

## A SURFACE FAMILY WITH A COMMON ASYMPTOTIC CURVE IN THE EUCLIDEAN 3-SPACE

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**ABSTRACT.** In this paper, we offer an approach for study the problem on how to construct a surface family from a given asymptotic curve and we deduce the necessary and sufficient condition for the given curve to be the asymptotic curve for the parametric surface. Meanwhile, some representative curves are chosen to construct the corresponding surfaces which possess these curves as asymptotic curves. The extension to ruled and developable surfaces is also outlined. Finally, we demonstrated some interesting ruled and developable surfaces about the subject.

### 1. Introduction

In the 3-dimensional Euclidean space, asymptotic curve on a surface is an intrinsic geometric feature that plays an important role in a diversity of applications. It is a useful tool in surface analysis for exhibiting variations of the principal direction and has been a long-standing research focus in Differential Geometry [1, 2]. Many studies on dealing with asymptotic curves have been reported. Hartman and Wintner [3] showed that the degree of smoothness is an important notion in the classical theory of asymptotic curves of non-positive Gaussian curvature, which is usually left unspecified. Kitagawa [4] proved that if  $M$  is a flat torus isometrically immersed in a unit 3-sphere, then all asymptotic curves on  $M$  are periodic. Garcia and Sotomayor [5] studied the simplest qualitative properties of asymptotic curves of a surface immersed in the Euclidean 3-space  $E^3$ . Garcia et al. [6] studied immersions of surfaces into  $E^3$  whose nets of asymptotic curves are topologically undisturbed under small perturbations of the immersion, which can then be described as structurally asymptotic stable. Contopoulos [7], proved that in order to find a set of escaping orbits of stars in a stellar system, it is necessary to find asymptotic curves of the Lyapunov orbits, because any small outward deviation from an asymptotic orbit will lead a star to escape from the system. The work of Contopoulos considered the asymptotic orbits of mainly unstable orbits, with a particular emphasis on the Lyapunov orbits, and found sets of escaping orbits with initial conditions on asymptotic curves. Efthymiopoulos et al. [8] concluded that the diffusion of any chaotic orbit inside the contours follows essentially the same path defined

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by the unstable asymptotic curves that emanate from unstable periodic orbits inside the contours.

In practical applications, the concept of family of surfaces having a given characteristic curve was first introduced by Wang et.al. [9] in Euclidean 3-space. The basic idea is to regard the wanted surface as an extension from the given characteristic curve, and represent it as a linear combination of the marching-scale functions  $u(s, t)$ ,  $v(s, t)$ ,  $w(s, t)$  and the three vector functions  $\mathbf{t}(s)$ ,  $\mathbf{n}(s)$ ,  $\mathbf{b}(s)$ , which are the unit tangent, the principal normal and the binormal vector of the curve respectively. With the given characteristic curve and isoparametric constraints, they derived the necessary and sufficient conditions for the correct parametric representation of the surface pencil. This principal has been used extensively in the works (see for example [10-16]).

However, the relevant work on surfaces through asymptotic curve is rare. It is an important and interesting problem in practical applications. Therefore, research on designing a surface from a given curve is attractive. Inspired by Wang et al. [10], in this paper, we offer an alternative approach for designing surfaces from a given asymptotic curve. We not only construct a surface possessing a given curve as an asymptotic curve, but also give a concrete expression of the surface. Moreover, the extension to the ruled and developable surfaces is also outlined. Also, some representative curves are chosen to construct the corresponding surfaces which possessing these curves as asymptotic curves.

## 2. PRELIMINARIES

In this section we list some notions, formulas and conclusions for space curves, and ruled surfaces in Euclidean 3-space  $E^3$  which can be found in the textbooks on differential geometry (See for instance Refs. [1-3]).

A curve is regular if it admits a tangent line at each point of the curve. In the following discussions, all curves are assumed to be regular. Given a spatial curve  $\alpha : s \rightarrow \alpha(s)$ , which is parameterized by arc length parameter  $s$ . We assume  $\ddot{\alpha}(s) \neq 0$  for all  $s \in [0, L]$ , since this would give us a straight line. In this paper,  $\dot{\alpha}(s)$  and  $\alpha'(r)$  denote the derivatives of  $\alpha$  with respect to arc-length parameter  $s$  and arbitrary parameter  $r$ , respectively. For each point of  $\alpha(s)$ , the set  $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$  is called the Serret–Frenet Frame along  $\alpha(s)$ , where  $\mathbf{t}(s) = \dot{\alpha}(s)$ ,  $\mathbf{n}(s) = \ddot{\alpha}(s) / \|\ddot{\alpha}(s)\|$  and  $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$  are the unit tangent, principal normal, and binormal vectors of the curve at the point  $\alpha(s)$ , respectively. The arc-length derivative of the Serret–Frenet frame is governed by the relations:

$$(1) \quad \begin{pmatrix} \dot{\mathbf{t}}(s) \\ \dot{\mathbf{n}}(s) \\ \dot{\mathbf{b}}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix},$$

where the curvature  $\kappa(s)$  and torsion  $\tau(s)$  of the curve  $\boldsymbol{\alpha}(s)$  are defined by

$$(2) \quad \kappa(s) = \|\ddot{\boldsymbol{\alpha}}(s)\|, \quad \tau(s) = \frac{\det(\dot{\boldsymbol{\alpha}}(s), \ddot{\boldsymbol{\alpha}}(s), \ddot{\boldsymbol{\alpha}}(s))}{\|\ddot{\boldsymbol{\alpha}}(s)\|^2}.$$

