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TRANSIENT SOLUTION OF AN $M^{[X]}/G/1$ QUEUEING MODEL WITH BERNOULLI FEED BACK, RANDOM BREAKDOWNS, BERNOULLI SCHEDULE SERVER VACATION AND RESTRICTED ADMISSIBILITY

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Abstract. This paper analyze an $M^{[X]}/G/1$ queue with Bernoulli feed back random server breakdowns, Bernoulli schedule server vacation and restricted admissibility policy. Customers arrive in batches following compound Poisson process and are served one by one according to first come first served basis. The service time follows general (arbitrary) distribution. The customer feedback to the tail of original queue for repeating the service until the service be successful with probability p or the customer departs the system if service be successful with probability 1-p. After completion of a service the server may go for a vacation with probability θ or continue staying in the system, to serve a next customer, if any with probability $1-\theta$. Unlike the usual batch arrival queueing model, there is a restriction over the admissibility of batch arrivals in which not all the arriving batches are allowed to join the system at all times. This restriction admissibility policy is different for different states of the system. The system may breakdown at random in accordance with Poisson process, while the repair time follows exponential distribution. We obtain the time dependent probability generating function in terms of their Laplace transforms and the corresponding steady state results explicitly. Also we derive the system performance measures like average number of customers in the queue and the average waiting time in closed form.

1.Introduction

The research on queueing theory has been extensively developed due to a lot of significance in the problems relating with decision making process and it has made a tremendous impact in industry and logistics sector apart from its significant applications in many other areas like air traffic, bio-sciences, population studies, health sectors, manufacturing and production sections etc. Numerous authors have put their efforts in studying queueing models by importing many aspects on them. Also the analysis of queueing models with Bernoulli feedback had a prominent role in queueing theory. Takacs (1963) introduced the concept of feedback in queueing model. Krishnakumar et al.(2001) studied M/G/1/1 model with feedback by considering both regular and optional service. Choudhury and Paul (2005) inspected the M/G/1 system with two phases of heterogeneous service and Bernoulli feedback. Choi et al.(2003) analyzed two phase queueing system with vacation

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and Bernoulli feedback. Badamchi Zadeh and Shahkar (2008) analyzed two phases queueing system with Bernoulli feedback and Bernoulli Schedule server vacation.

Most recently, research studies on queues with server breakdown have been attracted, as an important area of queueing theory and have been studied extensively and successfully due to their various applications in production, communication systems. Mostly in the queueing literature, the server may be assumed as an reliable one, so that it never fails. But in real scenario, mostly servers are unreliable, as we often encounter the cases where service stations do not last for ever and which can encounter break downs. Such phenomena always occur in the area of computer communication networks and flexible manufacturing system etc, since the performance of system may be severely affected by server breakdowns. Regarding restoration of server as the server has to resume its service, it needs a repair process immediately. such systems with break down and repair are well worth to investigate from the queueing theory point of view as well as reliability point of view. Recently, there are several contributions in the research study of non-Morkovian single server queueing system, in which the server may experience with break downs and repairs. Thus queueing systems with repairable server has been studied by many authors including Avi-Itzhak and Naor (1963), Graver(1960), Takine and Sengupta(1997), Tang(1997) etc.

Another aspect in queueing theory is the study of server vacations which play a significant role in queueing models. During the last three or four decades, queueing theorists are interested in studying queueing models with vacations elaborately, because of their applications and theoretical structures in real life situations such as manufacturing and production systems, computer and communication systems, service and distribution systems, etc. Queue systems with batch arrival and vacation time was analyzed by Baba (1986), he derived the queue size distribution, waiting time distribution and busy period distribution. The more detailed survey about vacation queues can be found in the survey of Doshi (1986). The most research works which have been done in recent past by some researchers on vacation models including Levy and Yechiali (1975), Keilson and servi (1987), Choudhary (2002), and Madan(2009) etc.

