ASIAN JOURNAL OF MATHEMATICS AND APPLICATIONS
Volume 2017, Article ID ama0387, 18 pages
ISSN 2307-7743
http://scienceasia.asia

BOUNDS FOR THE SOLUTION OF NONLOCAL FRACTIONAL DIFFERENTIAL EQUATION

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Abstract. In this paper we will find bounds for the nonlinear fractional differential equation with weighted initial and nonlocal conditions. These bounds we will find by extending the solution of a nonlinear fractional differential equation with weighted initial and nonlocal conditions to Maximal and Minimal solutions.

1. Introduction

In numerous fields of physical and technical sciences such as biomathematics, blood flow phenomena, ecology, environmental issues, visco-elasticity,aerodynamics, electrodynamics of complex medium, electrical circuits, electron-analytical chemistry, control theory, etc. the subject of Fractional Calculus is developing its attention,because the tools of Fractional Calculus have improved the mathematical modelling of many real world problems.The nonlocal nature of fractional order differential operator with variety of initial and boundary conditions in fractional differential equation can describe the hereditary properties of many important materials which overwhelm the interest in this subject.
Nonlinear fractional differential equation with weighted initial data has been studied by several authors. The weighted Cauchy-type problem

$$\begin{aligned} D^{\alpha}(u(t)) &= f(t,u(t)) \\ t^{1-\alpha}u(t)|_{t=0} &= b \end{aligned} \tag{1.1}$$

Studied by Khaled et al [15].
The solution of the periodic boundary value problem for a fractional differential equation involving a RiemannLiouville fractional derivative

$$\begin{aligned} D^{\alpha}(u(t)) &= f(t,u(t)) \\ t^{1-\alpha}u(t)|_{t=0} &= t^{1-\alpha}u(t)|_{t=T} \end{aligned} \tag{1.2}$$

Studied by Weia et al [16], Also the existence of solutions of fractional equations of Volterra type

Key words and phrases. Weighted nonlocal problem, nonlinear fractional differential equation, approximate solution.

with the RiemannLiouville derivative,

$$
\begin{aligned}
D^{\alpha}(u(t)) &= f(t, u(t), \int_0^t k(t,s)u(s)ds) \\
t^{1-\alpha}u(t)|_{t=0} &= r
\end{aligned}
\tag{1.3}
$$

Studied by Jankowski [17]. The weighted nonlocal fractional differential equation

$$
\begin{aligned}
{}^{c}D^{\alpha}(u(t)) &= f(t, u(t)) \\
\lim_{t\to 0^+} t^{1-\alpha}u(t) &= \sum_{i=1}^{m} a_i u(\tau_i)
\end{aligned}
\tag{1.4}
$$

studied by Holambe et al[13, 14] etc., and the references therein.Problems in nonlinear fractional differential equation were studied by various researchers.
The immportance of non-local problems appears to have been first noted in the literature by Bitsadze-Samarski[7].By Byszewski[18, 19],the nonlocal condition can be more useful than the standard initial contion to describe some physical phenomena.
Now here we consider the weighted nonlocal fractional differential equation

$$
\begin{aligned}
D^{\alpha}(u(t) - f(t, u(t))) + a(u(t) - f(t, u(t)) &= g(t, u(t)) \\
\lim_{t\to 0^+} t^{1-\alpha}u(t) &= u_0
\end{aligned}
\tag{1.5}
$$

where D^{α} is Riemann-Liouville fractional derivatives of order $0 < \alpha \leq 1$ and $0 < t \leq T < \infty$.

2. Auxiliary Results

let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is **regular** if $\{x_n\}_{n\in\mathbb{N}}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Clearly, the partially ordered Banach space $C(J, \mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space E may be found in Heikkilä and Lakshmikantham [**?**] and the references therein.

We need the following definitions.

Definition 2.1. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$.*

Definition 2.2 ([5])**.** *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.3. *A mapping $\mathcal{T} : E \to E$ is called* **partially bounded** *if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called* **uniformly partially bounded** *if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. $\mathcal{T}$ is called* **bounded** *if $\mathcal{T}(E)$ is a bounded subset of E.*

Definition 2.4. *A mapping $\mathcal{T} : E \to E$ is called* **partially compact** *if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **uniformly partially compact** *if $\mathcal{T}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Definition 2.5 ([5])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Definition 2.6 ([20])**.** *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(r) = 0$ iff $r = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called* ***partially nonlinear $\mathcal{D}$-Lipschitz*** *if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \le \psi(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathcal{T}$ is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \big\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\big\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } \ u, v \in E^+\}. \tag{2.2}$$

The elements of $\mathcal{K}$ are called the positive vectors of the normed linear algebra E. The following lemma follows immediately from the definition of the set $\mathcal{K}$ and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.7 ([3])**.** *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.8. *An operator $\mathcal{T} : E \to E$ is said to be positive if the range $R(\mathcal{T})$ of $\mathcal{T}$ is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.*

The method may be stated as "***the monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation***" which is a powerful tool in the existence theory of nonlinear analysis.

