NEW ASYMPTOTIC METHODS IN OPTIMIZATION AND APPLICATIONS

DRISS MENTAGUI

Ibn Tofail University, Faculty of Sciences, Department of Mathematics,

Laboratory of Operations Research and Optimization, Kénitra, PC 14000, Morocco

Correspondence: dri_mentagui@yahoo.fr

ABSTRACT. This paper aims to introduce a large class of new general stabilization methods in optimization and saddle point theory. The concept of well-posedness in several senses is also investigated in metric and normed spaces. New variational asymptotic methods and some variational approximation results have been displayed within the framework of variational analysis and the theory of variational convergence of functions and operators.

INTRODUCTION

Approximation theory is a central subject in pure and applied mathematics. Notably it is encountered in algorithmic optimization when we use various numerical methods to approximate a solution of class of extremum problems. In theoritical or parametric optimization, we very often study the sensitivity analysis of an initial minimization or maximization problem when the objective function and constraints are perturbed. Various types of perturbation schemes can be investigated, for instance if we consider an intial hard or ill-posed parametric problem (H_{θ_0}) :$\min_{x \in C(\theta_0)}$ f(x,θ_0), it is possible to replace it by a sequence of simple or well-posed problems (H_θ) :$\min_{x \in C(\theta)}$ g(x,θ) such that "(H_θ) approximates (H_{θ_0})", that is the sequences of functions $(g(.,\theta))_\theta$ and sets $(C(\theta))_\theta$ converge in a certain sense to f(.,θ_0) and C(θ_0) respectively when $\theta \to \theta_0$ (for instance see [3, 9, 34, 72, 74, 75, 78] and references therein) ,$(\theta, \theta_0) \in$ A×B, where A, B are abstract index sets and the convergence $\theta \to \theta_0$ is considered in a specified sense using the notion of filters, nets or grills in general topological spaces [3, 31]. Under a class of hypotheses, it is possible to show that the solutions x_θ of (H_θ) have a cluster point x' which is a solution of (H_{θ_0}) (see [3, 9, 34, 75] and references therein). If $((H_\theta)_\theta$ are well-posed in a certain sense, it is possible to apply a large class of numerical methods to each problem (H_θ) in order to derive a solution z_θ for it (see [34] and references therein).On the other hand the notion of well-posedness in optimization is strongly related to

the regularization methods considered as a logistic support for the theory of small parameter and asymptotic analysis. They play a crucial role in the stabilization and approximation of the solutions of a wide class of problems in pure and applied mathematics. Well-posedness has several definitions and characterizations in the literature [12, 13, 14, 34, 112, 113]. The concept of regularization or stabilization goes back to the works of Tikhonov (for instance, see [34, 100, 101, 102] and references therein) and has considerable applications as in variational analysis and optimization [3, 28, 68, 82, 93], partial differential equations and optimal control [30, 34, 50, 56, 63, 64, 65, 83], inverse problems [16, 21, 23, 38, 45, 51, 110], plasticity theory [98], calculus of variations [34, 36, 42], variational inequalities [2, 22, 43, 66, 67, 75, 85], fixed point theory and inclusion problems [41, 60, 62, 87, 103], minimax and saddle-value problems [69, 86, 88, 94, 95]. Besides the classical stabilization methods used in variational analysis and its related topics, other types of regularizations can be considered in functional analysis and operator theory [19, 20, 39, 45, 54, 88, 108], in statistics [37, 53, 55, 99] and serve as a very useful and flexible tool for establishing key results in these disciplines.

The central feature of problems we regularize is their ill-posedness. This means that the solution fails to be unique or most importantly, small changes in data of the model, which are closely related to the errors of experimental measurements or unexpected phenomena, could lead to uncontrollable errors. In other words, the gap between the solutions (if any) of the perturbed model and the ones of the original problem may be very large relatively to a specified metric; accordingly meaningless interpretations may occur in the course of physical or economical investigations or other fields of experimental sciences. A natural idea is to replace the initial problem by a sequence of well-posed problems guaranteeing robustness and stability of their solutions and providing a large choice of numerical methods for approximating them. Roughly speaking, a model is said to be robust if its solutions and performance results remain relatively unchanged when exposed to perturbations, random phenomena and uncertainties. In Phillipe Vincke's opinion [107], the uncertainty sources that have been considered as the most important are the following:

i)The decision problem specifications are usually very imprecise, unpredictable, not much known and not well defined.

ii) The environment in which the decision has to be taken could affect the conditions under which the decision could operate.

iii) The unstable and imprecise character of the value systems and the decision-maker preferences has priority in deciding the feasibility and the relative interest of the potential alternatives.

The terms robustness and stability are also very often used in many disciplines. For instance, in statistics they refer to certain desirable characteristics of statistical processes. A process is considered robust with respect to the deviations of the model hypothesis, when the process continues working in a suitable way, even though some of the initial assumptions are

not maintained [84]. In Hampel's opinion [49]: Robust statistics is the statistics process stability theory. It studies systematically the deviation effects from the initial model hypothesis to the known processes and, if necessary, it develops new and better processes. For more details about robustness and its philosophy see for instance [15, 17, 52, 57, 89, 92, 97, 105, 106] .