Control of queues is also one of the most remarkable and interesting areas of queueing theory. In this context, there are many researchers including Rue and Rosenberg (1981), Neuts (1984), Stidham (1985), Lee and Srinivasan (1989) and Huang and Mc-Donald (1998) dealt with control policies on arrivals in queues and queueing networks. One of the queue control is, control on admission of arriving customers which is an interesting control problem for queues. In some of queueing models, queue lengths are controlled by the rejection of some of the incoming arriving customers. Such a control policy is called restriction admissibility policy and which has been studied by many authors. Madan and Abu-Dayyeh (2002, 2003) proposed restricted admissibility policy on $M^X/G/1$ queueing system with Bernoulli server vacation. Madan and Choudhury (2004), Alnowibet and Tadj(2007) and Choudhury (2008) studied many queueing models in this perspective. Badamchi Zadeh (2012) studied a batch arrival queueing system with two phases of heterogenous service with optional second service and restricted admissibility with single vacation policy.

Transient state indices are very essential to track down the functioning of the system at any instant of time. Thus in this paper, we propose a study on transient behavior of a queueing model with feedback subjected to random break downs and Bernoulli vacation model under restriction policy for a single server batch arrival in put. If a customer feels that, his service is being unsatisfactory he require the service repeatedly with probability p or the customer departs the system if service be successful with probability 1-p. once the system gets break down, it enters a repair process and the customer whose service is interrupted goes back to the head of the queue. Also the server may go on vacation according to Bernoulli scheduled server vacation, that is, after a service completion, the server may go for a vacation with probability $\theta(0 \leq \theta \leq 1)$ or may continue to serve the next customer, if any, with probability $1 - \theta$. The service time and the vacation time are generally distributed, while the repair time is exponentially distributed. The customers arrive in batches to the system and served one by one on a "first come - first served" basis. Also in our model restricted Admissibility policy has been applied that, not all arriving batches are allowed to join the system at all times.

The rest of the paper has been organized as follows: in section 2, the mathematical description of our model has been found, in section 3, the transient solution of the system has been derived, in section 4, the steady state analysis has been discussed.

2. Mathematical description of the queueing model

To describe the required queueing model, we assume the following.

• Let $\lambda c_i dt; i = 1, 2, 3...$be the first order probability of arrival of 'i' customers in batches in the system during a short period of time(t,t+dt)where $0 \leq c_i \leq 1, \sum_{i=1}^{\infty} c_i = 1, \lambda > 0$ is the mean arrival rate of batches. Unlike the usual batch arrival queueing systems, considering the policy of restricted admissibility of batches in which not all batches are allowed to join the system at all times. Let $r_1(0 \leq r_1 \leq 1)$ be the probability that an arriving batch will be allowed to join the system during the period of server's non-vacation period which includes system's service time and repair time also let $r_2(0 \leq r_2 \leq 1)$ be the probability that an arriving batch will be allowed to join the system during server's vacation period.

• After completion of service, if customer received unsuccessful service for certain reasons, the customer may immediately join the tail of the original queue as a feedback customer for receiving another regular service with probability $p(0 \leq p \leq 1)$. Otherwise the customer may depart forever from the system with probability q (= 1 - p). The service discipline for feedback and newly customers are first come first served discipline. Also service time for a feedback customer is independent of its previous service times.

• There is a single server which provides service following a general(arbitrary) distribution with distribution function B(v)and density function b(v). Let$\mu(x)dx$

be the conditional probability density function of service completion during the interval (x,x+dx] given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)} \tag{1}$$

and therefore

$$b(v) = \mu(v) e^{-\int_0^v \mu(x)} dx \tag{2}$$

• As soon as a service is completed, the server may take a vacation of random length with probability θ (or) he may stay in the system providing service to a customer if any, with probability $1 - \theta$, where $0 \leq \theta \leq 1$.

• The vacation time of the server follows a general(arbitrary)with distribution function V(s)and the density function v(s). Let $\nu(x)$dx be the conditional probability of a completion of a vacation during the interval (x,x+dx] given that the elapsed vacation time is x so that

$$\nu(x) = \frac{v(x)}{1 - V(x)} \tag{3}$$

and therefore

$$v(s) = \nu(s) e^{-\int_0^s \nu(x)} dx \tag{4}$$

• The system may breakdown at random and the breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$. Further we assume that once the system breakdown, the customer whose service is interrupted comes back to the head of queue.

• Once the system breaks down it enters a repair process immediately. The repair times are exponentially distributed with mean repair rate $\beta > 0$.

• Various stochastic processes involved in the queueing system are assumed to be independent of each other.

