)$, $I(t)$ and $Q(t)$ solutions of system (1) are positive.

$$S(t) \leq \frac{\rho+\mu-\nu}{\mu_1+d}, \text{for all} t \leq 0.$$

Suppose that

$$N(0) \leq \frac{\rho+\mu-\nu}{\mu_1+d},\ \text{then } N(t) \leq \frac{\rho+\mu-\nu}{\mu_1+d}$$

On the contrary

If $N(0) > \frac{\rho+\mu-\nu}{\mu_1+d}$ then $N(t) < N(0)$, and $S(t) < N(0)$ for all $t > 0$. □

Theorem 5. *Let* $S(s) = S_0 > 0$, $Q(s) = Q_0 > 0$ *for all* $s \in [-\tau, 0]$ *and* $Q_0 > 0$. E^*_τ *is globally asymptotically stable for all* τ

$$\tau > \max\left\{\begin{array}{c} \frac{1}{\gamma}\ln\frac{\omega M+3\omega Q^*_\tau}{2\omega(\mu_1+d)}, \\ \frac{1}{\gamma}\ln\frac{\omega M+3\omega Q^*_\tau}{2\omega(\mu_2+d)+(\mu_2+d)-\beta M}, \\ \frac{1}{\gamma}\ln\frac{Q^*_\tau-3\omega M}{2(\mu_3+d)-\beta M} \end{array}\right\}$$

Where

$$\begin{aligned} M &= \max\left\{\frac{\rho+\mu-\nu}{\mu_1+d}, S_0+I_0+Q_0\right\}, \\ \omega &= \frac{\beta Q_\tau^*}{\mu_1+\mu_2+2d} \end{aligned}$$

Proof. We consider system (3) .

Let us introduce the functional

$$V(x,y,z) = \frac{1}{2}\omega(x+y)^2 + \frac{1}{2}\left(y^2+z^2\right),$$

The derivative $\dot{V}(x,y,z)$ is

$$\begin{aligned} \dot{V}(x,y,z) &= \omega(x+y)(\dot{x}+\dot{y}) + y\dot{y} + z\dot{z} \\ &= \omega(x+y)\begin{bmatrix} \left(-(\mu_1+d)-\beta Q_\tau^*\right)x - \beta S_\tau^* z + \left(\beta-\gamma e^{-\mu_2\tau}\right)Q_\tau^* x \\ -(\mu_2+d)y + \left(\beta-\gamma e^{-\mu_2\tau}\right)S_\tau^* z \end{bmatrix} \\ &\quad + y\left[\left(\beta-\gamma e^{-\mu_2\tau}\right)Q_\tau^* x - (\mu_2+d)y + \left(\beta-\gamma e^{-\mu_2\tau}\right)S_\tau^* z\right] + \\ &\quad + z\left[\beta\gamma e^{-\mu_2\tau}Q_\tau^* x + \left(\gamma e^{-\mu_2\tau}S_\tau^* - (\mu_3+d)\right)z\right] \\ &= -\omega(\mu_1+d)x^2 - \left[(\omega+1)(\mu_2+d)\right]y^2 \\ &\quad - \left(\gamma e^{-\mu_2\tau}S_\tau^* - (\mu_3+d)\right)z^2 \\ &\quad + \left[\beta Q_\tau^* - \omega(\mu_1+d) - \omega(\mu_2+d)\right]xy \\ &\quad + \beta S_\tau^* yz - \left[\omega Q_\tau^*\gamma e^{-\mu_2\tau}\right]xx(t-\tau) \\ &\quad - (\omega+1)Q_\tau^*\gamma e^{-\mu_2\tau}yx(t-\tau) \\ &\quad + Q_\tau^*\gamma e^{-\mu_2\tau}zx(t-\tau) - \omega S_\tau^*\gamma e^{-\mu_2\tau}xz(t-\tau) \\ &\quad - (\omega+1)S_\tau^*\gamma e^{-\mu_2\tau}yz(t-\tau) + S_\tau^*\gamma e^{-\mu_2\tau}zz(t-\tau). \end{aligned}$$

