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ON THE OSCILLATION OF THE DIFFERENCE EQUATIONS WITH REAL COEFFICIENTS

E. M. ELABBASY, M. BARSOUM AND A. A. EL-BIATY

ABSTRACT. We investigate some qualitative behavior of the solutions of the difference equation $x_{n+1} = a + \sum_{i=0}^{k} \alpha_{(2i+1)} x_{n-(2i+1)} / b x_{n-\ell} + \sum_{i=0}^{k} \beta_{(2i)} x_{n-(2i)}$ where the the initial conditions $x_{-r}, x_{-r+1}, ..., x_0$ are arbitrary positive real numbers such that $r = \max\{\ell, k\}$ where $i, r \in \{0, 1, ...\}$ and $a, \alpha_{(2i+1)}, \beta_{(2i)}$ are positive constants.

1. Introduction

In this paper we deal with some properties of the solutions of the difference equation

$$x_{n+1} = a + \frac{\sum_{i=0}^{k} \alpha_{(2i+1)} x_{n-(2i+1)}}{b x_{n-\ell} + \sum_{i=0}^{k} \beta_{(2i)} x_{n-(2i)}}, \quad n = 0, 1, 2, ..., \tag{1.1}$$

where the the initial conditions $x_{-r}, x_{-r+1}, ..., x_0$ are arbitrary positive real numbers such that $r = \max\{\ell, k\}$ where $i, r \in \{0, 1, ...\}$ and $a, \alpha_{(2i+1)}, \beta_{(2i)}$ are positive constants. There is a class of nonlinear difference equations, known as the rational difference equations, each of which consists of the ratio of two polynomials in the sequence terms in the same form. there has been a lot of work concenring the global asympototic of solutions of rational difference equations [2], [3], [4], [5], [6], [7], [8], [10] and [11].

Many reseaches have investigated the behavior of the solution of difference equation for example:

Amleh et al. [1] has studied the global stability, boundedness and the periodic character of solutions of the equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}.$$

Our aim in this paper is to extend and generalize the work in [1]. That is, we will investigate the global behavior of (1.1) including the asymptotical stability of equilibrium points, the existence of period two solution and investigate the oscillation property of the recursive sequence of Eq. (1.1).

Now we recall some well-known results, which will be useful in the investigation of (1.1) and which are given in [9].

Key words and phrases. difference equation; Stability; Periodicity; Oscillatory; global Stability.

Let I be an interval of real numbers and let

$$F : I^{k+1} \to I,$$

where F is a continuous function. Consider the difference equation

$$x_{n+1} = F\left(x_n, x_{n-1}, ..., x_{n-k}\right), \quad n = 0, 1, 2, ..., \tag{1.2}$$

with the initial condition $x_{-k}, x_{-k+1}, ..., x_0 \in I$.

Definition 1. *(Equilibrium Point)*

A point $\overline{x} \in I$ is called an equilibrium point of Eq. (1.2) if

$$\overline{x} = f\left(\overline{x}, \overline{x}, ..., \overline{x}\right).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq. (1.2), or equivalently, $\overline{x}$ is a fixsed point of f.

Definition 2. *(Stability)*

Let $\overline{x} \in (0, \infty)$ be an equilibrium point of Eq. (1.2). Then

(i) An equilibrium point $\overline{x}$ of Eq. (1.2) is called stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-r}, x_{-r+1}, ..., x_0 \in (0, \infty)$ with $|x_{-r} - \overline{x}| + |x_{-r+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta$, then $|x_n - \overline{x}| < \varepsilon$ for all $n \geqslant -r$.

(ii) An equilibrium point $\overline{x}$ of Eq. (1.2) is called locally asymptotically stable if $\overline{x}$ is locally stable and there exists $\gamma > 0$ such that, if $x_{-r}, x_{-r+1}, ..., x_0 \in (0, \infty)$ with $|x_{-r} - \overline{x}| + |x_{-r+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma$, then

$$\lim_{n - \infty} x_n = \overline{x}.$$

(iii) An equilibrium point $\overline{x}$ of Eq. (1.2) is called a global attractor if for every $x_{-r}, x_{-r+1}, ..., x_0 \in (0, \infty)$ we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iv) An equilibrium point $\overline{x}$ of Eq. (1.2) is called globally asymptotically stable if $\overline{x}$ is locally stable and a global attractor.