3.Definitions and Equations governing the system

We let,

(i)$P_n(x,t)$ = Probability that at time 't' the server is active providing service and there are 'n' ($n \geq 0$) customers in the queue excluding the one being served and the elapsed service time for this customer is x.
Consequently $p_n(t)$ denotes the probability that at time 't' there are 'n' customers

in the queue excluding the one customer in service irrespective of the value of x.

(ii)$V_n(x,t)$ = Probability that at time 't', the server is on vacation with elapsed vacation time x, and there are 'n' ($n \geq 0$) customers waiting in the queue for service. Consequently $V_n(t)$ denotes the probability that at time 't' there are 'n' customers in the queue and the server is on vacation irrespective of the value of x.

(iii)$R_n(t)$ = Probability that at time t, the server is inactive due to breakdown and the system is under repair while there are'n' ($n \geq 0$) customers in the queue.

(iv)Q(t)= Probability that at time 't' there are no customers in the system and the server is idle but available in the system .

3.Transient solution of the queueing model

The model is then, governed by he following set of differential-difference equations.

(5)
$$\frac{\partial}{\partial t}P_n(x,t)+\frac{\partial}{\partial x}P_n(x,t)+(\lambda+\mu(x)+\alpha)P_n(x,t) = \lambda(1-r_1)P_n(x,t)+\lambda r_1\sum_{i=1}^{n-1}c_iP_{n-i}(x,t); n \geq 1$$

(6)
$$\frac{\partial}{\partial t}P_0(x,t) + \frac{\partial}{\partial x}P_0(x,t) + (\lambda + \mu(x) + \alpha)P_0(x,t) = \lambda(1-r_1)P_0(x,t)$$

(7)
$$\frac{\partial}{\partial t}V_n(x,t)+\frac{\partial}{\partial x}V_n(x,t)+(\lambda+\nu(x))V_n(x,t) = \lambda(1-r_2)V_n(x,t)+\lambda r_2\sum_{i=1}^{n-1}c_iV_{n-i}(x,t); n \geq 1$$

(8)
$$\frac{\partial}{\partial t}V_0(x,t) + \frac{\partial}{\partial x}V_0(x,t) + (\lambda + \nu(x))V_0(x,t) = \lambda(1-r_2)V_0(x,t)$$

(9)
$$\frac{d}{dt}R_n(t) = -(\lambda+\beta)R_n(t)+\lambda(1-r_1)R_n(x,t)+\lambda r_1\sum_{i=1}^{n-1}c_iR_{n-i}(x,t)+\alpha\int_0^\infty P_{n-1}(x,t)dx$$

(10)
$$\frac{d}{dt}R_0(t) = -(\lambda+\beta)R_0(t) + \lambda(1-r_1)R_0(x,t)$$

$$\frac{d}{dt}Q(t) = -\lambda r_1Q(t) + \lambda r_1(1-r_2)Q(t) + r_2\beta R_0(t)$$

(11)
$$+r_1\int_0^\infty V_0(x,t)\nu(x)dx + r_2(1-\theta)q\int_0^\infty P_0(x,t)\mu(x)dx$$

The above equations are to be solved subject to the following boundary conditions

$$r_2 P_n(0,t) = r_2(1-\theta)\left\{p\int_0^\infty P_n(x,t)\mu(x)dx + q\int_0^\infty P_{n+1}(x,t)\mu(x)dx\right\}$$

$$+r_1\int_0^\infty V_{n+1}(x,t)\nu(x)dx + r_2\beta R_{n+1}(t) + \lambda r_1 r_2 c_{n+1}Q(t); n \geq 0 \tag{12}$$

$$r_1 V_n(0,t) = r_2\theta\int_0^\infty P_n(x,t)\mu(x)dx; n \geq 0 \tag{13}$$

Assuming there are no customers in the system initially so that the server is idle.

$$V_0(0) = 0; V_n(0) = 0; Q(0) = 1; P_n(0) = 0, n = 0,1,2,... \tag{14}$$

Generating functions of the queue length
The time dependent solution.