In the majority of practical cases, the parameter of the curve is usually not in arc-length representation. Given the parametric curve

$$\boldsymbol{\alpha}(r) = (\alpha_1(r), \alpha_2(r), \alpha_3(r)), 0 \leq r \leq H,$$

where the parameter  $r$  is not the arc length. The components of the Serret–Frenet frame are defined by [1-3]:

$$(3) \quad \mathbf{t}(r) = \frac{\boldsymbol{\alpha}'(r)}{\|\boldsymbol{\alpha}'(r)\|}, \quad \mathbf{b}(r) = \frac{\boldsymbol{\alpha}'(r) \times \boldsymbol{\alpha}''(r)}{\|\boldsymbol{\alpha}'(r) \times \boldsymbol{\alpha}''(r)\|}, \quad \mathbf{n}(r) = \mathbf{b}(r) \times \mathbf{t}(r),$$

and the corresponding Serret-Frenet frame is given by

$$(4) \quad \begin{pmatrix} \mathbf{t}'(r) \\ \mathbf{n}'(r) \\ \mathbf{b}'(r) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(r) \|\boldsymbol{\alpha}'(r)\| & 0 \\ -\kappa(r) \|\boldsymbol{\alpha}'(r)\| & 0 & \tau(r) \|\boldsymbol{\alpha}'(r)\| \\ 0 & -\tau(r) \|\boldsymbol{\alpha}'(r)\| & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(r) \\ \mathbf{n}(r) \\ \mathbf{b}(r) \end{pmatrix}.$$

### 3. A SURFACE FAMILY WITH AN ASYMPTOTIC CURVE

Consider the construction of a surface from a unit speed space curve  $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ ,  $0 \leq s \leq L$ , such that the surface tangent plane is coincident with the curve osculating plane. Expressing the surface in terms of Serret–Frenet frame  $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$  along  $\boldsymbol{\alpha}(s)$  as:

$$(5) \quad M : \mathbf{P}(s, t) = \boldsymbol{\alpha}(s) + u(s, t)\mathbf{t}(s) + v(s, t)\mathbf{n}(s); \quad 0 \leq t \leq T, \quad 0 \leq s \leq L,$$

where  $u(s, t)$ , and  $v(s, t)$  are all  $C^1$  functions. If the parameter  $t$  is seen as the time, the functions  $u(s, t)$ , and  $v(s, t)$  can then be viewed as directed marching distances of a point unit in the time  $t$  in the direction  $\mathbf{t}$ , and  $\mathbf{n}$ , respectively, and the position vector  $\boldsymbol{\alpha}(s)$  is seen as the initial location of this point.

It is easily checked that the two tangent vectors of  $M$  are given by:

$$(6) \quad \left. \begin{aligned} \mathbf{P}_s &= (1 + u_s - v\kappa)\mathbf{t} + (v_s + u\kappa)\mathbf{n} + v\tau\mathbf{b}, \\ \mathbf{P}_t &= u_t\mathbf{t} + v_t\mathbf{n}. \end{aligned} \right\}$$

The lowercase subscript letters  $s$ , and  $t$  denote partial derivatives corresponding to the indicated variable, e.g.,  $\mathbf{P}_s = \frac{\partial \mathbf{P}}{\partial s}$ ,  $\mathbf{P}_t = \frac{\partial \mathbf{P}}{\partial t}$ . Consequently, the normal vector of the surface is given by:

$$(7) \quad \mathbf{N}(s, t) := \mathbf{P}_s \times \mathbf{P}_t = \eta_1(s, t)\mathbf{t}(s) + \eta_2(s, t)\mathbf{n}(s) + \eta_3(s, t)\mathbf{b}(s),$$

where

$$(8) \quad \left. \begin{aligned} \eta_1(s, t) &= -v(s, t)\tau(s)v_t(s, t), \quad \eta_2(s, t) = v(s, t)\tau(s)u_t(s, t), \\ \eta_3(s, t) &= (1 + u_s(s, t) - v(s, t)\kappa(s))v_t(s, t) - (v_s(s, t) + u(s, t)\kappa(s))u_t(s, t). \end{aligned} \right\}$$

