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ON THE DYNAMICS OF A HIGHER-ORDER RATIONAL DIFFERENCE EQUATION

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ABSTRACT. The main objective of this paper is to study the qualitative behavior for a class of nonlinear rational difference equation. We study the local stability, periodicity, Oscillation, boundedness, and the global stability for the positive solutions of equation. Examples illustrate the importance of the results.

1. INTRODUCTION

In this paper, we aim to achieve qualitative study was of some of the behavior and solutions in the non-linear of differential equations

$$x_{n+1} = \alpha + \frac{a x_{n-k}^{\gamma}}{b x_{n-\ell}^{\gamma} + c x_n^{\gamma}},\ n = 0, 1, 2, ..., \tag{1.1}$$

where the coefficients a, b and $c \in (0, \infty)$ while k and ℓ are positive integers. The initial conditions $x_{-j}, x_{-j+1}, ..., x_0$ are arbitrary positive real numbers such that $j = -\max\{k, \ell\}$. Consider $\alpha \in [0, \infty), \gamma \in [1, \infty)$ Qualitative analysis of difference equation is not only interesting in its own right, but it can provide insights into their continuous counterpararts, namely, differential equations.

There is a set of nonlinear difference equations, known as the rational difference equations, all of which consists of the ratio of two polynomials in the sequence terms in the same from .there has been many work about the global asympototic of solutions of rational difference equations [3], [7], [10], [11],[12] and [13].

There has been much investigation and interest in difference equations by several authors such Amleh [1] has studied the global stability , boundedness and the periodic character of solutions of the equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}.$$

Camouzis, [2] investigated the periodic character, and global stability of all positive solution of the recursive sequece,

$$x_{n+1} = -1 + \frac{x_{n-1}}{x_n}$$

Hamza and Morsy in [4] investigated the global behavior of the

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$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^{\gamma}}$$

Owaidy at al [5] investigated local stability of positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}^{\gamma}}{x_n^{\gamma}}.$$

Saleh, [9] investigated the periodic character, invariant intervals, oscillation and global stability of all positive solution of the recursive sequece,

$$x_{n+1} = A + \frac{x_{n-k}}{x_n}.$$

In the following we present some basic definitions and known results which will be useful in our study.

Definition 1. *Consider a difference equation in the form*

$$x_{n+1} = F\left(x_n, x_{n-k}, x_{n-\ell}\right) \tag{1.2}$$

where F is a continuous function, while $k,\ell \in (0,\infty)$. Any equilibrium point $\overline{x}$ of this equation is a point that satisfies the condition $\overline{x} = F\left(\overline{x}, \overline{x}, \overline{x}\right)$. That is, the constant sequence $\{x_n\}$ with $x_n = \overline{x}$ for all $n \geqslant -k \geqslant -\ell$ is a solution of that equation.

Definition 2. *Let $\overline{x} \in (0,\infty)$ be an equilibrium point of Eq.(1.2). As well as we have the*

(i) An equilibrium point $\overline{x}$ of Eq. is said to be locally stable if for every $\varepsilon > 0$ there exists $\sigma > 0$ such that, if $x_{-j}, ..., x_{-1}, x_0 \in (0,\infty)$ with $|x_{-j} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \sigma$, then $|x_n - \overline{x}| < \varepsilon$ for all $n \geqslant -j$.

(ii) An equilibrium point $\overline{x}$ of Eq.(1.2) is said to be locally asymptotically stable if it is locally stable and there exists $y > 0$ such that, $x_{-j}, ..., x_{-1}, x_0 \in (0,\infty)$ with $|x_{-j} - \overline{x}| + ... + |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < y$, then $\lim_{n\to\infty} x_n = \overline{x}$.

(iii) An equilibrium point $\overline{x}$ of Eq.(1.2) is said to be a global attractor if for every $x_{-j}, ..., x_{-1}, x_0 \in (0,\infty)$ we have $\lim_{n\to\infty} x_n = \overline{x}$.

(iv) The Eq.(1.2) is said to be globally asymptotically stable if it is locally stable and a global attractor of the equilibrium point $\overline{x}$.

