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RICCI-QUADRATIC HOMOGENEOUS EINSTEIN METRICS

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Abstract. In this paper, we introduce the notation of Einstein-reversibility for Finsler metrics. A characteristic condition of Einstein-reversibility to be Ricci-quadratic is given, and we obtain theorem that characterizes Ricci-quaratic Einstein metric. We prove that a homogeneous Einstain metrics is Ricci-quadratic if and only if it is Berwald type.

1. Introduction

A Finsler space is a manifold M equipped with a family of smoothly varying Minkowsky norms, one on each tangent space. Riemannian metrics are examples of Finsler norms that arise from an inner-product. A Finsler metric function $L(x,y)$ is called an (α,β) -metric if L is a positively homogeneous function of a Riemannian metric $\alpha(x,y)=\sqrt{a_{ij}y^iy^j}$ and a differential 1-form $\beta(x,y)=b_i(x)y^i$ of degree one.

Finsler metrics are Riemann metrics without quadratic restriction. For a Finsler metric $F=F(x,y)$, its locally minimizing curves are characterized by a system of differential equations:

$$\frac{d^2x^i}{dt^2}+2G^i(x,\frac{dx}{dt})=0,$$

where the local functions $G_i=G^i(x,y)$ are called the spray coefficients. If $F=\sqrt{g_{ij}(x)y^iy^j}$ is Riemannian, then $G^i=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ are quadratic in $y\in T_xM$. This quadratic property is crucial in the regularity of the exponential map $exp_x:T_xM\to M$ at the origin of T_xM. Namely, exp_x is C^∞ at the origin $0\in T_xM$ at any point x if and only if the spray coefficients of F are quadratic in $y\in T_xM$ at any point x. There are non-Riemannian metrics whose spray coefficients still have this quadratic property. Finsler metrics with this property are

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called Berwald metrics. It is known that every Berwald metric has the same geodesics as a Riemannian metric [15]. Thus Berwald metrics can be identified with Riemannian metrics at geodesic level. The Riemann curvature is a family of linear maps

$R_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k : T_xM \longrightarrow T_xM$, given by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i \partial G^j}{\partial y^j \partial y^k}.$$

From the above formula, one can see that if F is a Berwald metric, then $R^i_k = R^i_k(x, y)$ are quadratic in $y \in T_xM$.

Guojun Yang in [5] study the reversibility of Einstein scalar in Finsler geometry, it is shown that Ricci curvature of Einstein metrics is reversible if and only if the Ricci curvature is quadratic. For a Finsler metric $F = F(x, y)$ on a manifold M, the Riemann curvature $R_y : T_xM \to T_xM$ is a family of linear transformations and the Ricci curvature $Ric(x, y) = trace(Ry), \forall y \in T_xM$.

We can always express the Ricci curvature $Ric(x, y)$ as

$$Ric = (n - 1)\lambda(x, y)F^2,$$

for some scalar $\lambda(x, y)$ on TM, where $\lambda(x, y)$ is called the Einstein scalar. A Finsler metric F is called of Einstein-reversibility if the Einstein scalar is reversible, namely,

$$\lambda(x, y) = \lambda(x, -y).$$

If the Einstein scalar $\lambda(x, y)$ is a scalar on M, namely, $\lambda(x, y) = \lambda(x)$, then F is called an Einstein metric. Einstein metrics are a natural extension of those in Riemann geometry and they have been shown to have similar good properties as in Riemann geometry for some special Finsler metrics (see more [13], [14], [16], [17], [18], and [19]).

A Finsler metric is called Ricci-quadratic if its Ricci curvature $Ric(x, y)$ is quadratic in y. It is clear that the notion of Ricci-quadratic metrics is weaker than that of R-quadratic metrics. A. Tayebi, T. Tabatabaeifar [21] proved that a Matsumoto metric is Ricci-reversible if and only if it is Ricci-quadratic. It is therefore obvious that any R-quadratic Finsler space must be Ricci-quadratic, in particular, any Berwald space must be Ricci-quadratic. However, there are many non-Berwald spaces which are Ricci-quadratic. In general, it is quite difficult to characterize Ricci-quadratic metrics. Li and Shen in [20] considered the case of Randers metrics in and obtained a characterization of Ricci-quadratic properties of such spaces.

Z.Hu and S. Deng in [6] prove homogeneous Randers spaces is Ricci quadratic if and only if is Berwald type. In this paper we obtain some results on characterizetion of Ricci-quaratic Einstein metric and we prove that homogeneous Einstein metrics is Ricci-quadratic if and only if it is of Berwald type.