Before Hadamard's theory on the stability in mathematical physics [29, 34, 47, 48, 59], most analysts payed special attention to stable models and consider the others as inefficient or enable to explain the results arising from experimental investigations in many areas of sciences.This way of thinking and analysing is one of the main characteristics of the scientific method prevailing during the 19th and the first half of 20th century, that of positive science based on the Cartesian method and characterized by the attempt of reducing the complexity to its elementary components. It turns out that this fabulous method which has brought great advances in sciences and adapted perfectly to the study of stable systems is no longer appropriate when considering the organized complexity such as that found in complex biological systems, physics, economics and sociology ... etc; and characterized by openness, fluctuation, chaos, disorder, blur, creativity, contradiction, ambiguity, pradoxe and instability. All these aspects that were once perceived as nonscientific by the prevailing positivisme are henceforth considered as crucial prerequisites for understanding the complexity of real-world phenomena and have been taken in consideration since the previous seventy years in a new approach socalled systemic approach or mathematical modelling. As far as one is interested for instance by stable or unstable characters of systems which may have many senses from phenomenon to another, there exists actually a wide class of unstable problems covring a large field of applications in many areas of sciences and technology [10, 34, 59, 101, 102] , and their rigourous mathematical analysis uses inescapably regularization methods adequately chosen for each specified problem. First let us consider some examples of stability and instability in physics and investigate by analogy the concept of stability in mathematical programming. Consider the known problem in classical mechanics that of physical pendulum which is a solid body of any shape form suspended from an horizontal axis at a support . Under the influence of gravity, the body will swing back and forth freely from an initial exitation θ_0. The position $K(\theta)$ of the pendulum in the space can be described by the oscillation angles θ of the body from its stable equilibrium position relatively to an appropriate vertical axis; and when θ goes to 0 the body reaches its unique stable position $K(0)$. From a purely mathematical point of view, $\theta \in [-\theta_0,\ \theta_0] \rightrightarrows K(\theta) \subset \mathbb{R}^3$ describes a special multifunction K, and it is easy to see that the sets $K(\theta)$ converge to the set $K(0)$ in the sense of the classical Hausdorff metric [3, 34] when θ goes to 0, that is $d_H(K(\theta),\ K(0)) = \max(e(K(\theta),\ K(0)),\ e(K(0), K(\theta))) \to 0$ if $\theta \to 0$; where $e(K(0),\ K(\theta)) = \sup_{x \in K(0)} d(x, K(\theta))$. This expresses simply the continuity of K at 0 relatively to the distance d_H . Now if we consider a multifunction of the kind $u \in C^1[c,d] \rightrightarrows T(u) = \{\alpha \in C[a,b]\ /\ \int_a^b \alpha(t)K(x,t)dt = u(x), \forall x \in [c,d]\}$, where $K:[c,d] \times [a,b] \to \mathbb{R}$ is a continuous kernel with a continuous partial derivitive $\frac{\partial K}{\partial x}$, it is wellknown that such multifunction models a large class of inverse problems as in signal theory,

automatics, control problems (see for instance [101] and references therein). Assume that for some $u \in C^1$ [c,d] T(u) is nonempty; we know [101] that T is unstable at u in the sense that there exist two sequences of functions $(u_n)_n$, $u_n \in C^1$ [c,d] and $(\alpha_n)_n$, $\alpha_n \in C$ [a,b] such that $\alpha_n \in T(u_n)$ with $d_1(u_n, u) \to 0$ if n$\to +\infty$ but $d_2(\alpha_{n_k}, \alpha) \not\to 0$ if k$\to +\infty$, for any subsequence $(n_k)_k$ and any $\alpha \in T(u)$, where d_1, d_2 belong in a class of practical useful metrics. This explains simply a type of discontinuity of T frequently encountered in many problems and expresses that small changes u_n in data u cannot lead in general to solutions α_n of the integral equation $\int_a^b \alpha_n(t)K(x,t)dt = u_n(x), \forall x \in$ [c,d] $\forall n \in \mathbb{N}$, which could be considered as good approximations of α in a desirable sense.

The concepts of stability and instability can be also easily extended to optimization problems. If we deal for instance with unstable minimization problems of the kind (P): $\min_{x \in C} f(x)$ with nonempty solution sets denoted by argmin(f, C), (i.e. there exists a sequence $(x_n)_n$ in C without any cluster point such that $f(x_n) \to \min_{x \in C} f(x)$) where f: X$\to$]-∞,+∞] is a proper convex lower semicontinuous function defined on a reflexive Banach space X renormed by an equivalent norm $\|.\|$ making it an E-space [34] and C is a weakly convex compact; we can stabilize it by a sequence of strongly well-posed problems in the Tikhonov sense (P_n): $\min_{x \in C}(f(x)+\epsilon_n g(x))$ where g: X$\to$ [0,+∞[is any lower semicontinuous uniformly convex function; that is for each $\epsilon_n > 0$, f+ϵ_ng has a unique minimizer x_n on C and every minimizing sequence of (P_n) converges srongly to x_n [34]. Moreover, if $\epsilon_n \to 0$, $\|x_n$-$\overline{x}\| \to 0$, where $\overline{x}$ is a solution of (P) satisfying remarkable properties [34, 61] (see also [78, 79] if C is the whole space X and g is a specified function). Consequently, every numerical method generating a minimizing sequence for (P_n) leads to an approximation of x_n and so to $\overline{x}$ for a suitable choice of ϵ_n. In the last example, stability and instability characters may be interpreted in terms of special multifunctions as follows: If we set for each fixed $\epsilon > 0$ the multifunction: $\alpha \in [0, +\infty[\rightrightarrows R^\epsilon(\alpha) = \alpha. \arg\min(f + \epsilon g, C) = \{x \in C/f(x) + \epsilon g(x) \leq \min(P_\epsilon) + \alpha\}$ and $R^\epsilon(0) = \arg\min(f + \epsilon g, C) = \{x_\epsilon\}$, we see that $R^\epsilon(0) \subset R^\epsilon(\alpha)$ and R^ϵ is stable at 0, that is, $\forall y_\alpha \in R^\epsilon(\alpha)$, $(y_\alpha)_\alpha$ converges to x_ϵ if $\alpha \to 0$. Now, if we consider the multifunction $\epsilon \in [0, +\infty[\rightrightarrows D(\epsilon) = \epsilon. \arg\min(f, C)$ and D(0)= argmin(f, C), we observe that D is unstable at 0 because there exists a minimizing sequence $(z_\epsilon)_\epsilon$, $z_\epsilon \in D(\epsilon)$ without any subsequence converging to a point in D(0); in other words, (P) is not well-posed in the generalized sense of Tikhonov [34]. Other types of well-posedness can be found in the literature as Levitin-Polyak well- posedness, Hadamard well-posedness,...etc [14, 34, 73, 74]. It is worth noting that the class of well-posed minimization problems enjoys many interesting generic properties expressing in general that most problems are well-posed or may be approximated in a certain sense by a sequence of well-posed problems involving specified regularization functions [13, 34,91]. Also, it should be pointed out that the regularization methods with their diversity and rich properties provide flexible tools for characterizing classes of variational convergences for functions and operators [3, 11, 81] in approximation theory and optimization. Finally, in the author's opinion, the regularization methods used