(v) Equation (1.2) is said to be unstable, if it is locally stable equilibrium point at $\overline{x}$.

Definition 3. *The sequence $\{x_n\}$ is said to be periodic with period p, if $x_{n+p} = x_n$ for $n = 0, 1, ...,$*

Definition 4. *Eq.(1.2) is said to be permanent and bounded if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-j}, ..., x_{-1}, x_0 \in (0,\infty)$ there exists a positive integer N which depends on these initial conditions such that $m \leq x_n \leq M$ for all $n \geq N$.*

Definition 5. *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be nonoscillatory about the point $\overline{x}$ if there is exists $N \geq -k$ such that either $x_n > \overline{x}$ for all $n \geq N$ or $x_n < \overline{x}$ for all $n \geq N$. Otherwise $\{x_n\}_{n=-k}^{\infty}$ is called oscillatory about $\overline{x}$.*

Definition 6. *The linearized equation of Eq.(1.2) about the equilibrium point $\overline{x}$ is defined by the equation.*

$$y_{n+1} = p_0 y_n + p_1 y_{n-k} + p_2 y_{n-\ell} \tag{1.3}$$

$$p_0 = \frac{\partial f}{\partial x_n}\left(\overline{x}, \overline{x}, \overline{x}\right) \ , \ p_1 = \frac{\partial f}{\partial x_{n-k}}\left(\overline{x}, \overline{x}, \overline{x}\right) \quad , \quad p_2 = \frac{\partial f}{\partial x_{n-\ell}}\left(\overline{x}, \overline{x}, \overline{x}\right)$$

The characteristic equation associated with Eq. (1.3) is

$$p\left(\lambda\right) = \lambda^{\ell+1} - p_o \lambda^{\ell} - p_1 \lambda^{\ell-k} - p_2 = 0 \tag{1.4}$$

Theorem 1.1. [6]*Assume that F is a C^1-function and let $\overline{x}$ be an equilibrium point of Eq.(1.2). We can say that the following statements are true*

(i) The equilibrium point $\overline{x}$ it's called locally asymptotically stable, if all roots of Eq.(1.4) lie in the open unit disk $|\lambda| < 1$.
(ii) The equilibrium point $\overline{x}$ it's called unstable, if at least one root of Eq.(1.4) has absolute value more than one.
(iii) The equilibrium point $\overline{x}$ it's called source, if all roots of Eq.(1.4) have absolute value more than one.

Theorem 1.2. [8]*Assume that p_0, p_1 and $p_2 \in R$. Then*

$$|p_0| + |p_1| + |p_2| < 1 \tag{1.5}$$

is a sufficient condition for the locally stability of Eq.(1.2).

2. Local stable of the equilibrium point

The equilibrium point $\overline{x}$ of Eq.(1.1) is the positive solution of the equation,

$$\overline{x} = \alpha + \frac{a\overline{x}^{\gamma}}{b\overline{x}^{\gamma} + c\overline{x}^{\gamma}}$$

which gives

$$\overline{x} = \alpha + \frac{a}{b+c} \tag{2.1}$$

Now let $f : (0, \infty)^3 \to (0, \infty)$ be a function defined by

$$f\left(u, v, w\right) = \alpha + \frac{av^{\gamma}}{bw^{\gamma} + cu^{\gamma}}.$$

Then, we have

$$\frac{\partial f}{\partial u} = \frac{-\gamma a c v^{\gamma} u^{\gamma-1}}{(bw^{\gamma}+cu^{\gamma})^{2}}, \tag{2.2}$$

$$\frac{\partial f}{\partial v} = \frac{\gamma a v^{\gamma-1}}{(bw^{\gamma}+cu^{\gamma})^{2}}, \tag{2.3}$$

and

$$\frac{\partial f}{\partial w} = \frac{-abpv^{\gamma} w^{\gamma-1}}{(bw^{\gamma}+cu^{\gamma})^{2}}. \tag{2.4}$$

Theorem 2.1. *If*

$$2a\gamma < \alpha b + \alpha c + a$$

then the equilibrium point $\overline{x} = \alpha + \frac{a}{b+c}$ *of eq (1.1) is local stable.*