(v) An equilibrium point $\overline{x}$ of Eq. (1.2) is called unstable if $\overline{x}$ is not locally stable.

Definition 3. *(Permanence)*

Eq. (1.2) is called permanent if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-r}, x_{-r+1}, ..., x_0 \in (0, \infty)$ there exists a positive integer N which depends on the initial conditions such that

$$m \leq x_n \leq M \quad \text{forall } n \geq -N.$$

Definition 4. *(Oscillation)*

A sequence $\{x_n\}_{n=-k}^{\infty}$ is called nonoscillatory about the point $\overline{x}$ if there is exists $N \geq -k$ such that either $x_n > \overline{x}$ for all $n \geq N$ or $x_n < \overline{x}$ for all $n \geq N$. Otherwise $\{x_n\}_{n=-k}^{\infty}$ is called oscillatory about $\overline{x}$.

Definition 5. *(Periodicity)*

A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -r$. A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

The linearized equation of Eq. (1.2) about the equilibrium point $\overline{x}$ is defined by the equation

$$z_{n+1} = \sum_{i=0}^{k} p_i z_{n-i}, \tag{1.3}$$

where

$$p_i = \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x},)}{\partial x_{n-i}}, \quad i = 0, 1, ..., k.$$

The characteristic equation associated with Eq. (1.3) is

$$\lambda^{k+1} - p_0\lambda^k - p_1\lambda^{k-1} - ... - p_{k-1}\lambda - p_k = 0. \tag{1.4}$$

Theorem 1.1. [9]. *Assume that F is a $C^1 -$ function and let $\overline{x}$ be an equilibrium point of Eq. (1.2). Then the following statements are true:*

(i) *If all roots of Eq. (1.4) lie in the open unit disk $|\lambda| < 1$, then he equilibrium point $\overline{x}$ is locally asymptotically stable.*
(ii) *If at least one root of Eq. (1.4) has absolute value greater than one, then the equilibrium point $\overline{x}$ is unstable.*
(iii) *If all roots of Eq. (1.4) have absolute value greater than one, then the equilibrium point $\overline{x}$ is a source.*

Theorem 1.2. [12] *Assume that $p_i \in R, i = 1, 2, ..., k$. Then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptoticcally stable of Eq. (1.5)

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \ \ n = 0, 1, ... \ . \tag{1.5}$$

2. Local stability of the equilibrium point of Eq.(1.1)

In this section we investigate the local stability character of the solutions of Eq. (1.1). Eq. (1.1) has a unique nonzero equilibrium point

$$\overline{x}=a+\frac{\sum\limits_{i=0}^{k}\alpha_{(2i+1)}\overline{x}}{b\overline{x}+\sum\limits_{i=0}^{k}\beta_{(2i)}\overline{x}},$$

then

$$\overline{x}=a+\frac{\sum\limits_{i=0}^{k}\alpha_{(2i+1)}}{b+\sum\limits_{i=0}^{k}\beta_{(2i)}}.$$

Let

$$G=\sum_{i=0}^{k}\alpha_{(2i+1)}\quad\text{and}\quad Q=\sum_{i=0}^{k}\beta_{(2i)}.$$

Then, we get

$$\overline{x}=a+\frac{G}{b+Q}.$$

Let $f:(0,\infty)^{k+1}\rightarrow(0,\infty)$ be a function defined by

$$f\left(v,u_0,u_1,...,u_k\right)=a+\frac{\sum\limits_{i=0}^{k}\alpha_{(2i+1)}u_{(2i+1)}}{bv+\sum\limits_{i=0}^{k}\beta_{(2i)}u_{(2i)}}.\tag{2.1}$$