in mathematics for theoretical or numerical goals have a physical analogy with the methods of correction and stabilization under the effects of turbulence and unpredictable phenomena in advanced technology systems in order to make them more robust face to the conditions of random nature, as in aerospace technology and ballistics [18, 27, 35], in the theory of machine regulation [1, 32, 40, 46, 90], or in the current problem of making more effective the missile-antimissile shield in military defense [111] .

The layout of this paper is as follows: Section 1 is devoted to a new characterization of well-posedness in metric and normed spaces. A new characterization via infimal-convolution operations is also given. In section 2 we introduce a new generalized stabilization method in a general topological space and prove a central theorem (Th 2.2) for a large class of minimization problems under suitable hypotheses. Afterwards, we observe through special cases of regularization functions that our assumptions are not restrictive and include most classical regularizations and more. Section 3 is devoted to the study of the stability of variational asymptotic developments by epi-convergence. Indeed, if the initial minimization problem is not easy to deal with and can be approximated in a variational sense by a sequence of simple problems $(P_n)_n$, we apply the regularization technique in theorem 2.2 to each (P_n) and derive variational asymptotic developments for the last problem; so by a diagonalization lemma established in [3] , we prove uniform asymptotic developments for a subsequence $(P_{n_k})_k$ and deduce the stability of the minimum of sum of functions under consideration without having necessarily the stability of this sum by epi-convergence, even in the nonconvex case. For the stability concept of sum of functions (and sets) by variational convergences and its crucial role in variational analysis and optimization we refer the reader to [4, 9, 12, 70, 72, 74, 75, 77, 78, 80] . In section 4, we investigate the convex case in reflexive Banach spaces and give applications to asymptotic developments for the Legendre-Fenchel transform. In section 5 we introduce new generalized regularizations of saddle functions and state a fondamental theorem (Th.5.1) in which we provide some approximation results and variational asymptotic developments for the regularizations of bivariate functions in a Hausdorff topological space. An application is also given to the conjugacy of bivariate functions. Well-posedness of such new regularizations is investigated in section 6. In section 7 we study the stability of variational asymptotic developments of the regularizations of saddle functions by epi/hypo-convergence.

1. NEW CHARACTERIZATION OF WELL-POSEDNESS IN METRIC AND NORMED SPACES

In this section, we will characterize some notions of generalized well- posedness in metric spaces. Let (X,d) be a metric space, C be a nonempty subset of X and f:X$\to$ $[-\infty, +\infty]$ is a function. We say that the minimization problem (f,C): $\min_{x \in C} f(x)$ is well-posed on in the sense of Levitin-Polyak, if it has a unique minimizer x'$\in$C such that f(x') is finite and every sequence $(x_{n)n}$ of X verifying d(x_n,C)$\to$ 0 and f(x_n)$\to$f(x') converges to x'. (f,C) is called