Since the curve  $\boldsymbol{\alpha}(s)$  is an isoparametric on the surface, there exists a parameter  $t = t_0 \in [0, T]$  such that  $\mathbf{P}(s, t_0) = \boldsymbol{\alpha}(s)$ ; that is,

$$(9) \quad u(s, t_0) = v(s, t_0) = 0, \quad 0 \leq t_0 \leq T, \quad 0 \leq s \leq L,$$

and thus when  $t = t_0$ —i.e., along the curve  $\boldsymbol{\alpha}(s)$ —the surface normal is

$$(10) \quad \mathbf{N}(s, t_0) = \eta_3(s, t_0)\mathbf{b}(s),$$

where

$$(11) \quad \left. \begin{aligned} \eta_1(s, t_0) &= 0, \\ \eta_2(s, t_0) &= 0, \\ \eta_3(s, t_0) &= v_t(s, t_0) \neq 0. \end{aligned} \right\}$$

Combining conditions (3.5) and (3.7), we have found the necessary and sufficient conditions for the surface  $\mathbf{P}(s, t)$  to have the curve  $\boldsymbol{\alpha}(s)$  as an isoasymptotic.

Hence, we can state the following theorem:

**Theorem 3.1.** The given spatial curve  $\boldsymbol{\alpha}(s)$  is an isoasymptotic curve on the surface  $\mathbf{P}(s, t)$  if and only if

$$(12) \quad \left. \begin{aligned} u(s, t_0) &= v(s, t_0) = 0, \\ v_t(s, t_0) &\neq 0, \quad 0 \leq t_0 \leq T, \quad 0 \leq s \leq L. \end{aligned} \right\}$$

Obviously, Eqs. (3.8) is more elegant and convenient for applications than those in [14]. The important point to note here is the technique we have used. In Ref. [9], for simplification and better analysis, the marching-scale functions  $u(s, t)$ , and  $v(s, t)$  can be decomposed into two factors:

$$(13) \quad \begin{aligned} u(s, t) &= l(s)U(t), \\ v(s, t) &= m(s)V(t). \end{aligned}$$

Here  $l(s)$ ,  $m(s)$ ,  $V(t)$  and  $W(t)$  are  $C^1$  functions and  $l(s)$ , and  $m(s)$  are not identically zero. Thus, from Theorem 3.1, we can get the following corollary.

**Corollary 3.1.** The necessary and sufficient condition of the curve  $\boldsymbol{\alpha}(s)$  being an isoasymptotic curve on the surface  $\mathbf{P}(s, t)$  is

$$(14) \quad \left. \begin{aligned} U(t_0) &= V(t_0) = 0, \quad 0 \leq t_0 \leq T, \quad 0 \leq s \leq L, \\ m(s) \frac{dV(t_0)}{dt} &\neq 0 \Leftrightarrow m(s) \neq 0, \text{ and } \frac{dV(t_0)}{dt} = \text{const} \neq 0. \end{aligned} \right\}$$

For the case when the marching-scale functions depend only on the parameter  $t$ ; that is,  $l(s) = m(s) = 1$ . By simplifying, condition (3.10) can be represented as

$$(15) \quad \left. \begin{aligned} U(t_0) &= V(t_0) = 0, \\ \frac{dV(t_0)}{dt} &\neq 0, \quad 0 \leq t_0 \leq T. \end{aligned} \right\}$$

The corresponding isoparametric asymptotic surface family becomes

$$(16) \quad M : \mathbf{P}(s, t) = \boldsymbol{\alpha}(s) + U(t)\mathbf{t}(s) + V(t)\mathbf{n}(s); \quad 0 \leq t \leq T, \quad 0 \leq s \leq L,$$

**3.1. Simple surfaces with a common asymptotic curve.** Now, we are going to deal with and construct some representative examples to verify the method. They also serve to verify the correctness of the formulae derived above.

**Example 1.** If  $\mathbf{p}_0 = (0, 0, 0)$ ,  $\mathbf{p}_1 = (0, 1, 1)$ , and  $\mathbf{p}_2 = (1, 2, 0)$  are the points in the Euclidean 3-space  $E^3$ , then the quadratic Bézier curve can be expressed as

$$\boldsymbol{\alpha}(r) = r(1-r)^2\mathbf{p}_0 + 2r(1-r)\mathbf{p}_1 + r^2\mathbf{p}_2, \quad 0 \leq r \leq 1.$$

It is easy to show that

$$\kappa(r) = \frac{\sqrt{6}}{2(5r^2 - 4r + 2)^{\frac{3}{2}}}, \quad \tau(r) = 0.$$