Theorem 2.9 ([6]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain of E. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ be nondecreasing operators such that*

(a) *$\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\psi_{\mathcal{A}}$.*
(b) *$\mathcal{B}$ is partially continuous and uniformly partially compact, and*
(c) *$M\psi_{\mathcal{A}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C$ is a chain in $E\}$, and*
(d) *there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0 + \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0 + \mathcal{B}x_0$.*

Then the operator equation

$$\mathcal{A}x + \mathcal{B}x = x \tag{2.3}$$

has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, $n = 0, 1, \ldots,$ converges monotonically to x^*.

Remark 2.10. The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E holds if every partially compact subset of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. Note that a subset S of the partially ordered Banach space $C(J, \mathbb{R})$ is called partially compact if every chain C in S is compact. This simple fact has been utilized to prove the main results of this paper.

3. Main Results

The equaivalent integral form of the problem 1.5 is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\le$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.1}$$

and

$$x \le y \iff x(t) \le y(t) \tag{3.2}$$

for all $t \in J$ respectively. Clearly, $C(J, \mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\le$. It is known that the partially ordered Banach algebra $C(J, \mathbb{R})$ has some nice properties concerning the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ in certain subsets of of it. The following lemma in this connection follows by an application of Arzelá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J,\mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(J,\mathbb{R})$.*

Proof. The lemma mentioned in Dhage [6], but the proof appears in Dhage [**?**]. □

We need the following definition in what follows.

Definition 3.2. *A function $u_l \in C(J,\mathbb{R})$ is said to be a lower solution of the problem (1.5) if it satisfies*

$$D^\alpha(u_l(t) - f(t,u_l(t))) + a(u_l(t) - f(t,u_l(t))) \leq g(t,u_l(t))$$
$$\lim_{t\to 0^+} t^{1-\alpha} u_l(t) \leq u_{l0}$$

for all $t \in J$. Similarly, a function $u_u \in C(J,\mathbb{R})$ is said to be an upper solution of the problem (1.5) if it satisfies the above inequalities with reverse sign.

Definition 3.3. *A function $f(t,u)$ is called Carathéodory if*

(i) *the map $t \mapsto f(t,u)$ is measurable for each $u \in \mathbb{R}$ and*
(ii) *the map $u \mapsto f(t,u)$ is continuous for each $t \in J$.*

A Carateoódory function f is called L^2-Carathéodory if

(iii) *there exists a function $h \in L^2(J,\mathbb{R})$ such that*

$$|f(t,u)| \leq h(t) \text{ a.e. } t \in J$$

We consider the following set of assumptions in what follows:

(A_1) The functions $f : J \times \mathbb{R} \to \mathbb{R}_+, \alpha : J \to \mathbb{R}_+$ where α is continuous function.
(A_2) There exist constants $M, M_f > 0$ such that $0 \leq t^{\alpha-1} \leq M$ and $0 \leq f(t,x) \leq M_f$ for all $t \in J$ and $x \in \mathbb{R}$.
(A_3) There exists a $\mathcal{D}$-function ψ_f such that

$$0 \leq f(t,x) - f(t,y) \leq \psi_f(x-y)$$

for all $t \in J$ and $x, y \in \mathbb{R}$,$x \leq y$.
(A_4) $f(t,x)$ is nondecreasing in x for all $t \in J$.
(A_5) The problem (1.5) has a lower solution $u_l \in C(J,\mathbb{R})$.

The following lemma is useful in what follows.