2. **Preliminaries**

Definition 2.1. *Let V be an n-dimensional real vector space. A Minkowski norm on V is a real function F on V which is smooth on $V-\{0\}$ and satisfies the following conditions:*

- $F(u)\geq 0, \forall u\in V$;
- $F(\lambda u)=\lambda F(u), \forall\lambda>0$;
- *Given any basis* $u_1, u_2, ..., u_n$ *of* V, *write* $F(y)=F(y^1, y^2, ..., y^n)$ *for* $y=y^1u_1+y^2u_2+ ... +y^nu_n$.

Then the Hessian matrix

$$(g_{ij}):=([\frac{1}{2}F^2]_{y^iy^j}),$$

is positive-definite at any point of $V-\{0\}$.

For example, let $\langle,\rangle$ *be an inner product on* V. *Define* $F(y)=\sqrt{<y,y>}$. *Then* F *is a Minkowski norm. In this case it is called Euclidean or coming from an inner product.*

It can be shown that for a Minkowski norm F, *we have* $F(u)>0$, $\forall u\neq 0$. *Furthermore*

$$F(u_1+u_2)\leq F(u_1)+F(u_2),$$

where the equality holds if and only if $u_2=\alpha u_1$ *or* $u_1=\alpha u_2$ *for some* $\alpha\geq 0$.

For any Minkowski norm F *on real vector space* V *we define*

$$C_{ijk}=\frac{1}{4}[F_2]y^iy^jy^k.$$

Then for any $y \neq 0$, we can define two tensors on V, namely,

$$g_y(u,v) = \sum_{i,j=n}^{n} g_{ij}(y)u^i v^j,$$

$$C_y(u,v,w) = \sum_{i,j,k=1}^{n} C_{ijk}(y)u^i v^j w^k.$$

They are called the fundamental tensor and the Cartan tensor, respectively.

For a Finsler metric F on an n-dimensional manifold M and G^i be the geodesic coefficients of F , which are defined by

$$G^i := \frac{1}{4}g^{il}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\}.$$

For any $x \in M$ and $y \in T_xM\backslash\{0\}$, the Riemann curvature $R_y := R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$is defined by

$$R^i_k := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k}y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i \partial G^m}{\partial y^m \partial y^k}.$$

Ricci curvature is the trace of the Riemann curvature, which is defined by

$$Ric \ := \ R^m_m.$$

For a finsler metric F, let

$$f(x,y) \ := \ \frac{R^m_m}{F^2} = \frac{Ric}{F^2}. \tag{2.1}$$

Then by the definition of Einstein-reversibility, we have

$$f(x,y) \ = \ f(x,-y).$$

By definition, an (α,β)-metric on M is expressed in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a positive definite Riemannian metric, $\beta = b_i(x)y^i$ a 1-form. It is known that (α,β)-metric with $\|\beta_x\|_\alpha < b_0$ is a Finsler metric if and only if $\phi = \phi(s)$ is a positive smooth function on an open interval $(-b_0, b_0)$ satisfying the following condition ([3], and [2]):

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi''(s) > 0, \quad \forall |s| \leq b < b_0,$$

For a pair of α and β, we define the following quantities:

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

where \ denotes the covariant derivative with respect to the Levi-Civita connection of α Denote

$$r^i_j := a^{ik} r_{kj}, \qquad r_j := b^i r_{ij}, \qquad s^i_j := a^{ik} s_{kj}, \qquad s_j := b^i s_{ij},$$

where

$$(a^{ij}) := (a_{ij})^{-1}, \qquad b^i := a^{ij} b_j,$$

and

$$\bigtriangledown\beta = b_{i|j} y^i dx^j,$$

denotes the covariant derivatives of β with respect to α. Denote

$$r^i := a^{ij} r_j, \quad s^i := a^{ij} s_j, \quad r_{i0} := r_{ij} y^j, \quad s_{i0} := s_{ij} y^j, \quad r_{00} := r_{ij} y^i y^j, \quad r_0 := r_i y^i \quad s_o := s_i y^i.$$

We use the following lemma proved by [5]:

Lemma 2.1. *An n-dimensional square metric $F = (\alpha+\beta)^2/\alpha$ is Einstein-reversible if and only if the following hold*

$$\begin{aligned} Ric_\alpha &= -c^2\{[(2n-5)b^2 + 5(n-1)]\alpha^2 - 6(n-2)\beta^2\} + \frac{t^k_k}{2(n-1)^2}\{[(n+1)(5n+3) + \\ &\quad +8(n+1)b^2 - \frac{(n-1)(n-3)}{b^2}]\alpha^2 + (n+1)(9n-17)(1+\frac{1}{b^2})\beta^2\}, \end{aligned} \tag{2.2}$$

$$r_{00} = c[(1+2b^2)\alpha^2 - 3\beta^2], \quad t_{00} = \frac{t^k_k}{(n-1)b^2}(b^2\alpha^2 - \beta^2), \quad s^k_{0|k} = -\frac{t^k_k}{b^2}\beta, \quad s_0 = 0,$$

$$c_i = -2\{\frac{(n+1)(1+b^2)}{(n-1)^2 b^2} t^k_k + c^2\} b_i, \quad (c_i := c_{x^i}), \tag{2.3}$$

where $c = c(x)$ is a scalar function.

3. **Computation of the quantities**

Some computations which used for the main result in this paper, then in this section we shall use Killing vector fields to present some formulas about the Levi-Civita connection of homogeneous Riemannian manifolds.

Let $(G/H, \alpha)$ be a homogeneous Riemannian manifold. Then the Lie algebra of G has a decomposition $g = \eta + m$, where h is the Lie algebra of H and $Ad(h)(m) \subset m, \forall h \in H$. We identify m with the tangent space $T_o(G/H)$ of the origin $o = H$. We shall use the notation $\langle , \rangle$ to denote the Riemannian metric on the manifold as well as its restriction to m. Note that it is an AdH-invariant inner product on m. Hence we have

$$\langle [x, u], v\rangle + \langle [x, v], u\rangle = 0, \quad \forall x \in \eta, \quad \forall u \in m,$$

which is equivalent to

$$\langle [x, u], u\rangle = 0, \quad \forall x \in \eta, \quad \forall u \in m.$$

Given $v \in g$, we can define the fundamental vector field $\hat{\mathrm{v}}$ generated by v, i.e.,

$$\hat{\mathrm{v}}_{gH} = \tfrac{d}{dt} exp(tv)gH|_{t=0}, \quad \forall g \in G.$$

Since the one-parameter transformation group $exp(tv)$ on G/H consists of isometries, $\hat{\mathrm{v}}$ is a Killing vector field.

Let $\hat{\mathrm{X}}$, $\hat{\mathrm{Y}}$, $\hat{\mathrm{Z}}$ be Killing vector fields on G/H and U, V be arbitrary smooth vector fields on G/H. Then we have in ([7], [8]and [9])

$$(3.1) \qquad [\hat{X}, \hat{Y}] = -\widehat{[X, Y]}$$

$$(3.2) \qquad \hat{X}\langle U, V\rangle = \langle [\hat{X}, U], V\rangle + \langle [\hat{X}, V], U\rangle,$$

$$(3.3) \qquad \langle \nabla_{\hat{X}} \hat{Y}, \hat{Z}\rangle = -\frac{1}{2}(\langle \widehat{[X, Y]}, \hat{Z}\rangle + \langle \widehat{[X, Z]}, \hat{Y}\rangle + \langle \widehat{[Y, Z]}, \hat{X}\rangle).$$

Let $u_1, u_2, ..., u_n$ be an orthonormal basis of m with respect to $\langle , \rangle$. We extend it to a basis $u_1, u_2, ..., u_m$ of g. By Helgason [22], there exists a local coordinate system on a neighborhood

V of o, which is defined by the mapping:

$$(exp(x^1u_1)exp(x^2u_2)...exp(x^nu_n))H \rightarrow (x^1, x^2, ..., x^n).$$

Let $gH = (x^1, x^2, ..., x^n) \in U$. Then

$$\frac{\partial}{\partial x^i} \mid_{gH} = f_i^a \hat{u}_a \mid_{gH},$$

where

$$f_i^a u_a = e^{x^1 adu_1} ... e^{x^{i-1} adu_{i-1}}(u_i). \tag{3.4}$$

In the following, the indices $a, b, c, ...$ range from 1 to m, the indices $i, j, k, ...$ range from 1 to n and the indices $\lambda, \mu, ...$ range from $n+1$ to m.