Therefore it follows that

$$\frac{\partial f\left(v,u_0,u_1,...,u_k\right)}{\partial u_{(2i+1)}}=\frac{\sum\limits_{i=1}^{k}\alpha_{(2i+1)}}{\left[bv+\sum\limits_{i=0}^{k}\beta_{(2i)}u_{(2i)}\right]},$$

$$\frac{\partial f\left(v,u_0,u_1,...,u_k\right)}{\partial u_{(2i)}}=\frac{-\sum\limits_{i=0}^{k}\beta_{(2i)}\left(\sum\limits_{i=0}^{k}\alpha_{(2i+1)}u_{(2i+1)}\right)}{\left[bv+\sum\limits_{i=0}^{k}\beta_{(2i)}u_{(2i)}\right]^2},$$

and

$$\frac{\partial f\left(v,u_0,u_1,...,u_k\right)}{\partial v}=\frac{-b\left(\sum\limits_{i=0}^{k}\alpha_{(2i+1)}u_{(2i+1)}\right)}{\left[bv+\sum\limits_{i=0}^{k}\beta_{(2i)}u_{(2i)}\right]^2}.$$

Then we see that

$$\frac{\partial f(\overline{x},\overline{x},...,\overline{x})}{\partial u_{(2i+1)}}=\frac{G}{a(b+Q)+G}=-P_{2i+1},$$

$$\frac{\partial f(\overline{x},\overline{x},...,\overline{x},\overline{x})}{\partial u_{(2i)}}=\frac{-GQ}{(b+Q)(a(b+Q)+G)}=P_{2i},$$

and

$$\frac{\partial f(\overline{x},\overline{x},...,\overline{x})}{\partial v}=\frac{-bG}{(b+Q)(a(b+Q)+G)}=P_0.$$

Then the linearized equation of (1.1) about $\overline{x}$ is

$$z_{n+1}=\sum_{i=0}^{k}p_i z_{n-i}. \tag{2.2}$$

Theorem 2.1. *Assume that*

$$G<a(b+Q).$$

Than the equilibrium point of Eq. (1.1) is locally stable.

Proof. It is follows by Theorem(1.2) that, Eq. (2.2) is locally stable if

$$|p_k|+...+|p_j|+...+|p_1|+|p_0|<1.$$

That is

$$\left|\frac{G}{a(b+Q)+G}\right|+\left|\frac{-GQ}{(b+Q)(a(b+Q)+G)}\right|+\left|\frac{-bG}{(b+Q)(a(b+Q)+G)}\right|<1,$$

this implies that

$$bG+G(b+Q)+GQ<(b+Q)(a(b+Q)+G).$$

Thus

$$G<a(b+Q).$$

Hence, the proof is completed. □

Example 2.1. *Consider the difference equation*

$$x_{n+1}=0.5+\frac{0.6x_{n-1}+0.2x_{n-3}}{2x_{n-2}+9x_n+0.5x_{n-2}},$$

where $k=1,\ell=2,a=0.5,b=2,\alpha_1=0.6,\alpha_3=0.2,\beta_0=9,\beta_2=0.5$. Figure(2.1), shows that the equilibrium point of Eq. (1.1) has locally stable, with initial data $x_{n-3}=2.2,x_{n-2}=$

$0.2, x_{n-1} = 3.2, x_0 = 0.6.$

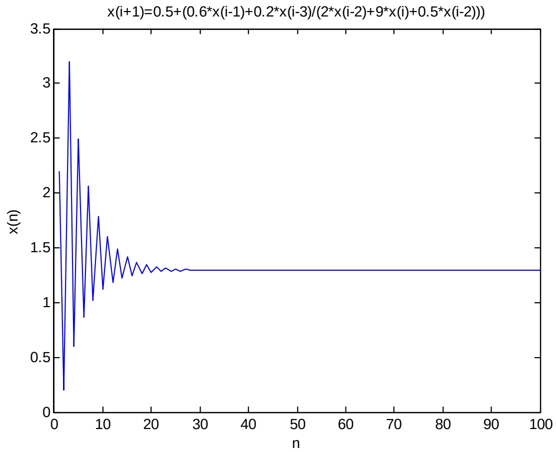

Figure *2.1*

3. Global Stability of Eq. (1.1)

Our aim in this section we investigate the global asymptotic stability of Eq. (1.1).