We define the probability generating functions

$P_q(x,z,t) = \sum_{n=0}^\infty z^n P_n(x,t)$

$P_q(z,t) = \sum_{n=0}^\infty z^n P_n(t)$

$V_q(x,z,t) = \sum_{n=0}^\infty z^n V_n(x,t)$

$V_q(z,t) = \sum_{n=0}^\infty z^n V_n(t)$

$R_q(z,t) = \sum_{n=0}^\infty z^n R_n(t)$

$$C(z) = \sum_{n=1}^\infty c_n z^n \tag{15}$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function f(t) as

$$\overline{f}(s) = \int_0^\infty f(t)e^{-st}dt \tag{16}$$

Taking Laplace transforms of equations (5) to (13)

$$\frac{\partial}{\partial x}\bar{P}_n(x,s) + (s+\lambda+\mu(x)+\alpha)\bar{P}_n(x,s) = \lambda(1-r_1)\bar{P}_n(x,s) + \lambda r_1\sum_{i=1}^{n-1} c_i\bar{P}_{n-i}(x,s) \tag{17}$$

$$\frac{\partial}{\partial x}\bar{P}_0(x,s) + (s+\lambda+\mu(x)+\alpha)\bar{P}_0(x,s) = \lambda(1-r_1)\bar{P}_0(x,s) \tag{18}$$

$$\frac{\partial}{\partial x}\bar{V}_n(x,s) + (s+\lambda+\nu(x))\bar{V}_n(x,s) = \lambda(1-r_2)\bar{V}_n(x,s) + \lambda r_2\sum_{i=1}^{n-1} c_i\bar{V}_{n-i}(x,s) \tag{19}$$

$$\frac{\partial}{\partial x}\bar{V}_0(x,s) + (s+\lambda+\nu(x))\bar{V}_0(x,s) = \lambda(1-r_2)\bar{V}_0(x,s) \tag{20}$$

$$(s+\lambda+\beta)\bar{R}_n(s)=\lambda(1-r_1)\bar{R}_n(x,s)+\lambda r_1\sum_{i=1}^{n-1}c_i\bar{R}_{n-i}(s)+\alpha\int_0^\infty\bar{P}_{n-1}(x,s)dx \tag{21}$$

$$(s+\lambda+\beta)\bar{R}_0(s)=\lambda(1-r_1)\bar{R}_o(x,s) \tag{22}$$

$$(s+\lambda r_1)\bar{Q}(s)=1+\lambda r_1(1-r_2)Q(s)+\beta r_2\bar{R}_0(s)$$

$$+r_2(1-\theta)q\int_0^\infty\bar{P}_0(x,s)\mu(x)dx+r_1\int_0^\infty\bar{V}_0(x,s)\nu(x)dx \tag{23}$$

for boundary conditions,

$$r_2\bar{P}_n(0,s)=(1-\theta)r_2\left\{p\int_0^\infty\bar{P}_n(x,s)\mu(x)dx+q\int_0^\infty\bar{P}_{n+1}(x,s)\mu(x)dx\right\}$$

$$+r_1\int_0^\infty\bar{V}_{n+1}(x,s)\nu(x)dx+r_2\beta\bar{R}_{n+1}(s)+\lambda r_1r_2\lambda c_{n+1}\bar{Q}(s);n\geq 0 \tag{24}$$

$$r_1\bar{V}_n(0,s)=r_2\theta\int_0^\infty\bar{P}_n(x,s)\mu(x)dx;n\geq 0 \tag{25}$$

multiply equation (17) by z^n and add (18) implies and using the probability generating function defined above.

$$\frac{\partial}{\partial x}\bar{P}_q(x,z,s)+(s+\lambda r_1(1-C(z))+\mu(x)+\alpha)\bar{P}_q(x,z,s)=0 \tag{26}$$

performing similar operations to equations (19) to (22).

$$\frac{\partial}{\partial x}\bar{V}_q(x,z,s)+(s+\lambda r_2(1-C(z))+\nu(x))\bar{V}_q(x,z,s)=0 \tag{27}$$

$$(s+\lambda r_1(1-C(z))+\beta)\bar{R}_q(z,s)=\alpha z\int_0^\infty\bar{P}_q(x,z,s)dx \tag{28}$$

For the boundary conditions, we multiply equation(24) by z^{n+1}, sum over n from 0 to ∞ and use generating function defined above, we get

$$zr_2\bar{P}_q(0,z,s)=r_2(1-\theta)(pz+q)\int_0^\infty\bar{P}_q(x,z,s)\mu(x)dx+r_1\int_0^\infty\bar{V}_q(x,z,s)\nu(x)dx$$

$$+r_2\beta\bar{R}_q(z,s)+(1-s\bar{Q}(s))+\lambda r_1r_2(C(z)-1)\bar{Q}(s) \tag{29}$$