By lemma 2 we have $S(t) \le M$ for all $t \ge 0$ and ω is an arbitrary real constant choosing as follows

$$\omega = \frac{\beta Q_\tau^*}{\mu_1+\mu_2+2d}$$

$$\begin{aligned} \dot{V}(x,y,z) &\le -\omega(\mu_1+d)x^2 - \left[(\omega+1)(\mu_2+d)\right]y^2 \\ &\quad - (\mu_3+d)z^2 \\ &\quad + \beta M yz - \left[\omega Q_\tau^*\gamma e^{-\mu_2\tau}\right]xx(t-\tau) \\ &\quad - (\omega+1)Q_\tau^*\gamma e^{-\mu_2\tau}yx(t-\tau) \\ &\quad + Q_\tau^*\gamma e^{-\mu_2\tau}zx(t-\tau) - \omega M\gamma e^{-\mu_2\tau}xz(t-\tau) \\ &\quad - (\omega+1)M\gamma e^{-\mu_2\tau}yz(t-\tau) \\ &\quad + M\gamma e^{-\mu_2\tau}zz(t-\tau). \end{aligned}$$

Applying Cauchy-Chwartz inequality; we obtain:

$$\begin{aligned}
\dot{V}(x,y,z) \leq\ & -\omega(\mu_1+d)x^2-[(\omega+1)(\mu_2+d)]y^2 \\
& -(\mu_3+d)z^2 \\
& -\frac{1}{2}\omega Q_\tau^*\gamma e^{-\mu_2\tau}\left[x^2+x^2(t-\tau)\right] \\
& -\frac{1}{2}(\omega+1)Q_\tau^*\gamma e^{-\mu_2\tau}\left[y^2+x^2(t-\tau)\right] \\
& +\frac{1}{2}Q_\tau^*\gamma e^{-\mu_2\tau}\left[z^2+x^2(t-\tau)\right] \\
& -\frac{1}{2}\omega M\gamma e^{-\mu_2\tau}\left[x^2+z^2(t-\tau)\right] \\
& -\frac{1}{2}(\omega+1)M\gamma e^{-\mu_2\tau}\left[y^2+z^2(t-\tau)\right] \\
& +\frac{1}{2}M\gamma e^{-\mu_2\tau}\left[z^2+z^2(t-\tau)\right] \\
& +\frac{1}{2}\beta M\left[y^2+z^2\right], \\
\leq\ & \left[-\omega(\mu_1+d)-\frac{1}{2}\omega\gamma e^{-\mu_2\tau}(M+Q_\tau^*)\right]x^2 \\
& +\left[\frac{1}{2}\beta M-(\omega+1)(\mu_2+d)-\frac{1}{2}(\omega+1)\gamma e^{-\mu_2\tau}(M+Q_\tau^*)\right]y^2 \\
& +\left[\frac{1}{2}\beta M-(\mu_3+d)+\frac{1}{2}\gamma e^{-\mu_2\tau}(M+Q_\tau^*)\right]z^2 \\
& -\left[\omega Q_\tau^*\gamma e^{-\mu_2\tau}\right]x^2(t-\tau)-\left[\frac{1}{2}(3\omega+1)\gamma M e^{-\mu_2\tau}\right]z^2(t-\tau)
\end{aligned}$$

Choose the Lyapunov functional

$$\begin{aligned}
V(x_t,y_t,z_t) =\ & V(x,y,z)-\omega Q_\tau^*\gamma e^{-\mu_2\tau}\int_{t-\tau}^{t}x^2(\theta)d\theta \\
& -\frac{1}{2}(3\omega+1)\gamma M e^{-\mu_2\tau}\int_{t-\tau}^{t}z^2(\theta)d\theta
\end{aligned}$$