Considar of the Eq. (1.2) we give the following theorem which is a minor modification of Theorem (A.0.6) in [9] and which will be useful for the investigation of the global attractivity of solutions of Eq. (1.1).

Theorem 3.1. *Let $[a, b]$ be an interval of real numbers and assume that*

$$f : [a, b]^{k+1} \to [a, b]$$

is a continuous function satisfying the following properties:

(a) *$f(x_1, x_2, ..., x_{k+1})$ is non-decreasing in the u_{2i+1} for each v and u_{2i} in $[a, b]$ and non-increasing in the v and u_{2i} for each $u_{2i+1} \in [a, b]$;*

(b) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$m = f(M, m, M, m, M, m, ..., M, m, M),$$

and

$$M = f(m, M, m, M, m, M, ..., m, M, m),$$

then

$$m = M.$$

Then Eq. (1.2) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq. (1.2) converges to $\overline{x}$.

Proof. Set

$$m_0 = a \quad \text{and} \quad M_0 = b$$

and for each $i = 0, 1, 2, ...Set$

$$m_i = f\left(M_{i-1}, m_{i-1}, M_{i-1}, m_{i-1}, M_{i-1}, m_{i-1}, ..., M_{i-1}, m_{i-1}, M_{i-1}\right),$$

and

$$M_i = f\left(m_{i-1}, M_{i-1}, m_{i-1}, M_{i-1}, m_{i-1}, M_{i-1}, ..., m_{i-1}, M_{i-1}, m_{i-1}\right).$$

Now observe that for each $i \geq 0$,

$$a = m_1 \leq m_3 \leq m_5 \leq ... \leq m_i \leq ... \leq M_i \leq ... \leq M_4 \leq M_2 \leq M_0 = b,$$

and

$$m_i \leq x_q \leq M_i \quad \text{for} \quad q \geq (k+1)\,i + 1.$$

Set

$$m = \lim_{i\to\infty} m_i \quad \text{and} \quad M = \lim_{i\to\infty} M_i.$$

Then

$$M \geq \lim_{i\to\infty} \sup x_i \geq \lim_{i\to\infty} \inf x_i \geq m,$$

and by the continuity of f,

$$m = f\left(M, m, M, m, M, m, ..., M, m, M\right),$$

and

$$M = f\left(m, M, m, M, m, M, ..., m, M, m\right).$$

In view of (b),

$$m = M,$$

from which the result follows. □

Theorem 3.2. *If $a\left(b + Q\right) \neq G$, then the equilibrium point $\overline{x}$ of Eq. (1.1) is global attractor.*

Proof. Let $f : (0, \infty)^{k+1} \to (0, \infty)$ be a function defined by

$$f\left(v, u_0, u_1, ..., u_k\right) = a + \frac{\sum_{i=0}^{k} \alpha_{(2i+1)} u_{(2i+1)}}{bv + \sum_{i=0}^{k} \beta_{(2i)} u_{(2i)}},$$

then we can see that the function $f\left(v, u_0, u_1, ..., u_k\right)$ is decreasing in the v, u_{2i} and increasing in u_{2i+1}.

Suppose that (m, M) is a solution of the system

$$m = f\left(M, m, M, m, M, m, ..., M, m, M\right),$$

and

$$M = f\left(m, M, m, M, m, M, ..., m, M, m\right).$$

Then from Eq. (2.1), we see that

$$\begin{aligned} m &= a+\frac{\sum_{i=0}^{k}\alpha_{(2i+1)}m}{bM+\sum_{i=0}^{k}\beta_{(2i)}M}, \quad M=a+\frac{\sum_{i=0}^{k}\alpha_{(2i+1)}M}{bm+\sum_{i=0}^{k}\beta_{(2i)}m}, \\ m &= a+\frac{Gm}{bM+QM}, \qquad M=a+\frac{GM}{bm+Qm}, \end{aligned}$$

$$\begin{aligned} bMm+QMm &= abM+aQM+Gm, \\ bMm+QMm &= abm+aQm+GM, \end{aligned}$$

than

$$ab\,(M-m)+aQ\,(M-m)-G\,(M-m)=0.$$

Thus

$$m=M.$$

It follows by Theorem(3.1) that $\overline{x}$ is a global attractor of Eq. (1.1) and then the proof is complete. □