Let Γ_{ij}^l be the Christoffel symbols in the coordinate system, i.e.,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

Then

$$\Gamma_{ij}^l \frac{\partial}{\partial x^l} = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \frac{\partial f_j^a}{\partial x^i} \hat{u}_a + f_i^b f_j^a \nabla_{\hat{u}_b} \hat{u}_a. \tag{3.5}$$

From (3.4), we see that f_i^a are functions of $x^1, ..., x^{i-1}$. Thus

$$\frac{\partial f_j^a}{\partial x^i} = 0, \qquad i \geq j.$$

Therefore (3.5) gives

$$\Gamma_{ij}^l \frac{\partial}{\partial x^l} = f_i^b f_j^a \nabla_{\hat{u}_b} \hat{u}_a, \qquad i \geq j.$$

Differentiating the above equation with respect to x_k, we get

$$\frac{\partial \Gamma_{ij}^l}{\partial x^k} \frac{\partial}{\partial x^l} + \Gamma_{ij}^s \Gamma_{ks}^l \frac{\partial}{\partial x^l} = \frac{\partial (f_i^b f_j^a)}{\partial x^k} \nabla_{\hat{u}_b} \hat{u}_a + f_i^b f_j^a f_k^c \nabla_{\hat{u}_c} \nabla_{\hat{u}_b} \nabla_{\hat{u}_a}, \quad i \geq j. \tag{3.6}$$

Differentiating (3.4) with respect to x_k and letting $(x^1, ..., x^n) \rightarrow 0$, we obtain

$$\frac{\partial f_i^a}{\partial x^k} = f(k, i) C_{ki}^a,$$

where C_{ki}^a are the structure constants of g which are defined by $[u_a, u_b] = C_{ab}^c u_c$ and $f(k, l)$ are defined by

$$f(k, i) = \begin{cases} 1, & k < i; \\ 0, & k \geq i. \end{cases}$$

From [10] we use the following:

Lemma 3.2. *Considering the value at the origin, we have*

$$\Gamma^l_{ij}(0) = f(i,j)C^l_{ij} + \langle \nabla_{\hat{u}_i}\hat{u}_j, \hat{u}_l\rangle, \tag{3.7}$$

$$\begin{aligned} \frac{\partial \Gamma^l_{ij}}{\partial x^k}|_0 &= -\Gamma^s_{ij}(\Gamma^l_{ks} + \langle \nabla_{\hat{u}_k}\hat{u}_l, \hat{u}_s\rangle) + f(k,j)C^a_{kj}\langle \nabla_{\hat{u}_i}\hat{u}_a, \hat{u}_l\rangle + \\ &\quad + f(k,i)C^s_{ki}\langle \nabla_{\hat{u}_s}\hat{u}_j, \hat{u}_l\rangle + \hat{u}_k\langle \nabla_{\hat{u}_i}\hat{u}_j, \hat{u}_l\rangle, \quad i \geq j. \end{aligned} \tag{3.8}$$

Lemma 3.3. *For $u_i, u_j, u_k, u_l \in m, u_\lambda \in \eta$, we have*

$$\langle \nabla_{\hat{u}_i}\hat{u}_j, \hat{u}_l\rangle|_0 = -\frac{1}{2}(C^l_{ij} + C^j_{il} + C^i_{jl}), \tag{3.9}$$

$$\langle \nabla_{\hat{u}_i}\hat{u}_\lambda, \hat{u}_j\rangle|_0 = \langle [u_j, u_\lambda]_m, u_i\rangle = C^i_{j\lambda}, \tag{3.10}$$

$$\begin{aligned} \hat{u}_k\langle \nabla_{\hat{u}_i}\hat{u}_j, \hat{u}_l\rangle|_0 &= \frac{1}{2}(C^l_{ka}C^a_{ij} + C^j_{ka}C^a_{il} + C^i_{ka}C^a_{jl} + C^s_{ij}C^t_{kl}\delta_{st} + \\ &\quad + C^s_{il}C^t_{kj}\delta_{st} + C^s_{jl}C^t_{ki}\delta_{st}), \end{aligned} \tag{3.11}$$

where $[v_i, v_j]_m$ denotes the projection of $[v_i, v_j]$ to m.