Lemma 3.4. *For any $f \in C(J \times \mathbb{R}, \mathbb{R}))$,if u is a solution of the problem*

$$D^\alpha(u(t) - f(t,u(t))) + a(u(t) - f(t,u(t)) = g(t,u(t))$$
$$\lim_{t\to 0^+} t^{1-\alpha} u(t) = u_0$$
$$0 < \alpha \leq 1, \quad 0 < t \leq T < \infty$$

then

$$u(t) = f(t,u(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s,u(s))ds \tag{3.3}$$

where $c = u_0 - f(0,u_0)$ *and* $E_{\alpha,\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha(k+1))}$ *is the classical Mittag-Leffler function and vice-versa.*

Proof. The solution followed by [1, 2] □

Theorem 3.5. *Assume that hypotheses* (A_1)-(A_5) *hold. Furthermore, assume that*

$$\left(cT^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + M_g T^{\alpha-1}\{1 - E_{\alpha,\alpha}(a(t)^{\alpha})\}\right)\psi_f(r) < r, \ r > 0, \tag{3.4}$$

then the FDE(1.5) has a solution x^* *defined on* J *and the sequence* $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ *of successive approximations defined by*

$$x_{n+1}(t) = f(t,x(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s,x_n(s))ds \tag{3.5}$$

for all $t \in J$, *where* $c = x_0 - f(0,x_0)$, *converges monotonically to* x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define three operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = f(t,x(t)), \ t \in J, \tag{3.6}$$

and

$$\mathcal{B}x(t) = ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s,x(s))ds, \ t \in J. \tag{3.7}$$

From the continuity of the integral and the hypotheses (A_1)-(A_5), it follows that $\mathcal{A}$ and $\mathcal{B}$ define the maps $\mathcal{A},\mathcal{B} : E \to \mathcal{K}$. Now by definitions of the operators $\mathcal{A}$ and $\mathcal{B}$, the FDE (1.5) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ \ t \in J. \tag{3.8}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.9. This is achieved in the series of following steps.

Step I: $\mathcal{A}$ *and* $\mathcal{B}$ *are nondecreasing on* E.

Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis , we obtain

$$\begin{aligned} \mathcal{A}x(t) &= f(t,x(t)), t \in J, \\ &\geq f(t,y(t)), t \in J, \end{aligned}$$

$$= \mathcal{A}y(t)$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operators on E into E. Similarly, using hypothesis (A_4),

$$\begin{aligned}
\mathcal{B}x(t) &= ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s,x(s))ds \\
&\geq ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s,y(s))ds \\
&= \mathcal{B}y(t)
\end{aligned}$$

for all $t \in J$. Hence, it is follows that the operator $\mathcal{B}$ is also a nondecreasing operator on E into itself. Thus, $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing positive operators on E into itself.

Step II: *$\mathcal{A}$ is partially bounded and partially $\mathcal{D}$-Lipschitz on E.*

Let $x \in E$ be arbitrary. Then by (A_2),taking supremum over t,we get

$$\begin{aligned}
\|\mathcal{A}x\| &\leq \sup_{t\in J} |\mathcal{A}x(t)| \\
&\leq \sup_{t\in J} |f(t,x(t))| \\
&\leq M_f
\end{aligned}$$

so $\mathcal{A}$ is bounded and is partially bounded on E.

Next, let $x, y \in E$ be such that $x \leq y$. Then, by hypothesis (A_3),

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| &= |f(t,x(t)) - f(t,y(t))| \\
&= f(t,x(t)) - f(t,y(t)) \\
&\leq \psi(x(t) - y(t)) \\
&\leq \psi|x(t) - y(t)| \\
&\leq \psi(|x-y|),
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \psi(\|x-y\|)$$

for all $x, y \in E$ with $x \leq y$. Hence $\mathcal{A}$ is partially nonlinear $\mathcal{D}$-Lipschitz operators on E which further implies it is also a partially continuous on E into itself.

Step III: *$\mathcal{B}$ is a partially continuous operator on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\lim_{n\to\infty} \mathcal{B}x_n(t) = \lim_{n\to\infty} \left\{ ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s,x_n(s))ds \right\}$$