well-posed in the Tikhonov sense if it has a unique minimizer t'$\in$C such that f(t') is finite and every sequence $(x_{n)n}$ of C such that f(x_n)$\to$f(t') converges to t' . For the interest of these two notions in theoritical and algorithmic optimization see for instance [34] and references therein. We say that (f,C) is strongly well-posed if it has a unique minimizer z'$\in$C such that f(z') is finite and every sequence $(z_{n)n}$ of X satisfying d(z_n,C)$\to$ 0 and $\overline{\lim}$f(z_n)$\leq$f(z') converges to z' [12, 14, 75]. In fact (f,C) may have many minimizers, so we need a generalized definition of well-posedness. Then (f,C) is called well-posed in the Tikhonov generalized sense [34] if the set of solutions denoted by argmin(f,C) is nonempty and every minimizing sequence $(x_{n)n}$ of C has a subsequence converging to an element of argmin(f,C). We say that (f,C) is well-posed in the generalized sense of Levitin-Polyak (resp.well-posed in the strong generalized sense) (see [12, 14, 34, 75] and references therein), if argmin(f,C) is nonempty and every sequence $(x_n)_n$ of X verifying d(x_n,C)$\to$ 0 and f(x_n)$\to$$\min_C$f (resp. if argmin(f,C) is nonempty and every sequence $(z_{n)n}$ of X satisfying d(z_n,C)$\to$ 0 and $\overline{\lim}$f(z_n)$\leq$$\min_C$f) has a subsequence converging to an element of argmin(f,C). In what follows, we consider some notations we use here: C_ϵ = {x$\in$X: d(x,C)$< \epsilon$}, υ(f,C) =inf{f(x):x$\in$C} assumed to be finite, ϵ.argmin(f,C)={x$\in$C:f(x)$\leq \upsilon$(f,C) + ϵ}, L(ϵ) = {x$\in$X:d(x,C)$\leq \epsilon$, f(x)$\leq \upsilon$(f,C)+ϵ},L'(ϵ) = {x$\in$X:d(x,C)$\leq \epsilon$, |f(x)-υ(f,C)| $\leq \epsilon$}.The Hausdorff distance between two nonempty subsets A,B of X is denoted by d_H(A,B)=max (e(A,B), e(B,A)) where e(A,B)=$\sup_{x\in A}$d(x,B).

DEFINITION 1.1. [34] . A function c:D$\to [0, +\infty[$,D$\subseteq \mathbb{R}$ is called a forcing function if 0$\in$D, c(0)=0 and $a_n \in$D, c(a_n) $\to$ 0 $\implies a_n \to$ 0.

DEFINITION 1.2. [34] .Let A be a bounded subset of X, the noncompctness degree of A, is the Kuratowski number of A, defined by α(A) = inf{$\epsilon \succ$ 0 : $\exists$ $(A_i)_{i=1,2...n}$ A$\subseteq \bigcup\limits_{i=1}^{n}A_i$ and diam$A_i \leq \epsilon$}.We can verify easly that α(A) $\leq \alpha$(B) if A$\subseteq$B, and α(A) = 0 if and only if A is relatively compact.

PROPOSITION 1.3. Let (X,d) be a metric space, C be a nonempty closed subset of X and f:X$\to [-\infty, +\infty]$ is a function such that υ(f,C) is finite. Then (f,C) is well-posed in the generalized sense of Levitin-Polyak (resp. well-posed in the strong generalized sense) if and only if argmin(f,C) is compact and the multifunction $\epsilon \rightrightarrows$L'($\epsilon$) (resp.$\epsilon \rightrightarrows$L($\epsilon$)) is upper semicontinuous at 0.

Proof. If (f,C) is well-posed in the generalized sense of Levitin-Polyak, it is clear that argmin(f,C) is compact. Now if $\epsilon \rightrightarrows$L'($\epsilon$) fails to be upper semicontinous (usc) at 0, there exist an open subset θ of X containing L'(0), a sequence $(t_n)_n$ of positive numbers converging to 0 and a sequence $(x_n)_n$, $x_n \in$L'(t_n) with $x_n \notin \theta$.But , d(x_n,C)$\to$ 0 and f(x_n)$\to \upsilon$(f,C), so $(x_n)_n$ has a subsequence converging to an element of L'(0)= argmin(f,C); this is a contradiction because $x_n \notin \theta$.Conservely, let $(x_n)_n$ be a sequence of X such that d(x_n,C)$\to$ 0, f(x_n)$\to \upsilon$(f,C) and pick $\epsilon \succ$ 0.The usc at 0 implies that $x_n \in$ (L'(0))$_\epsilon$ for all n sufficiently large, so d(x_n,argmin(f,C))$\to$ 0 and by compactness of argmin(f,C), $(x_n)_n$ has a subsequence

converging to an element of argmin(f,C). In the same way we show the second equivalence replacing $L'(\epsilon)$ by $L(\epsilon)$.

PROPOSITION 1.4. Let (X,d) be a metric space locally compact, C be a nonempty closed subset of X and f:X$\to$]$-\infty, +\infty$] is a proper semicontinuous function. Suppose that for every $\epsilon \succ 0$,$L(\epsilon)$ is connexe. The following assertions are equivalent:

(i) $L(\epsilon)$ is compact for some $\epsilon = \epsilon_0 \succ 0$.

(ii) (f,C) is well-posed in the strong generalized sense.

(iii) argmin(f,C) is an nonempty compact.

Proof. First we point out that (f,C) is well-posed in the strong generalized sense if and only if (g, X) is well-posed in the Tikhonov generalized sense with g(x)=max(f(x)-υ(f,C),d(x,C)).

Afterwards, we apply proposition 1.4.2 of [75, p.23] (and references therein) and some arguments of its proof to obtain the previous equivalences.

THEOREM 1.5. Let (X,d) be a metric complete space, C be a nonempty closed subset of X and f:X$\to \mathbb{R}$ is a continuous function. Then (f,C) is well-posed in the generalized sense of Levitin-Polyak if and only if $\alpha(L'(\epsilon)) \to 0$ if $\epsilon \to 0$.