From the above two lemmas, at the origin 0 we have

$$\Gamma^j_{ni} - \Gamma^i_{nj} = \langle \nabla_{\hat{u}_n}\hat{u}_i, \hat{u}_j\rangle - \langle \nabla_{\hat{u}_n}\hat{u}_j, \hat{u}_i\rangle = C^n_{ji}.$$

4. Ricci-quadratic homogenous Einstein metrics

We will recall that the group of isometries of a Finsler space (M, F) is a Lie transformation group of M [11]. A Finsler space M is called homogeneous if its isometry group acts transitively on M. A homogeneous Finsler space can be expressed as $(G/H, F)$, where G is a connected Lie group, H is a compact subgroup of G and F is invariant under the action of G. Moreover, the action of G on G/H is almost effective and the Lie algebra g of G has a reductive decomposition $g = \eta + m$, where h is the Lie algebra of H and m is a subspace of g satisfying $Ad(h)(m) \subset m, \forall h \in \eta$. Let

$$F = \frac{(\alpha+\beta)^2}{\alpha} = \frac{(\sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i)^2}{\sqrt{a_{ij}(x)y^iy^j}}$$

be a square metric. Let $\bigtriangledown\beta = b_{i|j}y^i dx^j$ denote the covariant derivative of β with respect to α:

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \qquad s_j := b^i s_{ij}, \qquad t_j := s_m s^m_j.$$

We use a_{ij} to raise and lower the indices of tensors defined by b_i and $b_{i|j}$. The index 0 means the contraction with y^i. For example, $s_0 = s_i y^i$ and $r_{00} = r_{ij}y^i y^j$, etc. For Ricci-quadratic metrics on square Einstein spaces, and based on lemma 2.1 we have the following.

Theorem 4.1. *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a Einstein metrics on an n-dimensional manifold. Then it is Ricci-quadratic if and only if*

$$r_{00} = \tilde{c}[(1+2b^2)\alpha^2 - 3\beta^2], \tag{4.1}$$

$$s^k_{0|k} = -\frac{t^k_k}{b^2}\beta, \tag{4.2}$$

where $\tilde{c} = \tilde{c}(x)$ is a scalar function. In this case,

$$\begin{aligned} Ric \;=\; & \overline{Ric} + c^2\{[(2n-5)b^2 + 5(n-1)]\alpha^2 - 6(n-2)\beta^2\} - \frac{t^k_k}{2(n-1)^2}\{[(n+1)(5n+3) + \\ & +8(n+1)b^2 - \frac{(n-1)(n-3)}{b^2}]\alpha^2 - (n+1)(9n-17)(1+\frac{1}{b^2})\beta^2\}. \end{aligned} \tag{4.3}$$

Proof. For the Finsler square metric $F = \frac{(\alpha+\beta)^2}{\alpha}$, let $f(x,y)$ defined by 2.1. First we can write in the following form

$$(4.4)\;(\ldots)[(1+2b^2)\alpha^2 - 3\beta^2] - 108\beta^2(\alpha^2-\beta^2)^2(2\alpha^2 s_0 + \beta r_{00}[4\alpha^2\beta s_0 + (\alpha^2+\beta^2)r_{00}] = 0,$$

where the omitted term in the bracket is a polynomial in (y^i). By (4.4) $(1+2b^2)\alpha^2 - 3\beta^2$ is divided by $2\alpha^2 s_0 + \beta r_{00}$, then

$$2\alpha^2 s_0 + \beta r_{00} = \theta[1+2b^2)\alpha^2 - 3\beta^2], \tag{4.5}$$

where θ is a 1-form. Clearly, (4.5) is equivalent to

$$(2s_0 - \Theta - 2b^2\Theta)\alpha^2) + \beta(r_{00} + 3\beta\Theta) = 0,$$

which implies

$$r_{00} = -3\beta\Theta + (1+2b^2)c\alpha^2, \qquad \Theta = c\beta + \frac{2s_0}{1+2b^2},$$

where $c = c(x)$ is a scalar function. Plugging Θ into r_{00}, we obtain

$$r_{00} = c[(1+2b^2)\alpha^2 - 3\beta^2] - \frac{6\beta s_0}{1+2b^2}. \tag{4.6}$$

Assume $s_0 = 0$. Then by the above equation we get r_{00} given in (4.1).

plug all the above quantities into (4.4) and then multiplied by $\frac{1}{[(1+2b^2)\alpha^2-3\beta^2]}$ then (4.4)can be written as

$$(...)[(1+2b^2)\alpha^2 - 3\beta^2] + (n-4)s_0^2\beta\alpha^6(\alpha^2+\beta^2)(\alpha^2-\beta^2)^3 = 0. \tag{4.7}$$

By (4.7), it is clear that $s_0 = 0$.Multiplied (4.7) by $\frac{1}{[(1+2b^2)\alpha^2-3\beta^2]}$we obtain

$$(...)\alpha^2 - \beta^2[4Ric_\alpha + (9n-17)c_0\beta - 2(3n-7)c^2\beta^2], \tag{4.8}$$