4. OSCILLATION OF Eq. (1.1)

Theorem 4.1. *If $k-odd,\ell-even$, then Eq. (1.1) has an oscillatory solution*

Proof. firstly if $k>\ell$ we assume that,

$$x_{-k},x_{-k+2},...,x_{-1}>\overline{x}, \quad \text{and} \quad x_{-k+1},x_{-k+3},...,x_{0}<\overline{x}. \tag{4.1}$$

So,

$$x_{1}=a+\frac{\sum_{i=0}^{k}\alpha_{(2i+1)}x_{-(2i+1)}}{bx_{-\ell}+\sum_{i=0}^{k}\beta_{(2i)}x_{-(2i)}},$$

then

$$x_{1}>a+\frac{\sum_{i=0}^{k}\alpha_{(2i+1)}\overline{x}}{b\overline{x}+\sum_{i=0}^{k}\beta_{(2i)}\overline{x}},$$

and

$$x_{1}>a+\frac{\sum_{i=0}^{k}\alpha_{(2i+1)}}{b+\sum_{i=0}^{k}\beta_{(2i)}}=\overline{x}.$$

So, we have

$$x_2 = a + \frac{\sum_{i=0}^{k} \alpha_{(2i+1)} x_{-(2i+1)+1}}{b x_{-\ell+1} + \sum_{i=0}^{k} \beta_{(2i)} x_{-(2i)+1}},$$

so,

$$x_2 < a + \frac{\sum_{i=0}^{k} \alpha_{(2i+1)} \overline{x}}{b \overline{x} + \sum_{i=0}^{k} \beta_{(2i)} \overline{x}},$$

then,

$$x_2 < a + \frac{\sum_{i=0}^{k} \alpha_{(2i+1)}}{b + \sum_{i=0}^{k} \beta_{(2i)}} = \overline{x}.$$

Secondily assume that,

$$x_{-k}, x_{-k+2}, ..., x_{-1} < \overline{x}, \quad \text{and} \quad x_{-k+1}, x_{-k+3}, ..., x_0 > \overline{x}.$$

So,

$$x_1 = a + \frac{\sum_{i=0}^{k} \alpha_{(2i+1)} x_{-(2i+1)}}{b x_{-\ell} + \sum_{i=0}^{k} \beta_{(2i)} x_{-(2i)}},$$

then,

$$x_1 < a + \frac{\sum_{i=0}^{k} \alpha_{(2i+1)} \overline{x}}{b \overline{x} + \sum_{i=0}^{k} \beta_{(2i)} \overline{x}},$$

and

$$x_1 < a + \frac{\sum_{i=0}^{k} \alpha_{(2i+1)}}{b + \sum_{i=0}^{k} \beta_{(2i)}} = \overline{x}.$$

So, we have

$$x_2 = a + \frac{\sum_{i=0}^{k} \alpha_{(2i+1)} x_{-(2i+1)+1}}{b x_{-\ell+1} + \sum_{i=0}^{k} \beta_{(2i)} x_{-(2i)+1}},$$

so,

$$x_2 > a + \frac{\sum\limits_{i=0}^{k} \alpha_{(2i+1)}\overline{x}}{b\overline{x} + \sum\limits_{i=0}^{k} \beta_{(2i)}\overline{x}},$$

then,

$$x_2 > a + \frac{\sum\limits_{i=0}^{k} \alpha_{(2i+1)}}{b + \sum\limits_{i=0}^{k} \beta_{(2i)}} = \overline{x}.$$