Proof. From (2.2)-(2.4), we get

$$\frac{\partial f}{\partial u}\left(\overline{x},\overline{x},\overline{x}\right) = \frac{-ac\gamma}{\left(\alpha b+\alpha c+a\right)\left(b+c\right)} = p_1$$

$$\frac{\partial f}{\partial v}\left(\overline{x},\overline{x},\overline{x}\right) = \frac{a\gamma}{\alpha b+\alpha c+a} = P_2$$

and

$$\frac{\partial f}{\partial w}\left(\overline{x},\overline{x},\overline{x}\right) = \frac{-ab\gamma}{\left(\alpha b+\alpha c+a\right)\left(b+c\right)} = P_3.$$

Thus, the linearized equation associated with Eq. (1.2) about $\overline{x}$, is

$$y_{n+1} = p_0 y_n + p_1 y_{n-k} + p_2 y_{n-l}.$$

It follows by Theorem 1.2 that Eq.(1.1) is localy stable if

$$\left|\frac{-ac\gamma}{\left(\alpha b+\alpha c+a\right)\left(b+c\right)}\right| + \left|\frac{a\gamma}{\left(\alpha b+\alpha c+a\right)}\right| + \left|-\frac{ab\gamma}{\left(\alpha b+\alpha c+a\right)\left(b+c\right)}\right| < 1,$$

so,

$$ac\gamma + ab\gamma + ac\gamma + ab\gamma < \left(\alpha b+\alpha c+a\right)\left(b+c\right),$$

which is true if

$$2a\gamma < \alpha b + \alpha c + a.$$

The proof is completed. □

3. Periodic solutions of Eq. (1.1)

In this part of the research we are studying the possibility of the existence of periodic solutions to the eq. (1.1).

Theorem 3.1. *In the all following cases, Equation (1.1) has no positive prime period-two solutions:*

(1) If k and ℓ are both even positive number.
(2) If k is odd and ℓ is even positive number.
(3)If k is even and ℓ is odd positive number.

Proof. Case(1) Suppose that there exists a prime period-two solution

$$..., p, q, p, q, p, q, , ...$$

If k, ℓ even then $x_n = x_{n-k} = x_{n-\ell} = q$

$$p = \alpha + \frac{a}{b+c}, \tag{3.1}$$

also,

$$q = \alpha + \frac{a}{b+c}. \tag{3.2}$$

By (3.1) and (3.2), we have

$$p - q = 0 \qquad \Longrightarrow \qquad p = q$$

Similarly, we can prove other cases which is omitted here for convenience. Hence, the proof is completed. □

The following theorem states the sufficient conditions that the Eq (1.1) has periodic solutions of prime period two.

Theorem 3.2. *Assume that k and ℓ are both odd positive integers and $\gamma = 1$. If*

$$a(c-b) > \alpha^2 b + \alpha^2 c + 2\alpha bc, \tag{3.3}$$

then Eq. (1.1) has prime period two solution.

Proof. Suppose that there exists a prime period-two solution

$$..., p, q, p, q, p, q, , ...$$

of (1.1). We will prove that condition (3.3) holds.

We see from (1.1) that if k and ℓ odd, then $x_{n-\ell} = x_{n-k}$

$$p = \alpha + \frac{ap^\gamma}{bp^\gamma + cq^\gamma},$$

and

$$q = \alpha + \frac{aq^\gamma}{bq^\gamma + cp^\gamma},$$

we have

$$bp^2 + cpq = \alpha bp + \alpha cq + ap, \tag{3.4}$$

and

$$bq^2 + cqp = \alpha bq + \alpha cp + aq. \tag{3.5}$$

By subtracting (3.4) and (3.5), we have

$$b\left(p^2 - q^2\right) = \alpha b\,(p-q) + \alpha c\,(q-p) + a\,(p-q)\,,$$

then,

$$(p+q) = \frac{\alpha b - \alpha c + a}{b}. \tag{3.6}$$

By Combining (3.4) and (3.5), we have

$$b\left(p^2 + q^2\right) + 2cpq = (\alpha b + \alpha c + a)(p+q), \tag{3.7}$$

then,

$$p^2 + q^2 = (p+q)^2 - 2pq. \tag{3.8}$$

Form (3.6), (3.7) and (3.8), we get

$$\begin{aligned} b\left[\frac{\alpha b - \alpha c + a}{b}\right]^2 + 2pq(c-b) &= (\alpha b + \alpha c + a)\left[\frac{\alpha b - \alpha c + a}{b}\right] \\ pq &= \frac{\alpha c\,(\alpha b - \alpha c + a)}{b(c-b)} \end{aligned}$$