Clearly, (4.8) shows

$$Ric_\alpha = d\alpha^2 + \frac{(3n-7)c^2}{2}\beta^2 - \frac{(9n-17)}{4}c_0\beta^2], \tag{4.9}$$

where $d = d(x)$ is a scalar function. Plugging (4.9) into (4.8) yields

$$\begin{aligned} &(\{[8(3-2n)b^2 - 22(n-1)]c^2 + 4(2t^k_k - 2b^k c_k - d)\}\beta + 4s^k_{0|k} - (n-1)c_o)\alpha^2 + \\ &+\{[8(3-2n)b^2 - 30(n-1)]c^2 - 4(2b^k c_k + d)\}\beta + [12s^k_{0|k} - 5(n-1)c_o)\beta + 16t_{00} = 0. \end{aligned} \tag{4.10}$$

By (4.10), we easily get

$$s^k_{0|k} = \{(2b^k c_k - 2t^k_k - 4\sigma + d) + [\frac{11}{2}(n-1) + 2(2n-3)b^2]c^2\}\beta + \frac{n-1}{4}c_0, \tag{4.11}$$

$$\begin{aligned} t_{00} &= \sigma\alpha^2 + \{[\frac{15}{8}(n-1) + \frac{1}{2}(2n-3)b^2]c^2 + \frac{1}{2}b^k c_k + \frac{1}{4}d\}\beta^2 + \\ &+ \{\frac{5}{16}(n-1)c_0 - \frac{3}{4}s^k_{0|k}\beta, \end{aligned} \tag{4.12}$$

where $\sigma = \sigma(x)$ is scalar function. Contracting (4.12) by a^{ij} we get

$$\begin{aligned} \sigma &= \frac{1}{8(n+3b^2)}(8b^2+1-n)b^k c_k + 2b^2[4(2n-3)b^2 + 9(n-1)]c^2 + \\ &+ 4b^2 d - 4(3b^2-2)t^k_k)\}. \end{aligned} \tag{4.13}$$

Contracting (4.12) by $b^i b^i$ we get

$$d = [2(3-2n)b^2 - \frac{9}{2}(n-1)]c^2 + \frac{n-1-8b^2}{4b^2}b^k c_k + \frac{2+3(n+1)b^2}{(n-1)b^2}t^k_k. \tag{4.14}$$

Plugging (4.13) and (4.13) into (4.12) and then contracting (4.12) by b^j and using $s_0 = 0$ we obtain

$$c_i \quad = \quad \frac{b^k c_k}{b^2} b_i. \tag{4.15}$$

Plugging (4.13)-(4.15) into (4.10) we have

$$s^k_{0|k} \quad = \quad \{(n-1)c^2 + \frac{n-1}{2b^2} b^k c_k + \frac{2+(n+1)b^2}{(n-1)b^2} t^k_k\}\beta. \tag{4.16}$$

contracting(4.16) by b^i and using $s_0 = 0$ we get

$$b^k c_k \quad = \quad -2b^2c^2 - \frac{2(n+1)(1+b^2)}{(n-1)^2} t^k_k. \tag{4.17}$$

Now using (4.17), it follows (4.16) that $s^k_{0|k}$ is given in (4.2). Finally, (4.3) follows from (4.9), (4.14), and (4.15).This completes the proof.

Now we consider homogeneous Einstein metrics. Let $(G/H, \alpha)$ and m be as above. If W is a G-invariant vector field on G/H, then the restriction of W to $T_o(G/H)$ must be fixed by the isotropy action of H. Under the identification of to (G/H) with m, W corresponds to a vector $w \in m$ which is fixed by $Ad(H)$. On the other hand, if $w \in m$ is fixed by $Ad(H)$, then we can define a vector field W on G/H by $W|_{gH} = \frac{d}{dt}(gexp(tw)H)|_{t=0}$. Therefore, G-invariant vector fields on G/H are one-to-one corresponding to vectors in m fixed by $Ad(H)$. Note that Einstein $F = \frac{(\alpha+\beta)^2}{\alpha}$ is G-invariant if and only if α and β are both invariant under G. Through α, β corresponds to a vector field $\tilde{U}$ which is invariant under G and satisfying $\alpha(\tilde{U}) < 1$ everywhere. This implies that there is a one-to-one correspondence between the invariant Einstein metrics on G/H with the underlying Riemannian metric and the set

$$V \quad = \quad \{u \in m \backslash Ad(h)u = u, \langle u, u\rangle < 1, \forall h \in H\}.$$

Also note that in this case the length c of β (or $\tilde{U}$) is constant.