One camproceed in prove manwer to show that $x_3 < \overline{x}$ and $x_4 > \overline{x}$ and soon. Hence, the proof is completed. □

Example 4.1. *Consider the difference equation*

$$x_{n+1} = 0.5 + \frac{0.001x_{n-1} + 0.5x_{n-3}}{0.1x_{n-2} + 0.8x_n + 0.001x_{n-2}},$$

where $k = 1, \ell = 2, a = 0.5, b = 0.1, \alpha_1 = 0.001, \alpha_3 = 0.5, \beta_0 = 0.8, \beta_2 = 0.001$. *figure(4.1) shows that the solution of Eq. (1.1) oscillates about* $\overline{x} = 1.0559$. *Where the initial data satisfies condition(4.1) of Theorem(4.1)* $x_{-3} = 0.4, x_{-2} = 2.1, x_{-1} = 1.3, x_0 = 0.2$. *(see Table 4.1)*

n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$
1	0.4000	21	1.2524	41	1.2785	61	1.4099	81	1.6305
2	2.1000	22	0.9259	42	0.8926	62	0.8348	82	0.7671
3	1.3000	23	1.1709	43	1.2902	63	1.4271	83	1.6598
4	0.2000	24	0.9486	44	0.8880	64	0.8283	84	0.7601
5	1.0388	25	1.2358	45	1.2998	65	1.4451	85	1.6908
6	1.5913	26	0.9192	46	0.8823	66	0.8218	86	0.7531
7	1.0037	27	1.2056	47	1.3122	67	1.4642	87	1.7235
8	0.6117	28	0.9363	48	0.8768	68	0.8152	88	0.7460
9	1.2999	29	1.2351	49	1.3234	69	1.4843	89	1.7581
10	1.1977	30	0.9150	50	0.8712	70	0.8085	90	0.7390
11	0.9933	31	1.2305	51	1.3366	71	1.5055	91	1.7947
12	0.8315	32	0.9230	52	0.8653	72	0.8018	92	0.7319
13	1.3277	33	1.2446	53	1.3494	73	1.5279	93	1.8333
14	1.0160	34	0.9094	54	0.8595	74	0.7950	94	0.7249
15	1.0551	35	1.2508	55	1.3636	75	1.5515	95	1.8742
16	0.9260	36	0.9106	56	0.8534	76	0.7881	96	0.7180
17	1.2881	37	1.2598	57	1.3781	77	1.5764	97	1.9173
18	0.9477	38	0.9018	58	0.8474	78	0.7811	98	0.7111
19	1.1207	39	1.2701	59	1.3937	79	1.6027	99	1.9630
20	0.9519	40	0.8991	60	0.8411	80	0.7742	100	0.7042

Table *4.1*

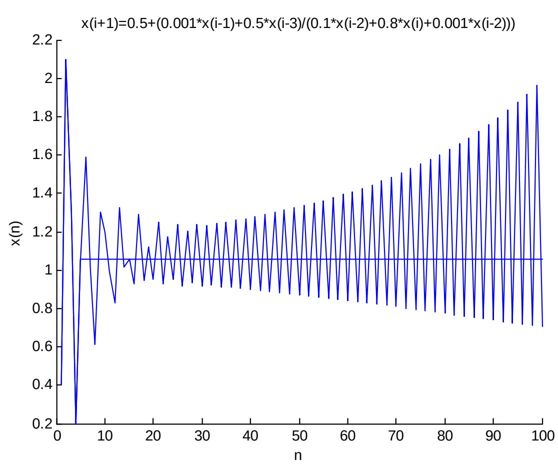

Figure *4.1*

5. Periodic solutions

In this section we investigate the periodic character of the positive solutions of Eq. (1.1).