we have,

$$u^2 + (p+q)\,u + pq = 0 \quad \text{and} \quad (p+q)^2 - 4pq > 0,$$

then,

$$\left(\frac{\alpha b - \alpha c + a}{b}\right)^2 - \frac{4\alpha c\,(\alpha b - \alpha c + a)}{b(c-b)} > 0,$$

which is true if

$$a(c-b) > \alpha^2 b + \alpha^2 c + 2\alpha bc$$

Hence, the proof is completed. $\square$

4. Global stability

Theorem 4.1. *Then the equilibrium point $\overline{x}$ of Eq. 1.1 is said to be global stability.*

Proof. We have the next function

$$f(u, v, w) = \alpha + \frac{av^{\gamma}}{bw^{\gamma} + cu^{\gamma}},$$

f non-decreasing for v and non-increasing for u, w

let $m = f(M, m, M)$ and $M = f(m, M, m)$

$$f(u, v, w) = \alpha + \frac{av^{\gamma}}{bw^{\gamma} + cu^{\gamma}},$$

$$m = \alpha + \frac{am^{\gamma}}{bM^{\gamma} + cM^{\gamma}}, \tag{4.1}$$

$$M = \alpha + \frac{aM^{\gamma}}{bm^{\gamma} + cm^{\gamma}}, \tag{4.2}$$

from (4.1)

$$M^{\gamma}(b + c)[m - \alpha] = am^{\gamma}, \tag{4.3}$$

from (4.2)

$$m^{\gamma}(b + c)[M - \alpha] = aM^{\gamma}. \tag{4.4}$$

Subtracting Equation (4.3) of (4.4) produces

$$(b + c)[M^{\gamma}(m - \alpha) - m^{\gamma}(M - \alpha)] - a(m^{\gamma} - M^{\gamma}) = 0,$$

then

$$M = m.$$

Hence, the proof is completed. □

5. Boundedness of the solutions

Theorem 5.1. *Let $\{x_n\}_{n=-\max\{k,\ell\}}^{\infty}$ be asolution of Eq (1.1), then the following statements are true :-*

(1) Assume that $a < b$ and let for some $N \geq 0, x_{N-j+1}, ..., x_{N-1}, x_N \in \left[\frac{a}{b}, 1\right]$ are valid, then we have

$$\frac{(b + c)\, a^{\gamma+1}}{b^{\gamma}} \leq x_n \leq \frac{ab^{\gamma}}{(b + c)\, a^{\gamma}}$$

(2) Assume that $a > b$ and for some $N \geq 0, x_{N-j+1}, ..., x_N \in \left[1, \frac{a}{b}\right]$ are valid , Then we have

$$\frac{ab^{\gamma}}{(b + c)\, a^{\gamma}} \leq x_n \leq \frac{(b + c)\, a^{\gamma+1}}{b^{\gamma}}$$

Proof. (1) If $a < b$ then $x_{N-\ell+1}, ..., x_{N-1}, x_N \in \left[\frac{a}{b}, 1\right]$

$$x_{n+1} = \alpha + \frac{ax_{n-k}^{\gamma}}{bx_{n-\ell}^{\gamma} + cx_n^{\gamma}},$$

then,

$$\begin{aligned} &\leq \quad \alpha + \frac{a}{(b+c)\left(\frac{a}{b}\right)^{\gamma}}, \\ &\leq \quad \frac{ab^{\gamma}}{a^{\gamma}(b+c)}, \end{aligned}$$

and

$$x_{n+1} = \alpha + \frac{ax_{n-k}^{\gamma}}{bx_{n-\ell}^{\gamma} + cx_n^{\gamma}},$$

then,

$$\begin{aligned} &\geq \quad \frac{a\left(\frac{a}{b}\right)^{\gamma}}{b+c}, \\ &\geq \quad \frac{(b+c)\,a^{\gamma+1}}{b^{\gamma}}. \end{aligned}$$