Then

$$\begin{aligned}
\dot{V}(x_t, y_t, z_t) &= \dot{V}(x, y, z) - \omega Q_\tau^* \gamma e^{-\mu_2 \tau} x^2(t) \\
&\quad + \omega Q_\tau^* \gamma e^{-\mu_2 \tau} x^2(t-\tau) \\
&\quad - \frac{1}{2}(3\omega + 1)\gamma M e^{-\mu_2 \tau} z^2(t) \\
&\quad + \frac{1}{2}(3\omega + 1)\gamma M e^{-\mu_2 \tau} z^2(t-\tau) \\
&\leq \left[-\omega(\mu_1 + d) - \frac{1}{2}\omega\gamma e^{-\mu_2 \tau}(M + Q_\tau^*)\right] x^2 \\
&\quad + \begin{bmatrix} \frac{1}{2}\beta M - (\omega + 1)(\mu_2 + d) \\ -\frac{1}{2}(\omega + 1)\gamma e^{-\mu_2 \tau}(M + Q_\tau^*) \end{bmatrix} y^2 \\
&\quad + \left[\frac{1}{2}\beta M - (\mu_3 + d) + \frac{1}{2}\gamma e^{-\mu_2 \tau}(M + Q_\tau^*)\right] z^2 \\
&\quad - \left[\omega Q_\tau^* \gamma e^{-\mu_2 \tau}\right] x^2 + \omega Q_\tau^* \gamma e^{-\mu_2 \tau} x^2(t-\tau) \\
&\quad - \left[\frac{1}{2}(3\omega + 1)\gamma M e^{-\mu_2 \tau}\right] z^2 \\
&\quad + \frac{1}{2}(3\omega + 1)\gamma M e^{-\mu_2 \tau} z^2(t-\tau), \\
&\leq \left[-\omega(\mu_1 + d) - \frac{1}{2}\omega\gamma e^{-\mu_2 \tau}(M + Q_\tau^*)\right] x^2 \\
&\quad + \left[\frac{1}{2}\beta M - (\omega + 1)(\mu_2 + d) - \frac{1}{2}(\omega + 1)\gamma e^{-\mu_2 \tau}(M + Q_\tau^*)\right] y^2 \\
&\quad + \left[\frac{1}{2}\beta M - (\mu_3 + d) + \frac{1}{2}\gamma e^{-\mu_2 \tau}(M + Q_\tau^*)\right] z^2 \\
&\quad - \left[\omega Q_\tau^* \gamma e^{-\mu_2 \tau}\right] x^2 - \left[\frac{1}{2}(3\omega + 1)\gamma M e^{-\mu_2 \tau}\right] z^2
\end{aligned}$$

Therefore

$$\begin{aligned}
\dot{V}(x_t, y_t, z_t) &\leq -\left[\omega(\mu_1 + d) + \frac{1}{2}\omega\gamma e^{-\mu_2 \tau}(M + 3Q_\tau^*)\right] x^2 \\
&\quad - \begin{bmatrix} (\omega + 1)(\mu_2 + d) - \frac{1}{2}\beta M \\ +\frac{1}{2}(\omega + 1)\gamma e^{-\mu_2 \tau}(M + Q_\tau^*) \end{bmatrix} y^2 \\
&\quad - \begin{bmatrix} (\mu_3 + d) - \frac{1}{2}\beta M \\ -\frac{1}{2}\gamma e^{-\mu_2 \tau}(Q_\tau^* - 3\omega M) \end{bmatrix} z^2.
\end{aligned}$$

While the above inequality is always negative provided that

$$\tau > \max\left\{ \begin{array}{c} \frac{1}{\gamma}\ln\frac{\omega M+3\omega Q_\tau^*}{2\omega(\mu_1+d)} \\ , \frac{1}{\gamma}\ln\frac{\omega M+3\omega Q_\tau^*}{2\omega(\mu_2+d)+(\mu_2+d)-\beta M}, \\ \frac{1}{\gamma}\ln\frac{Q_\tau^*-3\omega M}{2(\mu_3+d)-\beta M} \end{array} \right\}$$

With application of the Lyapunov-LaSalle type theorem in [10]

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0.$$

□

5. Stochastic Stability

We limit ourselves here to perturbing only the contact rate so we replace β by $\beta+\sigma W(t)$, where$W(t)$ is white noise (Brownian motion). The system (3) is transformed to the following Itô stochastic differential equations, with $\gamma_0 = \gamma e^{-\mu_2\tau}$

$$\left\{ \begin{array}{l} dS = [\rho+\mu-\nu-(\mu_1+d)\,S-\beta SQ]-\sigma SQdW, \\ dI = [\beta SQ-(\mu_2+d)\,I-\gamma_0 SQ]+\sigma SQdW, \\ dQ = [\gamma_0 SQ-(\mu_3+d)\,Q]\,, \end{array} \right. \tag{12}$$