Similarly multiply equation (25) by $z^n$and sum over n from 0 to ∞ and use generating function defined above

$$r_1\bar{V}_q(0,z,s)=r_2\theta\int_0^\infty\bar{P}_q(x,z,s)\mu(x)dx;n\geq 0 \tag{30}$$

Integrating equation(26) from 0 to x yields

$$\bar{P}_q(x,z,s) = \bar{P}_q(0,z,s)e^{-(s+\lambda r_1(1-C(z))+\alpha)x-\int_0^x \mu(t)dt} \tag{31}$$

where $\bar{P}_q(0,z,s)$ is given by equation(29). Again integrating equation (31) by parts with respect to x yields

$$\bar{P}_q(z,s) = \bar{P}_q(0,z,s)\left[\frac{1-\bar{B}(s+\lambda r_1(1-C(z))+\alpha)}{(s+\lambda r_1(1-C(z))+\alpha)}\right] \tag{32}$$

where

$$\bar{B}(s+\lambda r_1(1-C(z))+\alpha) = \int_0^\infty e^{-(s+\lambda r_1(1-C(z))+\alpha)x}dB(x) \tag{33}$$

is Laplace - Stieltjes transform of the service time B(x). Now multiplying both sides of equation (31) by $\mu(x)$ and integrating over x, we get

$$\int_0^\infty \bar{P}_q(x,z,s)\mu(x)dx = \bar{P}_q(0,z,s)\bar{B}(s+\lambda r_1(1-C(z))+\alpha) \tag{34}$$

Using equation (34), equation (30) becomes

$$r_1\bar{V}_q(0,z,s) = r_2\theta\bar{P}_q(0,z,s)\bar{B}(s+\lambda r_1(1-C(z))+\alpha) \tag{35}$$

Similarly integrate equation (27) from 0 to x, we get

$$\bar{V}_q(x,z,s) = \bar{V}_q(0,z,s)e^{-(s+\lambda r_2(1-C(z)))x-\int_0^x \nu(t)dt} \tag{36}$$

substituting the value of $\bar{V}_q(0,z,s)$ from equation(35) in equation (36) we get

(37)

$$\bar{V}_q(x,z,s) = \left(\frac{r_2}{r_1}\right)\theta\bar{P}_q(0,z,s)\bar{B}(s+\lambda r_1(1-C(z))+\alpha)e^{-(s+\lambda r_2(1-C(z)))x-\int_0^x \nu(t)dt}$$

Again integrating equation (37) by parts with respect to x

(38)

$$\bar{V}_q(z,s) = \left(\frac{r_2}{r_1}\right)\theta\bar{P}_q(0,z,s)\bar{B}(s+\lambda r_1(1-C(z))+\alpha)\left[\frac{1-\bar{V}(s+\lambda r_2(1-C(z)))}{(s+\lambda r_2(1-C(z))}\right]$$

where

$$\bar{V}(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-(s+\lambda r_2(1-C(z)))x}dV(x) \tag{39}$$

is Laplace - Stieltjes transform of the vacation time V(x). Now multiplying both sides of equation(37) by $\nu(x)$ and integrating over x, we get

(40)
$$\int_0^\infty \bar{V}_q(x,z,s)\nu(x)dx = \left(\frac{r_2}{r_1}\right)\theta\bar{P}_q(0,z,s)\bar{B}(s+\lambda r_1(1-C(z))+\alpha)\bar{V}(s+\lambda r_2(1-C(z)))$$

Using equation (32), equation (28) becomes

(41)
$$\bar{R}_q(z,s) = \frac{\alpha z\bar{P}_q(0,z,s)[1-\bar{B}(s+\lambda r_1(1-C(z))+\alpha)]}{[s+\lambda r_1(1-C(z))+\beta][s+\lambda r_1(1-C(z))+\alpha]}$$

Now using (34), (40) and (41) in equation (29) and solving for $\bar{P}_q(0,z,s)$ we get