Let $(G/H, F)$ be a homogeneous Einstein metrics and $(U, (x^1, ..., x^n))$ be a local coordinate system. We suppose the vector field $\tilde{U}$ which corresponds to the invariant 1-form β

corresponds to $u = cu_n (c < 1)$ under the Riemannian metric α. Thus

$$\begin{aligned}\tilde{U}|_{gH} &= \frac{d}{dt} gexp(tu)H|_{t=0},\\ &= \frac{d}{dt}(expx^1u_1expx^2u_2...exp(x^n+ct)u_n)H|_{t=0},\\ &= c\frac{\partial}{\partial x^n}|_{gH}.\end{aligned}$$

See [12] for more information on invariant metrics.

Then we have the following [4]:

$$\begin{aligned}b_i &= \beta(\frac{\partial}{\partial x^i}) = \langle \tilde{U}, \frac{\partial}{\partial x^i}\rangle = c\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i}\rangle = ca_{ni},\\ \frac{\partial b_i}{\partial x^j} &= c\frac{\partial a_{ni}}{\partial x^j} = c(\Gamma^k_{nj}a_{ki} + \Gamma^k_{ji}a_{kn}),\\ b_{i|j} &= \frac{\partial b_i}{\partial x^j} - b_l\Gamma^l_{ij} = c\Gamma^k_{nj}a_{ki},\\ r_{ij} &= \frac{1}{2}(b_{i|j} + b_{j|i}) = \frac{c}{2}(\Gamma^k_{nj}a_{ki} + \Gamma^k_{ni}a_{kj}),\\ s_{ij} &= \frac{1}{2}(b_{i|j} - b_{j|i}) = \frac{c}{2}(\Gamma^k_{nj}a_{ki} - \Gamma^k_{ni}a_{kj}),\\ s_j &= b^is_{ij} = a^{il}b_ls_{ij} = cs_{nj}.\end{aligned} \tag{4.18}$$

Lemma 4.4. *Let $(G/H, F)$ be a homogeneous Einstein metrics and β correspond to u. Then* (4.1) *implies that*

$$\langle [y,u]_m, y\rangle = 0, \quad \forall y \in m. \tag{4.19}$$

Proof. Considering the value at 0, by (4.3), (3.7), (3.9), we have

$$\begin{aligned}r_{00} &= \frac{c}{2}(\Gamma^k_{n0}a_{k0} + \Gamma^k_{n0}a_{k0})\\ &= \frac{c}{2}(\Gamma^0_{n0}a_{k0} + \Gamma^0_{n0}a_{k0})\\ &= cC^0_{n0}\\ &= \langle [y,u]_m, y\rangle.\end{aligned}$$

It is obvious that

$$\tilde{c}[(1+2b^2)\alpha^2 - 3\beta^2]|_0 = \tilde{c}[(1+2b^2)\langle y,y\rangle - 3\langle u,y\rangle.$$

Plugging the above two equation into (4.1), we get that at 0

$$\langle [y,u]_m, y\rangle = \tilde{c}[(1+2b^2)\langle y,y\rangle - 3\langle u,y\rangle].$$

Setting $y = u$ and taking into account the fact $\langle u,u\rangle < 1$, we get

$$\tilde{c}(0) \quad = \quad 0.$$

Thus

$$\langle [y,u]_m, y\rangle \quad = \quad 0.$$

Replacing y by $y+u$ in the above equation yields

$$\langle [y,u]_m, u+y\rangle \quad = \quad 0.$$

From the above two equations and the fact that $\langle u,u\rangle < 1$, we deduce that

$$\langle [y,u]_m, u\rangle \quad = \quad 0, \qquad \langle [y,u]_m, y\rangle = 0.$$

This proves the lemma.

Note. From the above lemma we have

$$\begin{aligned} C^j_{ni} + C^i nj &= 0 = C^n_{ni}, \\ s_i(0) &= cs_{ni} = \frac{c^2}{2}(\Gamma^n_{ni} - \Gamma^i_{nn}) = \frac{1}{2}\langle [u,u_i]_m, u\rangle = 0, \\ t_i(0) &= s_m s^m_i = 0. \end{aligned} \tag{4.20}$$

5. **Proof of the main theorem**

Let $(G/H,F)$ be a Homogeneous Einstein metrics and β correspond to u which satisfies (4.4) we have

$$\frac{\partial b_{0|0}}{\partial x^0} \quad = \quad 0. \tag{5.1}$$

$$\frac{\partial s_{k0}}{\partial x^i}|_0 \quad = \quad \frac{c}{2}(f(i,k)C^s_{ik}C^n_{s0} + f(i,0)C^s_{i0}C^n_{ks} + C^s_{0k}C^n_{is}). \tag{5.2}$$

For more details see [1].