$$
\begin{aligned}
&= ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \lim_{n\to\infty}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s,x_n(s))ds \\
&= ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})\ \Big[\lim_{n\to\infty} g(s,x_n(s))\Big]\ ds \\
&= ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s,x(s))ds \\
&= \mathcal{B}x(t),
\end{aligned}
$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges monotonically to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$
\begin{aligned}
&\Big|Bx_n(t_2) - Bx_n(t_1)\Big| \\
&\quad = \Big|ct_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})\Gamma(\alpha) + \int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})g(s,x_n(s))ds \\
&\qquad - ct_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})\Gamma(\alpha) - \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x_n(s))ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le c\Gamma(\alpha)|t_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha}) - t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})| \\
&\qquad + \Big|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})g(s,x_n(s))ds \\
&\qquad - \int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x_n(s))ds\Big| \\
&\quad + \Big|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x_n(s))ds \\
&\qquad - \int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x_n(s))ds\Big| \\
&\quad + \Big|\int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x_n(s))ds \\
&\qquad - \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x_n(s))ds\Big| \\
&\le c\Gamma(\alpha)|t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha}) - t_1^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha}) + t_1^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha}) - t_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})| \\
&\qquad + \int_0^{t_2}(t_2-s)^{\alpha-1}\left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|\left|g(s,x_n(s))\right| ds \\
&\quad + \Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x_n(s))ds\Big|
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&+\int_0^{t_1} \left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\, |g(s,x_n(s))|\, ds \\
&\le c\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^{\alpha})-E_{\alpha,\alpha}(-at_2^{\alpha})|+E_{\alpha,\alpha}(-at_2^{\alpha})|t_1^{\alpha-1}-t_2^{\alpha-1}|\right\} \\
&\qquad +\int_0^{T}(t_2-s)^{\alpha-1}\left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})-E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right| M_g ds \\
&+\int_0^{T}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})M_g ds \\
&+\int_0^{T}\left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})M_g ds \\
&\le c\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^{\alpha})-E_{\alpha,\alpha}(-at_2^{\alpha})|+E_{\alpha,\alpha}(-at_2^{\alpha})|t_1^{\alpha-1}-t_2^{\alpha-1}|\right\} \\
&\qquad + M_g\left(\int_0^T \left|(t_2-s)^{\alpha-1}\right|^2 ds\right)^{1/2}\left(\int_0^T \left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})-E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|^2 ds\right)^{1/2} \\
&+2\left(\int_0^T \left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^2 ds\right)^{1/2}\left(\int_0^T \left|E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|^2 ds\right)^{1/2} M_g
\end{aligned}
\tag{3.10}
$$

Since the functions $E_{\alpha,\alpha}$ and α are continuous on compact interval J so uniformly continuous there. Therefore, from the above inequality (3.9) it follows that

$$|\mathcal{B}x_n(t_2)-\mathcal{B}x_n(t_1)|\to 0 \quad \text{as} \quad n\to\infty$$

uniformly for all $n\in\mathbb{N}$. This shows that the convergence $\mathcal{B}x_n\to\mathcal{B}x$ is uniform and hence $\mathcal{B}$ is partially continuous on E.

Step IV: *$\mathcal{B}$ is uniformly partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y\in\mathcal{B}(C)$ be any element. Then there is an element $x\in C$ be such that $y=\mathcal{B}x$. Now, by hypothesis ,

$$
\begin{aligned}
|y(t)| &= |ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha)+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s,x(s))ds| \\
&\le |cT^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha)+M_gT^{\alpha-1}\{1-E_{\alpha,\alpha}(a(t)^{\alpha})\}| \\
&= r
\end{aligned}
$$

for all $t\in J$. Taking the supremum over t, we obtain $\|y\|\le\|\mathcal{B}x\|\le r$ for all $y\in\mathcal{B}(C)$. Hence, $\mathcal{B}(C)$ is a uniformly bounded subset of E. Moreover, $\|\mathcal{B}(C)\|\le r$ for all chains C in E. Hence, $\mathcal{B}$ is a uniformly partially bounded operator on E.

Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $y \in \mathcal{B}(C)$, one has