Theorem 5.1. *Eq. (1.1) has positive prime priod two solution only If*

$$\ell - odd \quad and \quad (\alpha-\beta)(a\beta - a\alpha + b) > 4a\alpha\beta. \tag{5.1}$$

Proof. Assume that there exists a prime period-two solution

$$..., p, q, p, q, ...$$

of (1.1). Let $x_n = q, x_{n+1} = p$. Since $\ell - odd$, we have $x_{n-\ell} = p$. Thus, from Eq. (1.1), we get

$$p = a + \frac{\alpha_1 p + \alpha_3 p + ... + \alpha_{(2i+1)} p}{bp + \beta_0 q + \beta_2 q + ... + \beta_{(2i)} q},$$

and

$$q = a + \frac{\alpha_1 q + \alpha_3 q + ... + \alpha_{(2i+1)} q}{bq + \beta_0 p + \beta_2 p + ... + \beta_{(2i)} p}.$$

Let

$$\alpha_1 + \alpha_3 + ... + \alpha_{(2i+1)} = \gamma,$$

and

$$\beta_0 + \beta_2 + ... + \beta_{(2i)} = \delta.$$

Than

$$p = a + \frac{\gamma p}{bp + \delta q},$$

and

$$q = a + \frac{\gamma q}{bq + \delta p}.$$

Than

$$bp^2 + \delta pq = abp + a\delta q + \gamma p, \tag{5.2}$$

and

$$bq^2 + \delta pq = abq + a\delta p + \gamma q. \tag{5.3}$$

Subtracting (5.2) from (5.3) gives

$$b\left(p^2 - q^2\right) = (ab - a\delta + \gamma)(p - q).$$

Since $p \neq q$, we have

$$p + q = \frac{ab - a\delta + \gamma}{b}. \tag{5.4}$$

Also, since p and q are positive, $(ab - a\delta + \gamma)$ shuold be positive. Again, adding (5.2) and (5.3) yields

$$b(p^2 + q^2) + 2\delta pq = (ab + a\delta + \gamma)(p + q). \tag{5.5}$$

It follows by (5.4), (5.5) and the relation

$$p^2 + q^2 = (p+q)^2 - 2pq, \quad \forall p, q \in \mathbb{R},$$

that

$$pq = \frac{a\delta(ab - a\delta + \gamma)}{b(\delta - b)}. \tag{5.6}$$

Assume that p and q are two distinct real roots of the quadratic equation

$$t^2 - \frac{ab - a\delta + \gamma}{b} t + \frac{a\delta(ab - a\delta + \gamma)}{b(\delta - b)} = 0,$$

and so

$$\left(\frac{ab - a\delta + \gamma}{b}\right)^2 - \frac{4a\delta(ab - a\delta + \gamma)}{b(\delta - b)} > 0,$$

which is equivalent to

$$(\delta - b)(ab - a\delta + \gamma) > 4ab\delta.$$

Thus, the proof is completed. □

Example 5.1. *Consider the difference equation*

$$x_{n+1} = 0.125 + \frac{4x_{n-1}}{x_{n-1} + 2x_n},$$

where $k = 0, \ell-odd, a = 0.125, b = 1, \alpha_1 = 4, \beta_0 = 2$. Figure(5.1), shows that Eq. (1.1) which is periodic with period two. Where the initial data satisfies condition(5.1) of Theorem(5.1)

$x_{-1} = 0.1, x_0 = 0.3.$*(see Table 5.1)*

n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$
1	0.1000	17	0.2928	33	0.2690	49	0.2686	65	0.2686
2	0.3000	18	3.5545	34	3.6055	50	3.6064	66	3.6064
3	0.3000	19	0.2832	35	0.2689	51	0.2686	67	0.2686
4	1.4583	20	3.5751	36	3.6059	52	3.6064	68	3.6064
5	0.4981	21	0.2774	37	0.2688	53	0.2686	69	0.2686
6	2.5016	22	3.5876	38	3.6061	54	3.6064	70	3.6064
7	0.4871	23	0.2739	39	0.2687	55	0.2686	71	0.2686
8	3.0038	24	3.5952	40	3.6062	56	3.6064	72	3.6064
9	0.4250	25	0.2718	41	0.2687	57	0.2686	73	0.2686
10	3.2427	26	3.5997	42	3.6063	58	3.6064	74	3.6064
11	0.3710	27	0.2705	43	0.2687	59	0.2686	75	0.2686
12	3.3801	28	3.6024	44	3.6063	60	3.6064	76	3.6064
13	0.3331	29	0.2697	45	0.2686	61	0.2686	77	0.2686
14	3.4664	30	3.6040	46	3.6063	62	3.6064	78	3.6064
15	0.3084	31	0.2693	47	0.2686	63	0.2686	79	0.2686
16	3.5208	32	3.6049	48	3.6064	64	3.6064	80	3.6064