After simple computation, we get:

$$\left. \begin{aligned} \mathbf{t}(r) &= \left( \frac{r, 1, -2r+1}{f(r)} \right), \\ \mathbf{n}(r) &= \left( \frac{2(1-r), 2-5r, -(2+r)}{\sqrt{6}f(r)} \right), \\ \mathbf{b}(r) &= \left( -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right), \end{aligned} \right\}$$

where  $f(r) = \sqrt{5r^2 - 4r + 2}$ . Choosing  $U(t) = \beta t$ , and  $V(t) = \gamma t$ ,  $\gamma \neq 0$ , and  $t \in [0, T]$ . Obviously, Eq. (3.11) is satisfied. Thus, we obtain the following isoparametric asymptotic surface family:

$$\mathbf{P}(r, t; \beta; \gamma) = (r^2, 2r, 2r - 2r^2) + t(\beta, \gamma, 0) \left( \begin{array}{ccc} \frac{r}{l(r)} & \frac{1}{l(r)} & \frac{-2r+1}{l(r)} \\ \frac{2(1-r)}{\sqrt{6}l(r)} & \frac{2-5r}{\sqrt{6}l(r)} & \frac{-(2+r)}{\sqrt{6}l(r)} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{array} \right),$$

where  $0 \leq t \leq 1$ , and  $0 \leq r \leq 1$ . The parameters  $\beta$  and  $\gamma$  can control the shape of the surface. For  $\beta = -1$ , and  $\gamma = -1$ , the surface is shown in Fig. 1. Figure 2 shows the surface with  $\beta = 1$ , and  $\gamma = 2$ .

**Example 2.** In this example, we construct surface family in which all the surfaces share an asymptotic circular helix represented as

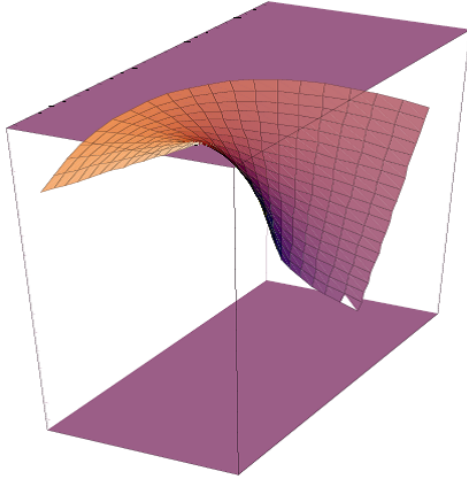
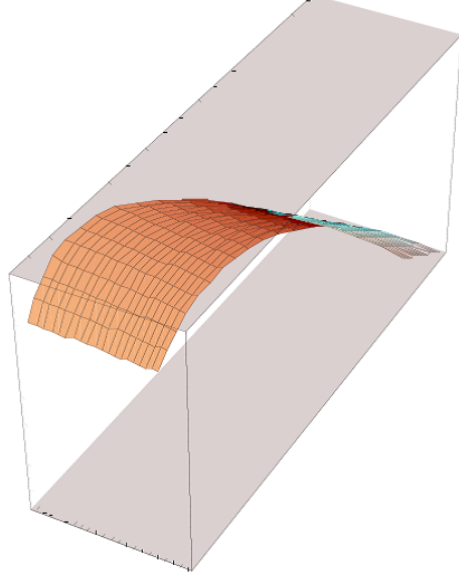
$$\boldsymbol{\alpha}(r) = (a \cos r, a \sin r, br), \quad a > 0, \quad b \neq 0, \quad 0 \leq r \leq 2\pi.$$

It is easy to show that:

$$\left. \begin{aligned} \mathbf{t}(r) &= \frac{1}{\sqrt{a^2+b^2}}(-a \sin r, a \cos r, b), \\ \mathbf{n}(r) &= (-\cos r, -\sin r, 0), \\ \mathbf{b}(r) &= \frac{1}{\sqrt{a^2+b^2}}(b \sin r, -b \cos r, a). \end{aligned} \right\}$$

A- Choosing  $U(t) = \beta t$ , and  $V(t) = \gamma t$ ,  $\gamma \neq 0$ , and  $t \in [0, T]$ , we obtain the following surface family:

$$\mathbf{P}(r, t; \beta; \gamma) = (a \cos r, a \sin r, br) + t(\beta, \gamma, 0) \left( \begin{array}{ccc} \frac{-a}{\sqrt{a^2+b^2}} \sin r & \frac{a}{\sqrt{a^2+b^2}} \cos r & \frac{b}{\sqrt{a^2+b^2}} \\ -\cos r & -\sin r & 0 \\ \frac{b}{\sqrt{a^2+b^2}} \sin r & \frac{-b}{\sqrt{a^2+b^2}} \cos r & \frac{a}{\sqrt{a^2+b^2}} \end{array} \right).$$

Figure 1.  $\mathbf{P}(r, t; -1; -1)$ .Figure 2.  $\mathbf{P}(r, t; 1; 2)$ .

Here, we chose  $t \in [0, 4]$ , and  $a = 2$ ,  $b = 1$ . For  $\beta = -1$ , and  $\gamma = -1$ , the surface is shown in Fig. 3. Figure 4 shows the surface with  $\beta = -\frac{\sqrt{5}}{4}$ , and  $\gamma = -\frac{\sqrt{5}}{2}$ .