$$
\begin{aligned}
&\Big|Bx(t_2) - Bx(t_1)\Big| \\
&\quad = \Big|ct_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})\Gamma(\alpha) + \int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})g(s,x(s))ds \\
&\qquad\quad - ct_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})\Gamma(\alpha) - \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x(s))ds\Big| \\
&\quad \le c\Gamma(\alpha)|t_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha}) - t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})| \\
&\qquad\quad + \Big|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})g(s,x(s))ds \\
&\qquad\quad - \int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x(s))ds\Big| \\
&\quad + \Big|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x(s))ds \\
&\qquad\quad - \int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x(s))ds\Big| \\
&\quad + \Big|\int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x(s))ds \\
&\qquad\quad - \int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x(s))ds\Big| \\
&\quad \le c\Gamma(\alpha)|t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha}) - t_1^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha}) + t_1^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha}) - t_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})| \\
&\qquad\quad + \int_0^{t_2}(t_2-s)^{\alpha-1}\left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|\left|g(s,x(s))\right|ds \\
&\quad + \Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s,x(s))ds\Big| \\
&\quad + \int_0^{t_1}\left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\left|g(s,x(s))\right|ds \\
&\quad \le c\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^{\alpha}) - E_{\alpha,\alpha}(-at_2^{\alpha})| + E_{\alpha,\alpha}(-at_2^{\alpha})|t_1^{\alpha-1} - t_2^{\alpha-1}|\right\} \\
&\qquad\quad + \int_0^{T}(t_2-s)^{\alpha-1}\left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha}) - E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|M_g ds \\
&\quad + \int_0^{T}\left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right|E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})M_g ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^T \left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^\alpha)M_g ds \\
&\leq c\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^\alpha)-E_{\alpha,\alpha}(-at_2^\alpha)|+E_{\alpha,\alpha}(-at_2^\alpha)|t_1^{\alpha-1}-t_2^{\alpha-1}|\right\} \\
&\qquad + M_g\left(\int_0^T \left|(t_2-s)^{\alpha-1}\right|^2 ds\right)^{1/2}\left(\int_0^T |E_{\alpha,\alpha}(-a(t_2-s)^\alpha)-E_{\alpha,\alpha}(-a(t_1-s)^\alpha)|^2 ds\right)^{1/2} \\
&+ 2\left(\int_0^T \left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^2 ds\right)^{1/2}\left(\int_0^T |E_{\alpha,\alpha}(-a(t_1-s)^\alpha)|^2 ds\right)^{1/2} M_g \\
&\longrightarrow 0 \quad \text{as} \quad t_1 \to t_2,
\end{aligned}
$$

uniformly for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathcal{B}$ is a uniformly partially compact operator on E into itself.

Step V: *u_l satisfies the operator inequality $u_l \leq \mathcal{A}u_l + \mathcal{B}u_l$.*

By hypothesis (A_5), the FDE 1.5 has a lower solution u_l defined on J. Then, we have

$$
u_l(t) \leq f(t,u_l(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)f(s,u_l(s))ds \tag{3.11}
$$

for all $t \in J$. From the definitions of the operators $\mathcal{A}$ and $\mathcal{B}$ it follows that $u_l(t) \leq \mathcal{A}u_l(t) + \mathcal{B}u_l(t)$ for all $t \in J$. Hence $u_l \leq \mathcal{A}u_l + \mathcal{B}u_l$.

Step VI: *The $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ satisfy the growth condition $M\psi_{\mathcal{A}}(r) < r$, for $r > 0$.*

Finally, the $\mathcal{D}$-function $\psi_{\mathcal{A}}$ of the operator $\mathcal{A}$ satisfy the inequality given in hypothesis (d) of Theorem 2.9, viz.,

$$
M\psi_{\mathcal{A}}(r) \leq r
$$

for all $r > 0$.

Thus $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.9 and we conclude that the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution. Consequently the FDE (1.5) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof. □

The conclusion of Theorems 3.5 also remains true if we replace the hypothesis (A_5) with the following one:

(A_5') The FDE (1.5) has an upper solution $u_u \in C(J,\mathbb{R})$.

The proof of Theorem 3.5 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications. We need the following definition in what follows.

Definition 3.6. *A function $r \in C(J, \mathbb{R})$ is said be a maximal solution of the FDE (1.5)if for any other solution x of the FDE (1.5), one has $x(t) \leq r(t)$ for all $t \in J$. Similarly, a minimal solution ρ of the FDE (1.5) can be defined in a similar way by reversing the above inequality.*

The following lemma is fundamental in the proof of maximal and minimal solutions for the FDE (1.5) on J.

Lemma 3.7. *Suppose that there exist two functions $y, z \in C(J, \mathbb{R})$ satisfying*

$$y(t) \leq f(t, y(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s, y(s))ds \tag{3.12}$$

and

$$z(t) \geq f(t, z(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s, z(s))ds \tag{3.13}$$

for all $t \in J$. If one of the inequalities (3.12) *and* (3.13) *is strict, then*

$$y(t) < z(t) \tag{3.14}$$

for all $t \in J$.