Table *5.1*

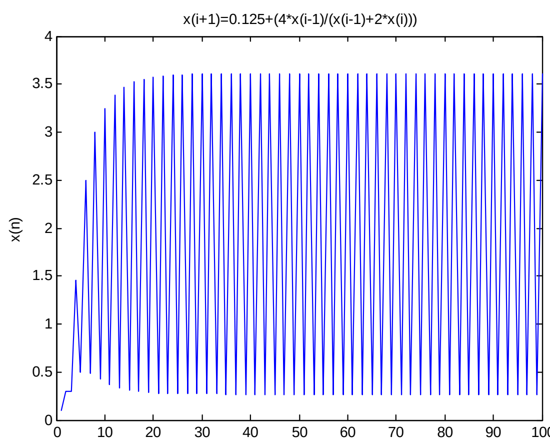

Figure *5.1*

Remark 5.1. *Note that the special cases of Eq. (1.1) have been studied in* [1] *when* $k = 0, b = 0, \alpha_1 = 1, \beta_0 = 1, \alpha_{2i+1} = 0, \beta_{2i} = 0, i \geq 1.$

References

[1] A. M. Amleh, E. A. Grove, D. A. Georgiou and G. Ladas, On the recursive sequence $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}$, J. Math. Anal. Appl. 233 (1999), 790-798.

[2] Camouzis, R. DeVault and G. Ladas, On the recursive sequence $x_{n+1} = -1 + \frac{x_{n-1}}{x_n}$, J. Differ. Equations Appl., 7 (2001), 477-482.

[3] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the periodic nature of some max-type difference equations, Int. J. Math. Sci., 14 (2005), 2227-2239.

[4] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed. On the difference equation $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}$. Adv. Difference Equ., pages Art. ID 82579, 10 (2006).

[5] E.M. Elabbasy, H. El-Metwally and E.M. Elsayed. Qualitative behavior of higher order difference equation. Soochow J. Math., 33(4) (2007), 861–873.

[6] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the difference equations $x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$, J. Conc. Appl. Math. 5(2) (2007), 101-113.

[7] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed. On the differencee quation $x_{n+1} = \frac{(a_0x_n + a_1x_{n-1} + \ldots + a_kx_{n-k})}{(b_0x_n + b_1x_{n-1} + \ldots + b_kx_{n-k})}$, Mathematica Bohemica, 133 (2008), No.2, 133-147.

[8] S. N. Elaydi, An Introduction to difference equations, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, (1996).

[9] E.A. Grove, G. Ladas, Periodicities in nonlinear difference equations, vol. 4, Chapman and Hall / CRC, (2005).

[10] G. Karakostas and S. Stevic, On the recursive sequence $x_{n+1} = A + \frac{f(x_n, \ldots, x_{n-k+1})}{x_{n-1}}$, Comm. Appl. Nonlinear Analysis, 11 (2004), 87-100.

[11] V. L. Kocic and G. Ladas, Global behavior of nonlinear difference equations of higher order with application, Kluwer Academic Publishers, Dordrecht, (1993).

[12] M.R.S. Kulenovic, G. Ladas, Dynamics of second order rational difference equations with open problems and conjectures, Chapman & Hall/CRC, Florida, (2001).

E. M. Elabbasy, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt

M. Barsoum, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt

A. A. El-biaty, Department of Mathematics, The Faculty of Education, University of Tikrit, Iraq