Proof. If (f,C) is well-posed in the generalized sense of Levitin-Polyak, we claim that $L'(\epsilon_0)$ is bounded for some positive ϵ_0; If this is not the case, $L'(\epsilon)$ is unbounded for every positive ϵ,so there exists a sequence $x_n \in L'(\frac{1}{n})$ such that $d(x_n,x_0) \geq n$, where x_0 is a fixed point in X; this is a contradiction because $(x_n)_n$ has a converging subsequence; accordingly $L'(\epsilon)$ is bounded for every small positive ϵ and $\alpha(L'(\epsilon))$ exists. We remark also that (f,C) is well-posed in the generalized sense of Levitin-Polyak if and only if (G,X) is well-posed in the generalized sense of Tikhonov where G(x)=|f(x)-inf$_C$f|+d(x,K), x$\in$X. Then the conclusion of the theorem is an immediate consequence of [34, Th.38, p.25] and the trivial inclusion $L'(\frac{\epsilon}{2}) \subset \epsilon$.argmin(G,X) $\subset L'(\epsilon)$.

THEOREM 1.6. Let (X,d) be a metric complete space, C be a nonempty closed subset of X and f:X$\to \mathbb{R}$ is a lower semicontinuous function. Then (f,C) is well-posed in the strong generalized sense if and only if $\alpha(L(\epsilon)) \to 0$ if $\epsilon \to 0$.

Proof. It is clear that (f,C) is well-posed in the strong generalized sense if and only if (g,X) is well-posed in the generalized sense of Tikhonov. On the other hand g is lower semicontinuous and X is complete, then [34, Th.38, p.25] permits to conclude, because for all ϵ, we have ϵ.argmin (g,X)=$L(\epsilon)$.

THEOREM.1.7. Let (X,d) be a complete metric space, C be a nonempty closed bounded subset of X and f:X$\to$]$-\infty, +\infty$] is a function.

1) If f is finite and continous,Then (f,C) is well-posed in the generalized sense of Levitin-Polyak if and only if there exits a forcing function c such that for every bounded subset A of X satifying $\sup_{x\in A} |f(x)| \prec +\infty$ one has :

$$(1.1) \qquad c(\alpha(A)) \leq \max(\sup_{x\in A} |f(x)-\upsilon(f,C)| ,\sup_{x\in A} d(x,C))$$

2) If f is lower semicontinuous, then (f,C) is well-posed in the strong generalized sense if and only if there exists a forcing function c verifying for every bounded subset A of X such that $\sup_{x \in A} f(x) \prec +\infty$:

$$(1.2) \qquad c(\alpha(A)) \leq \max(\sup_{x \in A} f(x) - \upsilon(f,C), \sup_{x \in A} d(x,C))$$

In particular (f,C) is well-posed in the strong generalized sense if there exits a forcing function c such that:

$$(1.3) \qquad c(\alpha(A)) \leq \sup_{x \in A}(f(x) - \upsilon(f,C))$$

for every bounded subset A of X such that $A \cap C \neq \varnothing$ and $\sup_{x \in A} f(x) \prec +\infty$.

Proof. 1). We will be inspired by the proof of theorem 39 of [34, p.26] .Assume that (f,C) is well-posed in the generalized sense of Levitin-Polyak, then $\alpha(L'(\epsilon)) \to 0$ if $\epsilon \to 0$ by theorem 1.5.Set $q(\epsilon) = \alpha(L'(\epsilon))$ and c(t)=inf{s:q(s)$\geq$ t}. By [34, Lemma 20] ,c is an increasing forcing function satisfying $c(q(\epsilon)) \leq \epsilon$ for every $\epsilon \geq 0$. Let A be a bounded subset of X such $\sup_{x \in A} |f(x)| \prec +\infty$ and consider p=max($\sup_{x \in A} |f(x) - \upsilon(f,C)|$,$\sup_{x \in A} d(x,C)$), we have p≥ 0 and A$\subset$L'(p), so c(α(A))$\leq$c(α(L'(p))=c(q(p))$\leq$p and then (1.1) is satisfied. Conversely if (1.1) holds for every bounded subset A of X such that $\sup_{x \in A} |f(x)| \prec +\infty$; set in particular A=L'(ϵ) which is clearly bounded so c(α(L'(ϵ)) $\leq \epsilon$, accordingly α(L'(ϵ)) $\to$ 0 if $\epsilon \to$ 0.Theorem 1.5 permits then to conclude.

2).This equivalence can be shown in the same way as point 1) using theorem 1.6. Now suppose that (1.3) is satisfied for every bounded set such that $A \cap C \neq \varnothing$ and $\sup_{x \in A} f(x) \prec +\infty$. We will show that (1.2) is satisfied. Let B be a bounded set with $\sup_{x \in B} f(x) \prec +\infty$. We may always assume that c is an increasing function. If $B \cap C \neq \varnothing$, we have always c($\alpha$(B)) $\leq \sup_{x \in B}(f(x) - \upsilon(f,C)) \leq \max(\sup_{x \in B} |f(x) - \upsilon(f,C)|$,$\sup_{x \in B} d(x,C)$) and (1.2) is satisfied. If $B \cap C = \varnothing$,we consider two cases: if there exists (a,b)$\in$C$\times$B such that f(a)$\leq$f(b), then set B'=B$\cup$\{a\}; we have c(α(B)) $\leq$c(α(B')) $\leq \sup_{x \in B'} f(x) - \upsilon(f,C) = \sup_{x \in B} f(x) - \upsilon(f,C)$ and (1.2) is satisfied. If for every (a,b)$\in$C$\times$B f(b)$\prec$f(a), set B(a)==B$\cup$\{a\} for every a$\in$C such that f(a) is finite, then by (1.3) we have c(α(B)) $\leq$c(α(B(a))) $\leq$

$\sup_{x \in B(a)} f(x) - \upsilon(f,C) = f(a) - \upsilon(f,C)$, so c($\alpha$(B)) = 0 and again (1.2) is satisfied which completes the proof. $\square$