(42)
$$\bar{P}_q(0,z,s) = \frac{f_1(z)f_2(z)\left[\left(\frac{1}{r_2}\right)(1-s\bar{Q}(s))+\lambda r_1(C(z)-1)\bar{Q}(s)\right]}{Dr}$$

where
Dr $= f_1(z)f_2(z)\{z-(1-\theta)(pz+q)\bar{B}[f_1(z)]-\theta\bar{V}(s+\lambda r_2(1-C(z)))\bar{B}[f_1(z)]\}$

$-\alpha\beta z\{1-\bar{B}[f_1(z)]\}$

$f_1(z) = s+\lambda r_1(1-C(z))+\alpha$

$f_2(z) = s+\lambda r_1(1-C(z))+\beta$

substituting the value of $\bar{P}_q(0,z,s)$ from equation (42)in to equations (32), (38) and (41)

(43)
$$\bar{P}_q(z,s) = \frac{f_2(z)[1-\bar{B}[f_1(z)]]\left[\left(\frac{1}{r_2}\right)(1-s\bar{Q}(s))+\lambda r_1(C(z)-1)\bar{Q}(s)\right]}{Dr}$$

(44)
$$\bar{V}_q(z,s) = \frac{f_1(z)f_2(z)\left(\frac{r_2}{r_1}\right)\theta\bar{B}[f_1(z)]\left[\left(\frac{1}{r_2}\right)(1-s\bar{Q}(s))+\lambda r_1(C(z)-1)\bar{Q}(s)\right]\left[\frac{1-\bar{V}(s+\lambda r_2(1-C(z)))}{(s+\lambda r_2(1-C(z)))}\right]}{Dr}$$

(45)
$$\bar{R}_q(z,s) = \frac{\alpha z[1-\bar{B}[f_1(z)]]\left[\left(\frac{1}{r_2}\right)(1-s\bar{Q}(s))+\lambda r_1(C(z)-1)\bar{Q}(s)\right]}{Dr}$$

where Dr is given above.

4.The steady state analysis

In this section we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, suppress the argument't' where ever it appears in the time dependent analysis.
By using well known Tauberian property,

(46)
$$Lt_{s\to 0}s\bar{f}(s) = Lt_{t\to\infty}f(t)$$

multiplying both sides of equation (43), (44), (45) and applying equation(46) and simplifying, we get

$$P_q(z) = \frac{f_2(z)(1-\bar{B}[f_1(z)])[\lambda r_1(C((z))-1)Q]}{Dr} \tag{47}$$

$$V_q(z) = \frac{\theta f_1(z)f_2(z)\bar{B}[f_1(z)][\bar{V}(\lambda r_2(1-C(z)))-1]Q}{Dr} \tag{48}$$

$$R_q(z) = \frac{\alpha z[1-\bar{B}[f_1(z)]][\lambda r_1(C((z))-1)Q]}{Dr} \tag{49}$$

where Dr and $f_1(z)$ and $f_2(z)$ are given in the previous section. Let $W_q(z)$ denotes the probability generating function of queue size irrespective of the state of the system. Then adding (47), (48) and (49) we get

$$W_q(z) = P_q(z) + V_q(z) + R_q(z) \tag{50}$$

$$W_q(z) = \frac{f_2(z)(1-\bar{B}[f_1(z)])[\lambda r_1(C((z))-1)Q]}{Dr}$$

$$+\frac{\theta f_1(z)f_2(z)\bar{B}[f_1(z)][\bar{V}(\lambda r_2(1-C(z)))-1]Q}{Dr}$$

$$+\frac{\alpha z[1-\bar{B}[f_1(z)]][\lambda r_1(C((z))-1)Q]}{Dr} \tag{51}$$

In order to obtain Q, using the normalization condition, as follows

$$W_q(1) + Q = 1 \tag{52}$$

We see that at z=1, $W_q(z)$ is indeterminate of the form 0/0. We apply L'Hospital rule in equation (51)

$$W_q(1) = \frac{\lambda r_1 Q E(I)(\alpha+\beta)[1-\bar{B}(\alpha)] + \lambda r_2 \theta\alpha\beta B(\alpha)E(I)E(V)}{dr} \tag{53}$$

where $dr = \alpha\beta(q+p\theta)\bar{B}(\alpha) - \lambda r_1 E(I)[(\alpha+\beta)(1-\bar{B}(\alpha))] - \theta\lambda r_2\alpha\beta\bar{B}(\alpha)E(I)E(V)$