Proof. Suppose that the inequality (3.13) is strict and let the conclusion (3.14) be false. Then there exists $t_1 \in J$ such that

$$y(t_1) = z(t_1), t_1 > 0,$$

and

$$y(t) < z(t), 0 < t < t_1.$$

From the monotonicity of $f(t, x)$ in x , we get

$$\begin{aligned} y(t_1) &\leq f(t_1, y(t_1)) + ct_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})\Gamma(\alpha) + \int_0^{t_1} (t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s, y(s))ds \\ &= f(t_1, z(t_1)) + ct_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})\Gamma(\alpha) + \int_0^{t_1} (t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g(s, z(s))ds \\ &< z(t_1) \qquad (3.15) \end{aligned}$$

which contradicts the fact that $y(t_1) = z(t_1)$. Hence, $y(t) < z(t)$ for all $t \in J$. □

Theorem 3.8. *Suppose that all the hypotheses of Theorem 3.5 hold. Then the FDE (1.5) has a maximal and a minimal solution on J.*

Proof. Let $\epsilon > 0$ be given. Now consider the fractional integral equation

$$x_\epsilon(t) = f_\epsilon(t, x_\epsilon(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g_\epsilon(s, x_\epsilon(s))ds \tag{3.16}$$

for all $t \in J$, where

$$f_\epsilon(t, x_\epsilon(t)) = f(t, x_\epsilon(t)) + \epsilon$$

and

$$g_\epsilon(t, x_\epsilon(t)) = g(t, x_\epsilon(t)) + \epsilon$$

Clearly the function $f_\epsilon(t, x_\epsilon(t))$,satisfy all the hypotheses (A_1)-(A_5)and therefore, by Theorem 3.5, FDE (1.5) has at least a solution $x_\epsilon(t) \in C(J, \mathbb{R})$.

Let ϵ_1 and ϵ_2 be two real numbers such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then, we have

$$\begin{aligned} x_{\epsilon_2}(t) &= f_{\epsilon_2}(t, x_{\epsilon_2}(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)g_{\epsilon_2}(s, x_{\epsilon_2}(s))ds \\ &= [f(t, x_{\epsilon_2}(t)) + \epsilon_2] + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)[g(s, x_{\epsilon_2}(s)) + \epsilon_2]ds \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} x_{\epsilon_1}(t) &= f_{\epsilon_1}(t, x_{\epsilon_1}(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)g_{\epsilon_1}(s, x_{\epsilon_1}(s))ds \\ &= [f(t, x_{\epsilon_1}(t)) + \epsilon_1] + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha_1) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)[g(s, x_{\epsilon_1}(s)) + \epsilon_1]ds \\ &> [f(t, x_{\epsilon_2}(t)) + \epsilon_2] + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)[g(s, x_{\epsilon_2}(s)) + \epsilon_2]ds \end{aligned} \tag{3.18}$$

for all $t \in J$. Now, applying the Lemma 3.7 to the inequalities (3.17) and (3.18), we obtain

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \tag{3.19}$$

for all $t \in J$.

Let $\epsilon_0 = \epsilon$ and define a decreasing sequence $\{\epsilon_n\}_{n=0}^\infty$ of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Then in view of the above facts $\{x_{\epsilon_n}\}$ is a decreasing sequence of functions in $C(J, \mathbb{R})$. We show that is is uniformly bounded and equicontinuous. Now, by hypotheses,

$$\begin{aligned} |x_{\epsilon_n}(t)| &\le |f_{\epsilon_n}(t, x_{\epsilon_n}(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)g_{\epsilon_n}(s, x_{\epsilon_n}(s))ds| \\ &\le (M_f + \epsilon + cME_{\alpha,\alpha}(-aT^\alpha)\Gamma(\alpha)) + \epsilon + \big(M_gT^{\alpha-1}(1 - E_{\alpha,\alpha}(aT^\alpha))\big) + \epsilon \\ &\le r \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain $\|x_{\epsilon_n}\| \le r$ for all $n \in \mathbb{N}$. This shows that the sequence $\{x_{\epsilon_n}\}$ is uniformly bounded.

Next we show that $\{x_{\epsilon_n}\}$ is an equicontinuous sequence of functions in $C(J, \mathbb{R})$. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$|x_{\epsilon_n}(t_1) - x_{\epsilon_n}(t_2)| \le |f_{\epsilon_n}(t_1, x_{\epsilon_n}(t_1)) + ct_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^\alpha)\Gamma(\alpha)$$

$$+\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds$$
$$-f_{\epsilon_n}(t_2,x_{\epsilon_n}(t_2))-ct_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})\Gamma(\alpha)$$
$$-\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds|$$

(3.20)
$$=|f_{\epsilon_n}(t_1,x_{\epsilon_n}(t_1))-f_{\epsilon_n}(t_2,x_{\epsilon_n}(t_2))|$$