B- By choosing marching-scale functions as

$$u(s, t) = (1 + \sin(t)) + \sum_{k=2}^4 a_{1k} (1 + \sin(t))^k,$$

$$v(s, t) = \cos(t) + \sum_{k=2}^4 a_{2k} \cos^k(t), t_0 = 0, t_0 = 3\pi/2, a_{1k}, a_{2k} \in \mathbb{R},$$

then Eqs. (3.11) are satisfied. If  $a = b = 1$ ,  $a_{1k} = 5$ , and  $a_{2k} = 0.5$ , then we obtain a member in the family as shown in in Figure 5. Figure 6 shows the surface with  $a = b = 1$ ,  $a_{1k} = 3$ , and  $a_{2k} = 0.5$ .

Note that we could continue this series of surfaces, by choosing yet another combination of characteristic curve, or number of curves to interpolate.

#### 4. RULED SURFACES WITH A COMMON ASYMPTOTIC CURVE

In the following, we will construct a ruled surface from a given asymptotic curve. For the ease of discussion, let us assume that  $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$  is a unit speed curve in the Euclidean 3-space  $\mathbb{E}^3$ , where  $s$  is the arc length parameter. Suppose  $\mathbf{P}(s, t)$  is a ruled surface with the directrix  $\boldsymbol{\alpha}(s)$  and  $\boldsymbol{\alpha}(s)$  is also an isoparametric curve of  $\mathbf{P}(s, t)$ , then there exists  $t_0$  such that  $\mathbf{P}(s, t_0) = \boldsymbol{\alpha}(s)$ . This follows that the surface can be expressed as

$$M : \mathbf{P}(s, t) = \mathbf{P}(s, t_0) + (t - t_0)\mathbf{d}(s), 0 \leq s \leq L, \text{ with } t, t_0 \in [0, T],$$

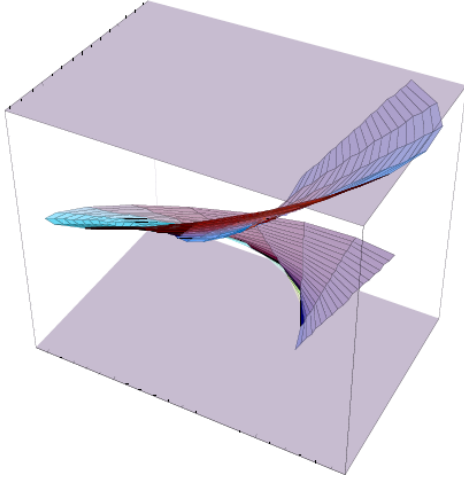


Figure 3.  $\mathbf{P}(r, t; -1; -1)$ .

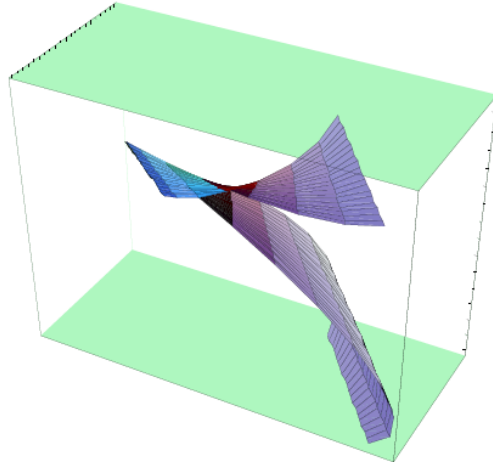


Figure 4.  $\mathbf{P}(r, t; -\frac{\sqrt{5}}{4}; -\frac{\sqrt{5}}{2})$ .

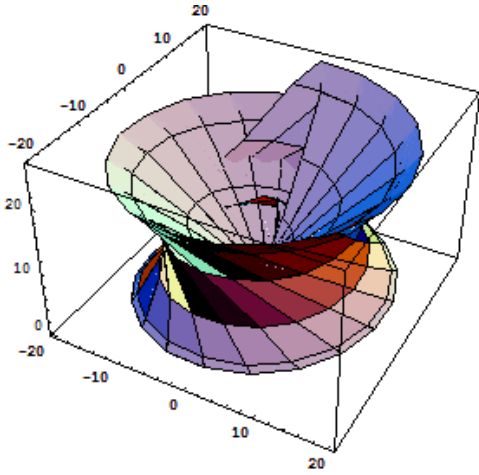


Figure 5.