In this section, we will proof, under some conditions, that E_0 is globally exponentially mean square and almost surely stable, and for this purpose, we need the following Theorem

Theorem 6. *The set Ω is almost surely invariant by the stochastic system* (12). *Thus if $(S_0, I_0, Q_0) \in \Omega$, then $P\left[(S, I, Q) \in \Omega\right] = 1$.*

Proof. The system (12) implies that $dN \leq [\rho+\mu-\nu-(\mu_1+d)\,N]\,dt$, then we have

$$N(t) \leq \frac{\rho+\mu-\nu}{\mu_1+d} + \left(N_0 - \frac{\rho}{\mu_1+d}\right), \text{ for all } t \geq 0.$$

Since $(S_0, I_0, Q_0) \in \Omega$, then

$$N(t) \leq \frac{\rho+\mu-\nu}{\mu_1+d}, \text{ for all } t \geq 0. \tag{13}$$

There exist $\varepsilon_0 > 0$, such that $S_0 > \varepsilon_0 > 0$, $I_0 > \varepsilon_0 > 0$ and $Q_0 > \varepsilon_0 > 0$.
Considering

$$\begin{array}{rcl} \upsilon_\varepsilon & = & \inf\{t \geq 0,\ S(t) \leq \varepsilon \text{ or } I(t) \leq \varepsilon \text{ or } Q(t) \leq \varepsilon,\}\,, \text{ for } \varepsilon \leq \varepsilon_0, \\ \upsilon & = & \lim_{t\to 0}\upsilon_\varepsilon = \inf\{t \geq 0,\ S(t) \leq 0 \text{ or } I(t) \leq 0 \text{ or } Q(t) \leq 0,\} \end{array} \tag{14}$$

Let

$$V(t) = \log\frac{\rho+\mu-\nu}{(\mu_1+d)\,S(t)} + \log\frac{\rho+\mu-\nu}{(\mu_1+d)\,I(t)} + \log\frac{\rho+\mu-\nu}{(\mu_1+d)\,Q(t)}.$$

Then, using Itô formula we have, for all $t \geq 0$ and $T \in [0, t \wedge \upsilon_\varepsilon]$,

$$\begin{aligned} dV(T) &= \left[\begin{array}{c} -\frac{\rho+\mu-\nu}{S} \\ +(\mu_1+d)+\beta Q+\frac{1}{2}I^2 \end{array} \right] dT + \sigma Q dW \\ &\quad + \left[\begin{array}{c} (\gamma_0-\beta)\frac{SQ}{I} \\ +(\mu_2+d)+\frac{1}{2}S^2 \end{array} \right] dT + \sigma \frac{SQ}{I} dW \\ &\quad + [-\gamma_0 S + (\mu_3+d)]\, dT, \end{aligned}$$

$$dV(T) \leq \left[\begin{array}{c} \mu_1+\mu_2+\mu_3+3d \\ +\beta Q+\frac{1}{2}I^2+\frac{1}{2}S^2 \end{array} \right] dT + \sigma \frac{Q}{I}(I-S)\, dW \tag{15}$$

With (13), we have S, I and $Q \in \left[0, \frac{\rho+\mu-\nu}{(\mu_1+d)}\right]$

Let

$$\begin{aligned} L &= \mu_1+\mu_2+\mu_3+3d \\ &\quad +\beta\frac{\rho+\mu-\nu}{\mu_1+d} + \left(\frac{\rho+\mu-\nu}{\mu_1+d}\right)^2, \\ f(I) &= \frac{Q}{I}, \end{aligned} \tag{16}$$

We remplace (16) into (15), we obtain

$$dV(T) \leq L dT + \sigma(I(T) - S(T))\, f(I(T))\, dW, \tag{17}$$

Then

$$V(T) \leq LT + \sigma \int_0^T f(I(u))(I(u) - S(u))\, dW(u), \tag{18}$$

With proposition 7.6 in [6], $\sigma \int_0^T f(I(u))(I(u) - S(u))\, dW(u)$ is mean zero process then,

$$E(V(T)) \leq LT \tag{19}$$

for all $t \geq 0$ and $T \in [0, t \wedge \upsilon_\varepsilon]$,

$$S(t \wedge \upsilon_\varepsilon), I(t \wedge \upsilon_\varepsilon), \text{ and } Q(t \wedge \upsilon_\varepsilon) \in \left[0, \frac{\rho}{(\mu_1+d)}\right],$$