and $\bar{B}(0) = 1, \bar{V}(0) = 1, -V'(0) = E[V]$ the mean vacation time. Using equation (53) in equation (52)

$$Q = 1 - \frac{\lambda E(I)}{(q+p\theta)}\left\{r_1\left[\frac{1}{\beta\bar{B}(\alpha)} + \frac{1}{\alpha\bar{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha}\right] + \lambda\theta r_2 E(I)E(V)\right\} \tag{54}$$

and the the utilization factor ρ of the system is given by

$$\rho = \frac{\lambda E(I)}{(q+p\theta)}\left\{r_1\left[\frac{1}{\beta\bar{B}(\alpha)} + \frac{1}{\alpha\bar{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha}\right] + \lambda\theta r_2 E(I)E(V)\right\} \tag{55}$$

where $\rho < 1$ is the stability condition under which the steady state exists, equation(54) gives the probability that the server is idle. Substitute Q from equation (54) in equation (50), $W_q(z)$ have been completely and explicitly determined which is the the probability generating function of the queue size.

The average queue size and average waiting time can be obtained as follows
Let $L_q$denote the mean number of customers in the queue under the steady state, then $L_q = \frac{d}{dz}W_q(z) \mid_{z=1}$,since this formula gives 0/0 form, then we write
$W_q(z) = \frac{N(z)}{D(z)}$ where N(z) and D(z) are the numerator and denominator of the right hand side of equation (49) respectively, then we use

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2[D'(1)]^2} \tag{56}$$

where primes and double primes in equation (56) denote first and second derivation at z=1 respectively. Carrying out the derivatives at z=1, we have

$$N'(1) = \lambda E(I)Q[r_1(\alpha+\beta) - \bar{B}(\alpha)(\theta r_2\alpha\beta E(V) - r_1(\alpha+\beta))] \tag{57}$$

$$N''(1) = 2Q[\lambda E(I)]^2 \Big\{ (\tfrac{\alpha r_1}{\lambda E(I)} - r_1^2) + \bar{B}(\alpha)[r_1^2 - \tfrac{\alpha r_1}{\lambda E(I)} - \theta(\alpha+\beta)r_1r_2E(V)]$$
$$+\tfrac{1}{2}\theta\alpha\beta r_2^2E(V^2) + \bar{B}'(\alpha)(r_1^2(\alpha+\beta) - \theta r_1r_2\alpha\beta E(V))\Big\}$$
$$+\lambda QE(I(I-1))\left\{r_1(\alpha+\beta) + \bar{B}(\alpha)(\theta r_2\alpha\beta E(V) - r_1(\alpha+\beta))\right\} \tag{58}$$

(59)
$$D'(1) = -\lambda E(I)r_1(\alpha+\beta) + \bar{B}(\alpha)\left\{r_1\alpha\beta(q+p\theta) + \lambda E(I)r_1(\alpha+\beta) - \theta\alpha\beta r_2E(V)\right\}$$

$$D''(1) = 2[\lambda E(I)]^2 \Big\{ (r_1^2 - \tfrac{(\alpha+\beta)r_1}{\lambda E(I)}) + \bar{B}(\alpha)$$
$$[-(q+p\theta)r_1^2 + \theta r_1r_2(\alpha+\beta)E(V) - \tfrac{1}{2}\alpha\beta\theta r_2^2E(V^2)]$$
$$+\bar{B}'(\alpha)[-(q+p\theta)r_1^2(\alpha+\beta) - \tfrac{r_1(\alpha\beta)}{\lambda E(I)} + \theta\alpha\beta r_1r_2E(V)]\Big\}$$
$$+\lambda E(I(I-1))\left\{-r_1(\alpha+\beta) + \bar{B}(\alpha)(r_1(\alpha+\beta) - \theta r_2\alpha\beta E(V))\right\} \tag{60}$$

where $E(V^2)$ is the second moment of the vacation time and Q has been found in equation (54). Then if we substitute the values of $N'(1), N''(1), D'(1)$ and $D''(1)$ from equations (57), (58), (59) and (60) in to (56) equation we obtain L_q in a closed form.
Mean waiting time of a customer could be found, as follows

$$W_q = \frac{L_q}{\lambda} \tag{61}$$

by using Little's formula.

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