THEOREM.1.8. Let (X,d) be a metric space, C be a nonempty subset of X and f:X$\to \mathbb{R}$ is a function. Assume that there exists a forcing function $\alpha:\mathbb{R} \to \mathbb{R}^+$ continuous at 0, $\alpha(0) = 0$ such that α(f-υ(f,C)) is uniformly continuous.   If (f,C) is well-posed in the generalized sense of Levitin-Polyak then we have the following implication (P): for every sequence of functions f_n:X$\to \mathbb{R}$ converging uniformly to f on X and for every sequences $(C_n)_n, (X_n)_n$ of subsets of X such that $d_H(C_n,C) \to 0$, $d_H(X_n,X) \to 0$ and $x_n \in$argmin(f_n+d(.,K_n), X_n) then d(x_n,argmin(f,C))$\to$ 0 when n$\to +\infty$.Conversely if the last implication is true and argmin(f,C) is compact then (f,C) is well-posed in the generalized sense of Levitin-Polyak.

Proof. If (f,C) is well-posed in the generalized sense of Levitin-Polyak, it is easy to see that (G,X) is well-posed in the generalized sense of Tikhonov and argmin(f,C)=argmin(G,X) with G(x)=α(f(x)-υ(f,C))+d(x,C).By hypothesis G(x) is uniformly continuous, so (G,X) is

well-posed in the generalized sense of Hadamard (see[75, p.26] and references therein) and (P) is satisfied. Conservely if (P) is satisfied, let be a sequence $(x_{n})_n$ of X verifying $d(x_n,C) \to 0$ and $f(x_n) \to \min_C f$. Set $f_n(x) = |\alpha(f(x)-\min_C f) - \alpha(f(x_n)-\min_C f)|$, $C_n = C \cup \{x_n\}$ and $X_n = X$. It is clear that $(f_n)_n$ converges uniformly to $\alpha(f(x)-\min_C f)$ on X, $d_H(C_n,C) \to 0$, $d_H(X_n,X) = 0$ and $x_n \in \text{argmin}(f_n+d(.,C_n), X)$; so $d(x_n,\text{argmin}(f,C)) \to 0$ when $n \to +\infty$ which completes the proof. $\square$

In what follows X will be a normed space with its norm $\|.\|$ and $d(x,y) = \|x-y\|$ is its associated metric. Considering two functions $f:K \to \,]-\infty, +\infty]$, $g:K' \to \,]-\infty, +\infty]$ where K,K' are two subsets of X. the fonction denoted by $f\nabla g$ called the infimal-convolution (or epi-sum) of f and g is defined on K+K' by $(f\nabla g)(z) = \inf\{f(x)+g(y)/ (x,y) \in K \times K' \text{ and } z=x+y\}$. If K=K'=X, we can also write $(f\nabla g)(z) = \inf_{x \in X}\{f(x)+g(z-x)\}$. This notion of convolution plays a crucial role in optimization and variational analysis (see for instance [58, 75, 76, 78] and references therein). We say that $f\nabla g$ is exact at a point $z \in$ K+K' if there exists $(x,y) \in K \times K'$ such that z=x+y and $(f\nabla g)(z) = f(x)+f(y)$.

THEOREM.1.9. The following assertions are equivalent:

(i) (f,K) anf (g,K) are well-posed in the Tikhonov sense with x_0,y_0 their unique solutions respectively.

(ii) $(f\nabla g, K+K')$ is well-posed in the Tikhonov sense with solution $z_0 = x_0+y_0$ and $f\nabla g$ is exact at z_0.

Proof. (i)$\Longrightarrow$(ii). For every $x \in K$, $y \in K'$ one has $f(x_0) \leq f(x)$ and $g(y_0) \leq g(y)$, so
$(f\nabla g)(x_0+y_0) \leq f(x_0)+g(y_0) \leq (f\nabla g)(z) \;\forall z \in K+K'$;
accordingly $z_0 = x_0+y_0 \in \text{argmin}(f\nabla g,K+K')$ and $(f\nabla g)(z_0) = f(x_0)+g(y_0)$ i.e $f\nabla g$ is exact at z_0. Now let $(z_n)_n$ be a sequence of K+K' such that $(f\nabla g)(z_n) \to (f\nabla g)(z_0)$, then there exists a sequence $(u_n,v_n) \in K \times K'$ such that $z_n = u_n+v_n$ and $f(u_n)+f(v_n) \to f(x_0)+g(y_0)$ so $f(u_n) \to f(x_0)$ and $f(v_n) \to f(y_0)$ and $(u_n,v_n) \to (x_0,y_0)$ i.e $z_n \to z_0$.

(ii)$\Longrightarrow$(i). The exactness of $f\nabla g$ at $z_0 = x_0+y_0$ implies that $(f\nabla g)(z_0) = f(x_0)+g(y_0)$
$\leq (f\nabla g)(z)$ for every $z \in$ K+K', so $f(x_0)+g(y_0) \leq f(x)+g(y) \;\forall(x,y) \in K \times K'$ and $x_0 \in \text{argmin}(f,K)$, $y_0 \in \text{argmin}(g,K')$. Now considering a sequence $(x_n)_n$ in K such that $f(x_n) \to f(x_0)$. For every n, we have $f(x_0)+g(y_0) \leq (f\nabla g)(x_n+y_0) \leq f(x_n)+g(y_0)$ so $(f\nabla g)(x_n+y_0) \to (f\nabla g)(z_0)$ and by hypothesis $x_n+y_0 \to x_0+y_0$ i.e $x_n \to x_0$ and (f,K) well-posed in the Tikhonov sense with x_0 its solution. In the same way (g,K) is well-posed in the Tikhonov sense with y_0 its solution.