Then

$$E(V(t \wedge \upsilon_\varepsilon)) \leq L(t \wedge \upsilon_\varepsilon) \leq Lt,$$

$V(t \wedge \upsilon_\varepsilon) \geq 0$,

$$E(V(t \wedge \upsilon_\varepsilon)) \geq E(V(t)) \times \kappa_{[\upsilon_\varepsilon \leq t]} \geq P(\upsilon_\varepsilon \leq t) \log \frac{\rho+\mu-\nu}{(\mu_1+d)\varepsilon} \tag{20}$$

Where $\kappa_{[\upsilon_\varepsilon \leq t]}$ is the indicator function of a subset $[\upsilon_\varepsilon \leq t]$,

Combining (19) , and (20) , we obtain

$$P\left(\upsilon_{\varepsilon} \leq t\right) \leq \frac{Lt}{\log \frac{\rho+\mu-\nu}{(\mu_1+d)\varepsilon}}, \text{ for all } t \geq 0; \tag{21}$$

for all $t \geq 0$, and $\varepsilon \to 0$, we obtain $P\left(\upsilon \leq t\right) = 0$;

From where

$$P\left(\upsilon \leq \infty\right) = 0$$

□

6. The Model with Age

The age distributions of the numbers in the classes are denoted by $S(a,t)$, $I(a,t)$, and $Q(a,t)$, denote the sizes of the population susceptible to disease, and infectious members, quarantine members with the possibility of infection through temporary immunity, respectively of age a, at time t, $d(a)$ is the age-specific death rate,

The system of partial equations for the age distributions is

$$\begin{cases} \frac{\partial S}{\partial t} + \frac{\partial S}{\partial a} = -\left(\mu_1 + d(a)\right) S(a,t) + \beta_1(t) S(a,t), \\ \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a} = -\beta_1(t) S(a,t) - \left(\mu_2 + d(a)\right) I(a,t) + \gamma_1(t-\tau) S(a,t-\tau), \\ \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial a} = -\gamma_1(t-\tau) S(a,t-\tau) - \left(\mu_3 + d(a)\right) Q(a,t), \end{cases} \tag{22}$$

With

$$\begin{aligned} \beta_1(t) &= -\beta \int Q(a,t) da \\ \gamma_1(t-\tau) &= -\gamma \int e^{-\mu_2 \tau} Q(a,t-\tau) da \end{cases} \tag{23}$$

6.1. **Equilibrium and stability.** Assume that sub population does not depend on the time when the system (22) is written as follows

$$\begin{cases} \frac{dS}{da} = \left(\beta_1 - \mu_1 - d(a)\right) S(a), \\ \frac{dI}{da} = \left(\gamma_1 - \beta_1\right) S(a) - \left(\mu_2 + d(a)\right) I(a), \\ \frac{dQ}{da} = -\gamma_1 S(a) - \left(\mu_3 + d(a)\right) Q(a), \end{cases} \tag{24}$$

The initial condition of (24) is given as

$$S(0) = S_1, \quad I(0) = I_1, \quad Q(0) = Q_1 \tag{25}$$

Differential equations of the system (24) are solved with different methods of resolutions and with (25), so

$$S(a) = S_1 e^{-(\mu_1 - \beta_1)a}\ \Phi(a), \tag{26}$$

$$I(a) = I_1 \Phi(a) e^{-\mu_2 a} - \frac{\left(\gamma_1 - \beta_1\right) S_1 \Phi(a)}{\mu_1 - \beta_1 - \mu_2} \left(e^{-(\mu_1 - \beta_1)a} - e^{-\mu_2 a}\right), \tag{27}$$

$$Q(a) = Q_1 \Phi(a) e^{-\mu_3 a} - \frac{\gamma_1 S_1 \Phi(a)}{\mu_1 - \beta_1 - \mu_3} \left(e^{-(\mu_1 - \beta_1)a} - e^{-\mu_3 a}\right) \tag{28}$$