Then

$$\frac{(b+c)\,a^{\gamma+1}}{b^{\gamma}} \leq x_n \leq \frac{ab^{\gamma}}{a^{\gamma}(b+c)}$$

Similarly, we can prove other cases which is omitted here for convenience. Hence, the proof is completed. □

6. Oscillatory solution

Theorem 6.1. *Eq.(1.1) has an oscilatory solution If k is odd and ℓ is even and let $k < \ell$,*

Proof. First assume that,

$$x_{-k}, x_{-k+2}, x_{-k+4}, ..., x_{-1} > \overline{x} \quad \textit{and} \quad x_{-k+1}, x_{-k+3}, ..., x_0 < \overline{x}$$

so

$$x_1 = \alpha + \frac{ax_{-k}^{\gamma}}{bx_{-\ell}^{\gamma} + cx_0^{\gamma}},$$

then

$$x_1 > \alpha + \frac{a\overline{x}^{\gamma}}{b\overline{x}^{\gamma} + c\overline{x}^{\gamma}},$$

and

$$x_1 > \alpha + \frac{a}{b+c} = \overline{x}.$$

So, we have

$$x_2 = \alpha + \frac{ax_{-k+1}^{\gamma}}{bx_{-\ell+1}^{\gamma} + cx_1^{\gamma}},$$

so,

$$x_2 < \alpha + \frac{a\overline{x}^{\gamma}}{b\overline{x}^{\gamma} + c\overline{x}^{\gamma}},$$

then,

$$x_2 < \alpha + \frac{a}{b+c} = \overline{x}.$$

Secandiy assume that,

$$x_{-k}, x_{-k+2}, x_{-k+4}, ..., x_{-1} < \overline{x} \quad \text{and} \quad x_{-k+1}, x_{-k+3}, ..., x_0 > \overline{x},$$

$$x_1 = \alpha + \frac{ax_{-k}^{\gamma}}{bx_{-\ell}^{\gamma} + cx_0^{\gamma}},$$

then,

$$x_1 < \alpha + \frac{a\overline{x}^{\gamma}}{b\overline{x}^{\gamma} + c\overline{x}^{\gamma}},$$

and

$$x_1 < \alpha + \frac{a}{b+c} = \overline{x}.$$

So, we have

$$x_2 = \alpha + \frac{ax_{-k+1}^{\gamma}}{bx_{-\ell+1}^{\gamma} + cx_1^{\gamma}},$$

so,

$$x_2 > \alpha + \frac{a\overline{x}^{\gamma}}{b\overline{x}^{\gamma} + c\overline{x}^{\gamma}},$$

then,

$$x_2 > \alpha + \frac{a}{b+c} = \overline{x}.$$

One camproceed in prove manwer to show that $x_3 < \overline{x}$ and $x_4 > \overline{x}$ and soon. Hence, the proof is completed. □

7. Numerical examples

In order to clarify the results obtained, we are offering some numerical examples, as follows

Example 7.1. *Fig. 1, shows that Eq. (1.1) has Local stable solutions if* $a = b = \alpha = \ell = 2, c = k = \gamma = 1, x_0 = 12, x_{-1} = 5, x_{-2} = 2, \overline{x} = 2\frac{2}{3}$

Fig.1.