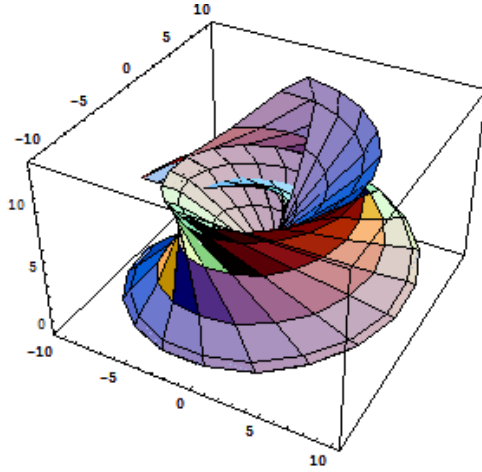


Figure 6.

where  $\mathbf{d}(s)$  denotes the direction of the rulings. According to the Eq. (3.12), we have

$$(17) \quad (t - t_0)\mathbf{d}(s) = \boldsymbol{\alpha}(s) + u(s, t)\mathbf{t}(s) + v(s, t)\mathbf{n}(s), \quad 0 \leq s \leq L, \quad \text{with } t, t_0 \in [0, T],$$

which is a system of two equations with two unknown functions  $u(s, t)$ , and  $v(s, t)$ . The solutions of the above system can be deduced as

$$(18) \quad \begin{aligned} u(s, t) &= (t - t_0) \langle \mathbf{d}(s), \mathbf{t}(s) \rangle = \det(\mathbf{d}(s), \mathbf{n}(s), \mathbf{b}(s)) \\ v(s, t) &= (t - t_0) \langle \mathbf{d}(s), \mathbf{n}(s) \rangle = \det(\mathbf{d}(s), \mathbf{b}(s), \mathbf{t}(s)). \end{aligned}$$

The above equations are just the necessary and sufficient conditions for which  $\mathbf{P}(s, t)$  is a ruled surface with a directrix  $\boldsymbol{\alpha}(s)$ .

Next, we need to check if the curve  $\boldsymbol{\alpha}(s)$  is also asymptotic on the surface  $\mathbf{P}(s, t)$  by using the conditions given in Theorem 3.1. It is evident that in this case, these conditions become

$$(19) \quad \begin{aligned} \det(\mathbf{d}(s), \mathbf{n}(s), \mathbf{b}(s)) &= 0, \\ \det(\mathbf{d}(s), \mathbf{b}(s), \mathbf{t}(s)) &\neq 0. \end{aligned}$$

It follows that at any point on the curve  $\boldsymbol{\alpha}(s)$ , the ruling direction  $\mathbf{d}(s)$  must be in the plane spanned by  $\mathbf{t}(s)$ , and  $\mathbf{n}(s)$ . On the other hand, the ruling direction  $\mathbf{d}(s)$ , and the vector  $\mathbf{t}(s)$  must not be parallel. This implies

$$(20) \quad \mathbf{d}(s) = \beta(s)\mathbf{t}(s) + \gamma(s)\mathbf{n}(s), \quad \gamma \neq 0, \quad 0 \leq s \leq L,$$

for some real functions  $\beta(s)$ , and  $\gamma(s)$ . Substituting it into the expressions in Eq. (4.3), we get

$$(21) \quad \beta(s)t = u(s, t), \quad \gamma(s)t = v(s, t), \quad \gamma \neq 0, \quad 0 \leq s \leq L.$$

Hence, the isoparametric asymptotic ruled surface family with the common asymptotic directrix  $\boldsymbol{\alpha}(s)$  can be expressed as

$$(22) \quad M : \mathbf{P}(s, t; \beta; \gamma) = \boldsymbol{\alpha}(s) + t(\beta(s)\mathbf{t}(s) + \gamma(s)\mathbf{n}(s)), \quad \gamma \neq 0, \quad 0 \leq s \leq L, \quad 0 \leq t \leq T,$$

where the functions  $\beta(s)$ , and  $\gamma(s)$  can control the shape of the surface pencil. Moreover, the normal vector to the ruled surface  $\mathbf{P}(s, t; \beta; \gamma)$  is

$$(23) \quad \mathbf{N}(s, t) = -t\gamma\tau(\gamma\mathbf{t} + \beta\mathbf{n}) + [\gamma(1 + t(\beta - \gamma\kappa) - t\beta(\dot{\gamma} + \beta\kappa))\mathbf{b},$$

and thus when  $t = 0$ — i.e., along the curve  $\boldsymbol{\alpha}(s)$ — the surface normal is

$$(24) \quad \mathbf{N}(s, 0) = -\gamma\mathbf{b}.$$

Coincidence of the osculating plane of  $\boldsymbol{\alpha}(s)$  with the tangent plane of  $\mathbf{P}(s, t; \beta; \gamma)$  identifies the curve as an asymptotic curve of the surface.

It should be pointed out that in this model, there exist two asymptotic curves passing through every point on the curve  $\boldsymbol{\alpha}(s)$ : one is  $\boldsymbol{\alpha}(s)$  itself and the other is a straight line in the direction  $\mathbf{d}(s)$  as given in Eq. (4.4). Every member of the isoasymptotic ruled surface pencil is decided by two pencil parameters  $\beta(s)$ , and  $\gamma(s)$ , i.e., by the direction vector function  $\mathbf{d}(s)$ . In Example 2: For  $\beta(r) = \gamma(r) = r$ , with  $t \in [0, 2]$ , and  $a = 2$ ,  $b = 1$  the corresponding ruled surface is shown in Fig. 7. Figure 8 shows the surface with  $\beta(r) = \gamma(r) = \sqrt{r}$ .