(3.21)
$$+c\Gamma(\alpha)|t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})-t_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})|$$
$$+|\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds$$
$$-\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds|$$

(3.22)
$$\leq|f_{\epsilon_n}(t_1,x_{\epsilon_n}(t_1))-f_{\epsilon_n}(t_2,x_{\epsilon_n}(t_2))|$$
$$+c\Gamma(\alpha)|t_1^{\alpha-1}E_{\alpha,\alpha}(-at_1^{\alpha})-t_1^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})+t_1^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})-t_2^{\alpha-1}E_{\alpha,\alpha}(-at_2^{\alpha})|$$
$$+\left|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds\right.$$
$$\left.-\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds\right|$$
$$+\left|\int_0^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds\right.$$
$$\left.-\int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds\right|$$
$$+\left|\int_0^{t_1}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds\right.$$
$$\left.-\int_0^{t_1}(t_1-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds\right|$$

(3.23)
$$\leq|f_{\epsilon_n}(t_1,x_{\epsilon_n}(t_1))-f_{\epsilon_n}(t_2,x_{\epsilon_n}(t_2))|$$
$$+c\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^{\alpha})-E_{\alpha,\alpha}(-at_2^{\alpha})|+E_{\alpha,\alpha}(-at_2^{\alpha})|t_1^{\alpha-1}-t_2^{\alpha-1}|\right\}$$
$$+\int_0^{t_2}(t_2-s)^{\alpha-1}\left|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})-E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\right|\left|g_{\epsilon_n}(s,x_{\epsilon_n}(s))\right|ds$$
$$+\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds\right|$$

$$+\int_0^{t_1} \left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})\,|g_{\epsilon_n}(s,x_{\epsilon_n}(s))|\,ds \tag{3.24}$$

$$\begin{aligned}
&\leq |f_{\epsilon_n}(t_1,x_{\epsilon_n}(t_1))-f_{\epsilon_n}(t_2,x_{\epsilon_n}(t_2))| \\
&\qquad + c\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^{\alpha})-E_{\alpha,\alpha}(-at_2^{\alpha})|+E_{\alpha,\alpha}(-at_2^{\alpha})|t_1^{\alpha-1}-t_2^{\alpha-1}|\right\} \\
&\quad + \int_0^T (t_2-s)^{\alpha-1}\,|E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})-E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})|\,[M_g+\epsilon_n]ds \\
&+\int_0^T \left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})[M_g+\epsilon_n]ds \\
&+\int_0^T \left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right| E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})[M_g+\epsilon_n]ds
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&\leq |f_{\epsilon_n}(t_1,x_{\epsilon_n}(t_1))-f_{\epsilon_n}(t_2,x_{\epsilon_n}(t_2))| \\
&\qquad + c\Gamma(\alpha)\left\{t_1^{\alpha-1}|E_{\alpha,\alpha}(-at_1^{\alpha})-E_{\alpha,\alpha}(-at_2^{\alpha})|+E_{\alpha,\alpha}(-at_2^{\alpha})|t_1^{\alpha-1}-t_2^{\alpha-1}|\right\} \\
&+[M_g+\epsilon_n]\left(\int_0^T \left|(t_2-s)^{\alpha-1}\right|^2 ds\right)^{1/2}\left(\int_0^T |E_{\alpha,\alpha}(-a(t_2-s)^{\alpha})-E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})|^2\,ds\right)^{1/2} \\
&+2\left(\int_0^T \left|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right|^2 ds\right)^{1/2}\left(\int_0^T |E_{\alpha,\alpha}(-a(t_1-s)^{\alpha})|^2\,ds\right)^{1/2}[M_g+\epsilon_n]
\end{aligned} \tag{3.26}$$

Since the functions f and $E_{\alpha,\alpha}$ are continuous on compact $[0,T]\times[-r,r]\times[-r,r]$, $(t-s)^{1-\alpha}$ is continuous on compact $[0,T]\times[0,T]$, so uniformly continuous there. Hence, from (3.20) it follows that

$$|x_{\epsilon_n}(t_1)-x_{\epsilon_n}(t_2)|\to 0 \quad \text{as} \quad t_1\to t_2$$

uniformly for all $n\in\mathbb{N}$. As a result $\{x_{\epsilon_n}\}$ is an equicontinuous sequence of functions in $C(J,\mathbb{R})$. Now the sequence $\{x_{\epsilon_n}\}$ is uniformly bounded and equicontinuous, so it is compact in view of Arzelá-Ascoli theorem. By Lemma 3.1, $\{x_{\epsilon_n}\}$ converges uniformly to a function say $r\in C(J,\mathbb{R})$, i.e. $\lim_{n\to\infty} x_{\epsilon_n}(t)=r(t)$ uniformly on J.