Where

$$\Phi(a) = \exp\left(-\int d(a)da\right) \tag{29}$$

The system (24) has the unique positive equilibrium point P_1,

$$P_1 = \left(\hat{S}_1,\ \hat{I}_1,\ \hat{Q}_1\right)^T = (0,\ 0,\ 0)^T.$$

We calculate the Jacobian matrix according to the system (24) with P_1

$$J(P_1) = \begin{bmatrix} \beta_1 - \mu_1 - d(a) & 0 & 0 \\ \lambda - \gamma_0 & -(\mu_2 + d(a)) & 0 \\ -\gamma_0 & 0 & -(\mu_3 + d(a)) \end{bmatrix}$$

The epidemic is locally asymptotically stable if and only if all eigenvalues of the Jacobian matrix $J(P_1)$ have negative real part. The eigenvalues can be determined by solving the characteristic equation of the linearization of (25) near P_1 is

$$\det\begin{pmatrix} \beta_1 - \mu_1 - d(a) - A & 0 & 0 \\ \lambda - \gamma_0 & -(\mu_2 + d(a)) - A & 0 \\ -\gamma_0 & 0 & -(\mu_3 + d(a)) - A \end{pmatrix} = 0 \tag{31}$$

So the eigenvalues are

$$A_1 = \beta_1 - \mu_1 - d(a), \quad A_2 = -(\mu_2 + d(a)), \quad A_3 = -(\mu_3 + d(a))$$

In order to A_1, A_2,and A_3 will be negative, it is required that

$$\beta_1 < \mu_1 + d(a)$$

The basic reproduction number R_0 is defined as the total number of infected population in the resulting sub-infected population where almost all of the uninfected. The basic reproduction number of the infection R_0 is defined as follows:

$$R_0 = \frac{\beta_1}{\mu_1 + d(a)} \tag{32}$$

The time during which people remain infective is defined as

$$T = \frac{1}{\mu_1 + d(a)}$$

The doubling time t_d of the epidemic can be obtained as

$$t_d = \frac{(\ln 2)\, T}{R_0 - 1} \tag{33}$$

Theorem 1 The disease-free equilibrium P_1 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

Let (26), so if $R_0 < 1$ then $\mu_1 - \beta_1 > 0$, so $S(a)$ converges to zero.

Let (27) , so

$$I(a) \leq \left[I_1\Phi(a) - \frac{(\gamma_1 - \beta_1)\, S_1\Phi(a)}{\mu_1 - \beta_1 - \mu_2} \right] e^{-m_1 a},\ m_1 = \min\{\mu_1 - \beta_1, \mu_2\} \tag{34}$$

If $R_0 < 1$, $i(a)$ converges to zero.

Let (28) , so

$$Q(a) = \left[Q_1\Phi(a) - \frac{\gamma_1 S_1\Phi(a)}{\mu_1 - \beta_1 - \mu_3} \right] e^{-m_2 a},\ m_1 = \min\{\mu_1 - \beta_1, \mu_3\} \tag{35}$$

If $R_0 < 1$, $Q(a)$ converges to zero.

Conclusion 7. *This paper addresses a epidemic model with temporary immunity, whenever the quarantine individuals will return to the susceptible. The endemic equilibrium is globally asymptotically stable, then under some conditions, study the stochastic stability. Finally, the equilibrium and stability of the epidemic model with age.*

References

[1] Anderson R. M and Medley R M and Jhonson A K. A Preliminary Study of the Transmission Dynamics of the Human Immunodeficiency Virus (HIV), the Causative Agent of AIDS. IMA. J. Math. Appl. Med. Biol 3, 229-263.(1986) .

[2] Abta A and Kaddar A and Talibi H. A. Global Stability for Delay SIR and SEIR Epidemic Models With Saturated Incidence Rates. Electronic Journal of Differential Equations, 23,1-13.(2012).

[3] Bailley. N.T.J. Some Stochastic Models for Small Epidemics in Large Population. Appl. Statist.13, 9-19.(1964) .

[4] Bailley. N.T.J. The Mathematical Theory of Infection Diseases and its Application. Applied Statistics, 26, N1, 85-87.(1977) .