2. NEW GENERALIZED REGULARIZATIONS IN THE TIKHONOV SENSE

The goal of this section is to introduce a new generalized regularization method in the Tikhonov sense in a general topological space and generalize a result established in reflexive spaces for convex functions [78, 79] and concerns the classical Tikhonov regularization

method used in variational analysis and its related topics. More precisely, considering a reflexive space X renormed by a strictly convex norm $\|.\|$ making it an E-space [3, 33, 34, 104]. Let $F: X \to \mathbb{R} \bigcup \{+\infty\}$ be a proper convex lower semicontinuous function and set for each $\epsilon > 0$, $F_\epsilon(x) = F(x) + \epsilon \|x-x_0\|^2$ where x_0 is any given point in X. Assume that $S = \mathrm{argmin}(F, X) = \{\overline{x} \in X / F(\overline{x}) = \min_{x \in X} F(x)\}$ is nonempty, then we have the following result:

THEOREM 2.1 [78, 79]. (a) F_ϵ has a unique minimizer x_ϵ over X for each ϵ; (b) the sequence $(x_\epsilon)_\epsilon$ converges strongly to $t_0 = \mathrm{proj}_S x_0$ when $\epsilon \to 0$; accordingly $\frac{F(x_\epsilon)\text{-}F(t_0)}{\epsilon} \to 0$, $\frac{F_\epsilon(x_\epsilon)\text{-}F(t_0)-\varepsilon\|x_0-t_0\|^2}{\epsilon} \to 0$, if $\epsilon \to 0$.

Throughout, unless otherwise stated, X stands for a general Hausdorff topological space, $f: X \to \mathbb{R} \bigcup \{+\infty\}$, $g: X \to \mathbb{R}$ be two lower semicontinuous (lsc) functions with f is proper and C be a nonempty closed subset of X. Along the paper we are concerned by the minimization problem (P): $\min_{x \in C} f(x)$. We will denote by $S = \mathrm{Argmin}(f, C)$ the solution set of (P) assumed to be nonempty and by $\mathrm{dom}f = \{x \in X / f(x) < +\infty\}$ the effective domain of f such that $\mathrm{dom}f \bigcap C \neq \emptyset$. Now consider a sequence $h_k: X \to \mathbb{R}$ of functions such that $r_k = \inf_{x \in C} h_k(x)$ is finite for all $k \geq k_0$. To (P) we associate the following generalized regularization problem (P_k): $\min_{x \in C} F_k(x)$ where $F_k(x) = f(x) + \epsilon_k g(x) + h_k(x)$, $\epsilon_k > 0$ and we suppose that $\epsilon_k \to 0$ if $k \to +\infty$.

THEOREM 2.2. Assume that the following conditions hold:

(a) $i_k = \inf_C F_k$ is finite for every $k \geq k_0$ and $(z_k)_k$ be a sequence of C relatively compact satisfying:

$$(2.1) \qquad \frac{F_k(z_k)\text{-}i_k}{\epsilon_k} \to 0, \; k \to +\infty$$

(b) $\qquad (2.2) \qquad \frac{h_k(s)-r_k}{\epsilon_k} \to 0, \; k \to +\infty, \; \forall \, s \in S$

Then:

(1) any cluster point $\overline{z} \in C$ of $(z_k)_k$ verifies $\overline{z} \in \mathrm{argmin}(g, S)$.

(2) $f(z_k) \to f(\overline{z})$ and $g(z_k) \to g(\overline{z})$ when $k \to +\infty$.

(3) there exist sequences $(\delta_k)_k, (\delta'_k)_k, (\theta_k)_k, (\theta'_k)_k$ of scalars converging to 0 such that we have the following asymptotic developments $F_k(z_k) = \min_{x \in C} f(x) + \epsilon_k \min_{x \in S} g(x) + \inf_{x \in C} h_k(x) + \epsilon_k \delta_k = \min_{x \in C} f(x) + \epsilon_k \min_{x \in S} g(x) + \inf_{x \in S} h_k(x) + \epsilon_k \delta'_k$; and $\inf_C F_k = \min_{x \in C} f(x) + \epsilon_k \min_{x \in S} g(x) + \inf_{x \in C} h_k(x) + \epsilon_k \theta_k = \min_{x \in C} f(x) + \epsilon_k \min_{x \in S} g(x) + \inf_{x \in S} h_k(x) + \epsilon_k \theta'_k$.

REMARK 2.3. In theorem 2.2, g is not necessarily positive as it is always supposed in the literature. On the other hand, it is clear that we can find always a sequence $(z_k)_k$ in C such that $F_k(z_k) \leq i_k + \epsilon_k^2$ if i_k is finite, so $\lim_{k \to +\infty} \frac{F_k(z_k)\text{-}i_k}{\epsilon_k} = 0$, but in general $(z_k)_k$ is not relatively compact . If (P_k) has a solution x_k for each $k \geq k_0$ and we take $z_k = x_k$ then (a) is straightforward satisfied if $(x_k)_k$ is relatively compact, so from (3) there exists a sequence $(\alpha_k)_k$ of numbers converging to 0 such that $F_k(x_k) = \min_{x \in C}(f(x) + \epsilon_k g(x) + h_k(x)) = \min_{x \in C} f(x) + \epsilon_k \min_{x \in S} g(x) + \inf_{x \in C} h_k(x) + \epsilon_k \alpha_k$. The proof we display here is very different from the one used in [78, 79].