Theorem 5.2. *A homogeneous Einstein metrics is Ricci- quadratic if and only if it is of Berwald type.*

Proof. Let $(G/H, F)$ be a Homogeneous Einstein metrics. Taking the local coordinate system as in section 3, we have seen that (4.4) holds. In particular, we have $C^n_{ni} = 0$. By (5.2) we have

$$\frac{\partial s_{n0}}{\partial x^0}|_0 = \frac{c}{2}(f(0,n)C^s_{0n}C^n_{s0} + f(0,0)C^s_{00}C^n_{ns} + C^s_{0n}C^n_{0s}) = 0.$$

Differentiating (4.1) and taking into account the fact that $\tilde{c}(o) = 0 = s_0(0)$, we deduce from (4.3) that

$$c\frac{\partial b_{k0}}{\partial x^0}|_0 = \frac{\partial \tilde{c}}{\partial x^0}[(1+2b^2)\alpha^2 - 3\beta^2],$$

$$\tilde{c}_0[(1+2b^2)\alpha^2 - 3\beta^2] = \frac{\partial b_{0|0}}{\partial x^0} = 0.$$

By lemma (2.3) since β is closed which yields

$$t^k_k = 0.$$

On the other hand , we have

$$s_{ij}(0) = \frac{c}{2}(\Gamma^i_{nj} - \Gamma^j_{ni}) = \frac{c}{2}C^n_{ij}.$$

When

$$s^k_{0|k}(0) = \frac{c}{2}\sum_{k=1}^{n}(f(k,0)-1)C^s_{k0}C^n_{ks} + \sum_{1\le k,1\le n}\frac{c}{2}(C^n_{l0}C^k_{kl} + \frac{1}{2}C^n_{kl}(C^l_{0k} + C^k_{0l} + C^0_{kl})),$$

$$s^k_{0|k} = 0, \quad s_0(0) = 0 \quad and \quad let \quad y = u,$$

then

$$\sum_{1\le k,1\le n} C^n_{kl}(C^l_{nk} + C^k_{nl} + C^n_{kl}) = 0 = \sum_{1\le k,1\le n} C^n_{kl}(-2)\langle\nabla_{\hat{u}_n}\hat{u}_k, \hat{u}_l\rangle,$$

$$= -2\sum_{1\le k,1\le n} C^n_{kl}(\langle\nabla_{\hat{u}_n}\hat{u}_k, \hat{u}_l\rangle + f(n,k)C^l_n k) = -2\sum_{1\le k,1\le n} C^n_{kl}\Gamma^l_{nk},$$

$$= -\sum_{1\le k,1\le n}(C^n_{kl}\Gamma^l_{nk} + C^n_{lk}\Gamma^k_{nl}) = -\sum_{1\le k,1\le n} C^n_{kl}(\Gamma^l_{nk} - \Gamma^k_{nl}) = \sum_{1\le k,1\le n}(C^n_{kl})^2.$$

Then by (4.2) we have

$$C^n_{kl} = 0, \quad k = 1, ..., n,$$

then

$$\langle [u_k, u_l]_m, u\rangle = 0, \quad k, l = 1, ..., n. \tag{5.3}$$

Which is prove that a Homogeneous square Einstein space is of Berwald type if and only if (4.1) and (5.3) are hold. This completes the proof.

6. **Conclusion**

A Finsler space is called Ricci-quadratic if its Ricci curvature $Ric(x, y)$ is quadratic in y. It is called a Berwald space if its Chern connection defines a linear connection directly on the underlying manifold M.

We use the notation of Einstein-reversibility for Finsler metrics. A characteristic condition of Einstein-reversibility to be Ricci-quadratic . And we use the most important characteristic and we obtain theorem that characterzes Ricci-quaratic Einstein metric. Then it can verified directly that β is closed.

We prove that a Homogeneous Einstein metrics is Ricci-quadratic if and only if it is of Berwald type by using a characteristic condition of Einstein-reversibility to be Ricci-quadratic.

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