[5] Batiha, M. S. M. Noorani and I. Hashim. Numerical solutions of the nonlinear integro-differential equations, Int. J. Open Probl. Compt. Math, 34-42.(2008) .

[6] Becker.N.G. The Uses of Epidemic Models**.** *Biometrics **35**, 295-305.* (**1979**).

[7] Billard.L. A Stochastic General Epidemic in m Sub-Population. *J. Appl. Prob. **13**, 567-572.* (1976) .

[8] Jinliang W, Xinxin Tian. Global Stabilty of a Delay Differential Equation Of Hepatitis B Virus Infection With Immune Response. Electronic Journal of Differential Equations,94,1-11. (2013).

[9] Jin. Z, Zhien. M and Maoan. H. Globale stability of an SIRS epidemic model with delay, Acta Matimatica Scientia. 26 B. 291-306.(2006) .

[10] Kuang Y. Delay-Differential Equations with application in population biology. Academic Press, new york.(1993) .

[11] Lounes. R and Arazoza. H. Modeling HIV Epidemic Under Contact Tracing. The Cuban Case. Journal of theoritical Medecine Vol 2, 267-274.(2000) .

[12] Lounes. R, Arazoza. H. A Non-Linear Model for a Sexually Transmitted Disease with contact tracing. IMA. J. MJath. Appl. Med. Biol.19, 221-234.(2002) .

[13] Lahrouz A and El Maroufy H. Qualitative Behaviour of a Model of an SIRS Epidemic:Stability and Permanence. Applied Mathematics & Information Sciences. An International Journal 5 (2), 220-238.(2011).

[14] Luo Q and Mao X. Stochastic population dynamics under regime switching. J. Math. Anal. Appl.334, 69-84.(2007)

[15] Michael Steel J. Stochastic calculus and finantial applications. Springer-Verlag. (2003) .

[16] Naresh R, and Omar S. An epidemic model for the transmission dynamics of HIV/AIDS and another infection. International Journal of Mathematical Archive-1(3) , 68-72. (2010) .

[17] Perto. L. Differential Equations and Dynamical Systems. 2nd edition, Springer, New York.(1996) .

[18] Ray Waston. A useful Random Time-Seal Transformation For The Standard Epidemic Model. J. Appl. Prob.17, 324-332. (1980) .

[19] Ray Waston, 1980. On The Size Distribution For Some Epidemic Models. J. Appl. Prob.17, 912-921.(1980) .

[20] Robert N and May. Population Biology of infectious diseases I. International centre of theoritical physics.1-9. (1982) .

[21] Ruoyan Sun. Global stability of the endemic equilibrium of multigroup SIR models with nonlinear incidence. Computers and Mathematics with Applications 60. 2286-2291. (2010) .

[22] Takeuchi and W. Ma. Stability analysis on a delayed SIR epidemic model with density dependent birth process, Dy-nam. Contin. Discrete Impuls. Systems, 5 . 171-184.(1999).

[23] Volodymyr Makarov, Denis Dragunov. A numeric-analytical method for solving the Cauchy problem for ordinary diferential equations. Applied Mathematics and Computation,1-26. (2010) .

[24] W. Ma, Y. Takeuchi, T. Hara and E. Beretta, Permanence of are SIR epidemic model with distributed time delays, Tohoku Math. J. 54, 581-591.(2002)

[25] W. Wang, Global behavior of an SEIR epidemic model with time delay, Appl. Math. Letters.15, 423-428.(2002).

[26] Wen L and Yang X. Global stability of a delayed SIRS model with temporary immunity. Chaos, Solitons and Fractals 38, 221-226. (2008)

[27] Xiao, L Chen and F. ven den Bosch, Dynamical behavior for a stage-structured SIR infectious disease model, Nonlinear Anal. Real World Appl 3,175-190.(2002).

[28] Z. Ma, J. Liu and J. Li, Stability analysis for differential infectivity epidemic models, Nonlinear Anal. Real World Appl 4, 841-856. (2003).

[29] Zhang F and Zhen Li and Zhang F. Global stability of an SIRepidemic model with constant infectious period. Applied Mathematics and Computation 199, 285-291.(2008) .

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