We show that the function r is a solution of the FDE (1.5) on J. Now, $\{x_{\epsilon_n}\}$ is a solution of the FDE

$$x_{\epsilon_n}(t)=f_{\epsilon_n}(t,x_{\epsilon_n}(t))+ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha)+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g_{\epsilon_n}(s,x_{\epsilon_n}(s))ds$$

(3.27)

(3.28)

$$= [f(t, x_{\epsilon_n}(t)) + \epsilon_n] + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})[g(s, x_{\epsilon_n}(s)) + \epsilon_n]ds$$

for all $t \in J$. Now, taking the limit as by hypotheses $n \to \infty$ in the above inequality (3.27), we obtain

$$r(t) = f(t, r(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s, r(s))ds$$

for all $t \in J$. This shows that r is a solution of the FDE (1.5) defined on J.

Finally, we shall show that $r(t)$ is the maximal solution of the FDE (1.5) defined on J. To do this, let $x(t)$ be any solution of the FDE (1.5) defined on J. Then,we have

$$x(t) = f(t, x(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s, x(s))ds \tag{3.29}$$

for all $t \in J$. Similarly, if x_ϵ is any solution of the FDE

(3.30)

$$x_\epsilon(t) = [f(t, x_\epsilon(t)) + \epsilon] + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})[g(s, x_\epsilon(s)) + \epsilon]ds$$

then,

$$x_\epsilon(t) > f(t, x_\epsilon(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s, x_\epsilon(s))ds \tag{3.31}$$

for all $t \in J$. From the inequalities (3.29) and (3.31) it follows that $x(t) \leq x_\varepsilon(t),\ \ t\ J$. Taking the limit as $\epsilon \to 0$, we obtain $x(t) \leq r(t)$ for all $t \in J$. Hence r is a maximal solution of the FDE (1.5) defined on J. In the same way Minimal solution of the FDE can be obtained ☐

Further we prove now that the maximal and minimal solutions serve as the bounds for the solutions of the related differential inequality to FDE (1.5) on $J = [0, T]$.

Theorem 3.9. *Suppose that all the hypotheses of Theorem 3.5 hold. Further, if there exists a function $u \in C(J, \mathbb{R})$ such that*

$$u(t) \leq f(t, u(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^{\alpha})g(s, u(s))ds \tag{3.32}$$

for all $t \in J$, then,

$$u(t) \leq r(t) \tag{3.33}$$

for all $t \in J$, where r is a maximal solution of the FDE (1.5) on J.

Proof. Let $\epsilon > 0$ be arbitrary small. Then, by Theorem 3.5, $r_\epsilon(t)$ is a solution of the FDE and that the limit

$$r(t) = \lim_{\epsilon \to 0} r_\epsilon(t) \tag{3.34}$$

is uniform on J and is a maximal solution of the FDE (1.5) on J. Hence, we obtain

$$r_\epsilon(t) = [f(t, r_\epsilon(t)) + \epsilon] + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)[g(s, r_\epsilon(s)) + \epsilon]ds \tag{3.35}$$

for all $t \in J$. From the above inequality it follows that

$$r_\epsilon(t) > f(t, r_\epsilon(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)g(s, r_\epsilon(s))ds \tag{3.36}$$

Now we apply Lemma 3.7 to the inequalities (3.32) and (3.36) and conclude that

$$u(t) < r_\epsilon(t) \tag{3.37}$$

for all $t \in J$. This further in view of limit (3.34) implies that the inequality (3.33) holds on J. This completes the proof. □

Similarly, we have the following result for the FDE (1.5) on J.

Theorem 3.10. *Suppose that all the hypotheses of Theorem 3.5 hold. Further, if there exists a function $v \in C(J, \mathbb{R})$ such that*

$$v(t) \geq f(t, v(t)) + ct^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha)\Gamma(\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)g(s, v(s))ds \tag{3.38}$$

for all $t \in J$, then,

$$v(t) \geq \rho(t) \tag{3.39}$$

for all $t \in J$, where ρ is a minimal solution of the FDE (1.5) on J.

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