**4.1. Developable surfaces.** A special class of ruled surfaces, the developable surfaces, is characterized by a constant surface normal along each ruling. A surface is developable if and only if it is the envelope of a one-parameter family of planes. Developable surface and asymptotic curve play an important role in geometric design and surface analysis, see for example [11-16].

As is customary in the literature, there are three types of developable surfaces, the given curve can be classified into three kinds correspondingly. In what follows, we will discuss the



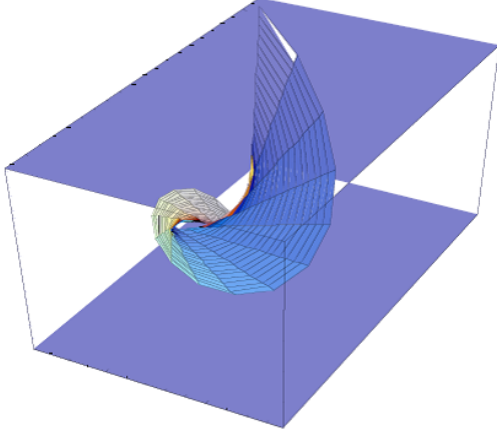


Figure 7.  $\mathbf{P}(r, t; r; r)$ .

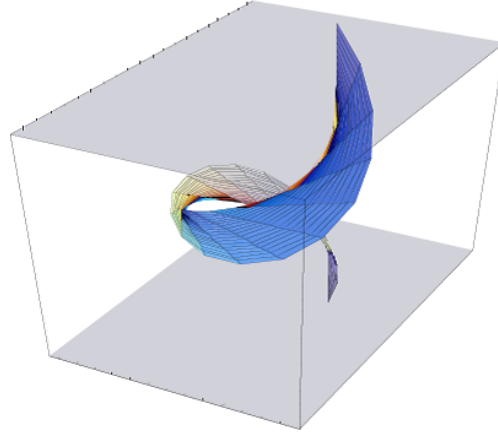


Figure 8.  $\mathbf{P}(r, t; \sqrt{r}; \sqrt{r})$ .

relationship between the given curve  $\alpha(s)$  and its isoparametric developable surface. In the light of above statements, we have:

$$(25) \quad \langle \mathbf{d} \times \dot{\mathbf{d}}, \dot{\alpha} \rangle = 0.$$

From Eq. (4.9), we note that there are two possible cases, as presented in the following:

The first case is when

$$(26) \quad \mathbf{d} \times \dot{\mathbf{d}} = \mathbf{0} \Leftrightarrow \dot{\beta} - \gamma\kappa = \mu\beta, \quad \dot{\gamma} + \beta\kappa = \mu\gamma, \quad \tau\gamma = 0,$$

where  $\mu = \mu(s)$  is a differentiable function. In this case, the ruled surface is referred to as a cylindrical surface. We may rewrite Eq. (4.10) as follows:

$$\frac{\dot{\beta} - \gamma\kappa}{\beta} = \frac{\dot{\gamma} + \beta\kappa}{\gamma} = \mu, \quad \tau\gamma = 0.$$

For  $\gamma \neq 0$ , it follows that  $\tau = 0$ , and we have:

$$(27) \quad \beta(s) = \gamma \tan\left(\int_0^s \gamma^2 \kappa ds\right).$$

Therefore, the cylindrical surface  $M$  is uniquely defined by the asymptotic curve  $\alpha = \alpha(s)$  and the differentiable function  $\gamma(s)$ .

**Corollary 4.1.** Planar curves are isoasymptotic to cylindrical surfaces.

By similar argument, we can also have the following:

$$\mathbf{d} \times \dot{\mathbf{d}} \neq \mathbf{0}.$$

This implies that  $M$  is a non-cylindrical surface. Therefore, the first derivative of the directrix is

$$(28) \quad \dot{\alpha}(s) = \dot{\mathbf{C}}(s) + \mu(s)\dot{\mathbf{d}}(s) + \dot{\mu}(s)\mathbf{d}(s),$$

where  $\dot{\mathbf{C}}$  is the first derivative of the striction curve,  $\mu(s)$  is a smooth function [1, 2]. Substituting Eq. (4.12) into Eq. (4.9) gives

$$(29) \quad \langle \mathbf{d} \times \dot{\mathbf{d}}, \dot{\mathbf{C}} \rangle = 0.$$

Similarly, there are two possible cases which satisfy Eq. (4.13), as presented in the following: The first case is when the first derivative of the striction curve is  $\dot{\mathbf{C}} = 0$ . Geometrically this condition implies that the striction curve degenerates to a point, and the ruled surface becomes a cone; the striction point of a cone is commonly referred to as the vertex. Therefore, the surface represented by Eq. (4.6) is a cone if and only if there exists a fixed point  $\mathbf{C}$  and a function  $\mu(s)$  such that

$$1 + \mu\kappa\gamma = \frac{d}{ds}(\mu\beta), \quad 0 = \mu\kappa\beta + \frac{d}{ds}(\mu\gamma), \quad \gamma\mu\tau = 0.$$