Example 7.2. *Fig. 2, shows that Eq. (1.1) has prime period two solutions if* $\ell = k = 1$, $\alpha = (1/16)$, $a = c = 2$, $b = 1$, $x_{-2} = 3.3$, $x_{-1} = 0.5$.(*see Table 7.2*)

n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$	n	$x(n)$
1	*3.3000*	*17*	*1.7741*	*33*	*1.8027*	*49*	*1.8032*	*65*	*1.8032*
2	*0.5000*	*18*	*0.1425*	*34*	*0.1344*	*50*	*0.1343*	*66*	*0.1343*
3	*1.5974*	*19*	*1.7857*	*35*	*1.8029*	*51*	*1.8032*	*67*	*1.8032*
4	*0.3332*	*20*	*0.1392*	*36*	*0.1344*	*52*	*0.1343*	*68*	*0.1343*
5	*1.4738*	*21*	*1.7927*	*37*	*1.8030*	*53*	*1.8032*	*69*	*1.8032*
6	*0.2656*	*22*	*0.1373*	*38*	*0.1344*	*54*	*0.1343*	*70*	*0.1343*
7	*1.5326*	*23*	*1.7969*	*39*	*1.8031*	*55*	*1.8032*	*71*	*1.8032*
8	*0.2220*	*24*	*0.1361*	*40*	*0.1343*	*56*	*0.1343*	*72*	*0.1343*
9	*1.6133*	*25*	*1.7994*	*41*	*1.8031*	*57*	*1.8032*	*73*	*1.8032*
10	*0.1912*	*26*	*0.1354*	*42*	*0.1343*	*58*	*0.1343*	*74*	*0.1343*
11	*1.6792*	*27*	*1.8009*	*43*	*1.8032*	*59*	*1.8032*	*75*	*1.8032*
12	*0.1702*	*28*	*0.1349*	*44*	*0.1343*	*60*	*0.1343*	*76*	*0.1343*
13	*1.7253*	*29*	*1.8018*	*45*	*1.8032*	*61*	*1.8032*	*77*	*1.8032*
14	*0.1565*	*30*	*0.1347*	*46*	*0.1343*	*62*	*0.1343*	*78*	*0.1343*
15	*1.7553*	*31*	*1.8024*	*47*	*1.8032*	*63*	*1.8032*	*79*	*1.8032*
16	*0.1479*	*32*	*0.1345*	*48*	*0.1343*	*64*	*0.1343*	*80*	*0.1343*

Tabe *7.2*

Fig.2.

Example 7.3. *Fig. 3, shows that Eq.(1.1) has 0scilatory solution if* $a = b = \alpha = \ell = \gamma = 2$, $c = k = 1$, $x_0 = 2$, $x_{-1} = 2$, $x_{-2} = 2$, $\overline{x} = 2.6666666667$.(*see Table 7.3*)

1	2.0000	17	2.4925	33	2.4858	49	2.4783	65	2.4698
2	2.0000	18	2.9032	34	2.9157	50	2.9302	66	2.9473
3	2.0000	19	2.4916	35	2.4849	51	2.4773	67	2.4686
4	2.6667	20	2.9047	36	2.9174	52	2.9322	68	2.9497
5	2.5294	21	2.4909	37	2.4840	53	2.4763	69	2.4674
6	2.9878	22	2.9062	38	2.9191	54	2.9342	70	2.9521
7	2.5528	23	2.4901	39	2.4831	55	2.4753	71	2.4662
8	2.9245	24	2.9077	40	2.9209	56	2.9363	72	2.9546
9	2.4936	25	2.4892	41	2.4822	57	2.4742	73	2.4650
10	2.8885	26	2.9093	42	2.9227	58	2.9384	74	2.9571
11	2.4887	27	2.4884	43	2.4813	59	2.4731	75	2.4638
12	2.8958	28	2.9108	44	2.9245	60	2.9405	76	2.9598
13	2.4940	29	2.4876	45	2.4803	61	2.4720	77	2.4625
14	2.9013	30	2.9124	46	2.9263	62	2.9427	78	2.9624
15	2.4939	31	2.4867	47	2.4793	63	2.4709	79	2.4612
16	2.9022	32	2.9141	48	2.9283	64	2.9450	80	2.9652

Table *7.3*

Fig.3.

Remark 7.1. *Special cases of Equation 1.1 discussed in the* [1] *when* $a = c = \gamma = k = 1$, $b = 0$ *and in* [2] *when* $a = b = \gamma = 1$, $c = 0$ *and in* [5] *when* $a = b = k = 1$, $c = \ell = 0$ *and in* [9] *when* $a = b = \gamma = 1$, $c = \ell = 0$.

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