For  $\gamma = 0$ , it follows that  $\tau = 0$  ( $\alpha$  is planar curve), which implies the following conditions:

$$1 + \mu\kappa\gamma = \frac{d}{ds}(\mu\beta), \quad 0 = \mu\kappa\beta + \frac{d}{ds}(\mu\gamma).$$

Let  $\beta = 0$ , where  $\mu(s)$  is a non-constant function. In this case, according to  $\alpha$  is planar curve, we have

$$1 + \mu\kappa\gamma = 0, \quad 0 = \frac{d}{ds}(\mu\gamma),$$

it follows that  $\alpha$  is a planar curve with constant curvature, i.e.,

$$\mu\gamma = -\frac{1}{\kappa} = \text{const.},$$

**Corollary 4.2.** In Euclidean 3-space, there is no a cone whose base curve is a planar asymptotic curve.

The second case is when  $\dot{\mathbf{C}} \neq 0$ . Since  $\det(\dot{\mathbf{C}}, \mathbf{d}, \dot{\mathbf{d}}) = 0$ ,  $\langle \dot{\mathbf{C}}, \mathbf{d} \rangle = 0$ , and  $\langle \mathbf{d}, \dot{\mathbf{d}} \rangle = 0$ , we can get  $\dot{\mathbf{C}} \parallel \mathbf{d}$ . This means the surface is a tangent surface parameterized by

$$\mathbf{P}(s, t) = \alpha(s) + t\dot{\alpha}(s)$$

where  $\alpha(s)$  is called the curve of regression. Generally, when the given curve does not satisfy Corollaries 4.1 and 4.3, it is isoasymptotic to a tangent surface (Fig. 9).

## 5. CONCLUSION

In this study, we have presented a method for finding a surface family whose members all share a given asymptotic curve as an isoparametric curve. By representing the surface by the combination of the given curve, and the three vectors decomposed along the directions of the Frenet–Serret frame, we derive the necessary and sufficient conditions for the given curve to be the asymptotic curve for the parametric surface. In the process of derivation, we define two controlling functions  $\beta(s)$ , and  $\gamma(s)$ . From the examples, we find that these two functions can control the shape of the surface flexibly. We also present a new method for designing a developable surface by constructing a surface family passing through the

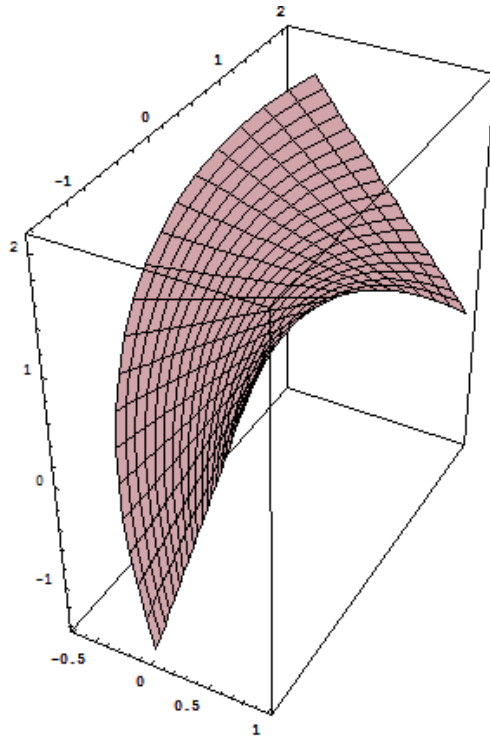


Figure 9. Construction of a cylinder  
 $\mathbf{P}(r, t; \sqrt{\sqrt{f(r)}/6})$

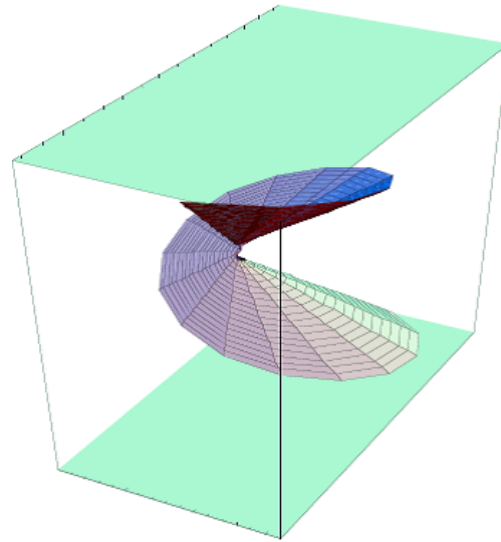


Figure 10. Construction of a tangent surface  $\mathbf{P}(r, t)$  passing through the given helix.

given asymptotic curve, which is quite in accord with the practice in industry design and manufacture.

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