Proof. Pick $\epsilon > 0$. By (2.1) we have for k large enough, $F_k(z_k) = f(z_k) + \epsilon_k g(z_k) + h_k(z_k) \leq \inf_C F_k + \epsilon_k \epsilon \leq f(s) + \epsilon_k g(s) + h_k(s) + \epsilon_k \epsilon \; \forall \, s \in S$. Then,

$$(2.3) \qquad 0 \leq f(z_k)\text{-}f(s) \leq \epsilon_k(g(s)\text{-}g(z_k)) + h_k(s)\text{-}r_k + \epsilon_k\epsilon \ , \ \forall \ s \in S$$

Hence we deduce that

$$(2.4) \qquad g(z_k) \leq g(s) + \frac{h_k(s)-r_k}{\epsilon_k} + \epsilon \ , \ \forall \ s \in S$$

and $\underline{\lim}_k g(z_k) \leq \overline{\lim}_k g(z_k) \leq g(s) + \epsilon$ by (2.2) for every $\epsilon > 0$ and $s \in S$. So,

$$(2.5) \qquad \underline{\lim}_k g(z_k) \leq \overline{\lim}_k g(z_k) \leq g(s), \ \forall \ s \in S$$

Now by lower semicontinuity of g and relative compactness of $(z_k)_k$ we can find $m \in \mathbb{R}$ such that $m \leq g(z_k) \ \forall \ k$. We derive from (2.5) that $(g(z_k))_k$ is bounded and $f(z_k) \rightarrow \min_{x \in C} f(x)$ by (2.2) and (2.3); so by lower semicontinuity of f we get $f(\overline{z}) = \min_{x \in C} f(x)$ for any cluster point $\overline{z} \in C$ of $(z_k)_k$ i.e $\overline{z} \in S$. Since g is lower semicontinuous, (2.4) and (2.2) imply that $\overline{z} \in \text{argmin}(g, S)$. Again by lower semicontinuity of g, boundedness of $(g(z_k))_k$ and (2.5) we may check easily that $(g(z_k))_k$ has the unique cluster point $g(\overline{z}) = \min_{s \in S} g(s)$ to which the sequence converges. This ends the proof of (1) and (2). Now take $s = \overline{z}$ in (2.3); we have $0 \leq \frac{f(z_k)-\min_C f}{\epsilon_k} \leq g(\overline{z})\text{-}g(z_k) + \frac{h_k(\overline{z})-r_k}{\epsilon_k} + \epsilon$, hence $\alpha_k = \frac{f(z_k)-\min_C f}{\epsilon_k} \rightarrow 0$ when $k \rightarrow +\infty$. Keeping in mind that $h_k(z_k) \leq h_k(\overline{z}) + \epsilon_k(g(\overline{z})\text{-}g(z_k)) + \epsilon_k\epsilon$, we derive that $0 \leq \frac{h_k(z_k)-r_k}{\epsilon_k} \leq \frac{h_k(\overline{z})-r_k}{\epsilon_k} + g(\overline{z})\text{-}g(z_k) + \epsilon$ and then $\frac{h_k(z_k)-r_k}{\epsilon_k} \rightarrow 0$ when $k \rightarrow +\infty$ by (2.2) and (2). On the other hand, $\frac{F_k(z_k)-f(z_k)-r_k}{\epsilon_k} = \frac{\epsilon_k g(z_k)+h_k(z_k)-r_k}{\epsilon_k} = g(z_k) + \frac{h_k(z_k)-r_k}{\epsilon_k}$. Set $\vartheta_k = \frac{F_k(z_k)-f(z_k)-r_k}{\epsilon_k} - g(\overline{z})$. It is clear that ϑ_k and $\delta_k = \vartheta_k + \alpha_k = \frac{F_k(z_k)-f(\overline{z})-r_k}{\epsilon_k} - g(\overline{z})$ converge to 0 if $k \rightarrow +\infty$ and $F_k(z_k) = f(\overline{z}) + \epsilon_k g(\overline{z}) + r_k + \epsilon_k\delta_k = \min_{x \in C} f(x) + \epsilon_k \min_{x \in S} g(x) + \inf_{x \in C} h_k(x) + \epsilon_k\delta_k$. From (b) it is easy to see that $\frac{\inf_{s \in S} h_k(s)-r_k}{\epsilon_k} = d_k \rightarrow 0$, $k \rightarrow +\infty$, so $r_k = \inf_{s \in S} h_k(s) - \epsilon_k d_k$ and $F_k(z_k) = \min_{x \in C} f(x) + \epsilon_k \min_{x \in S} g(x) + \inf_S h_k + \epsilon_k\delta'_k$ with $\delta'_k = \delta_k - d_k$. The asymptotic development in $\inf_C F_k$ is an immediate consequence of the last developments and (2.1) which completes the proof of (3). $\quad\square$

REMARK 2.4. Hypothesis (b) in the above theorem is not restrictive. Indeed, consider the wide class of functions h_k given by $h_k(x) = \sum_{i=1}^p \beta_{k,i} g_i(x)$ where $g_i \colon X \rightarrow \mathbb{R}$ is any function bounded below by a scalar m_i on C, $\beta_{k,i} \geq 0$ for every $k \in \mathbb{N}$, $1 \leq i \leq p$ and $\frac{\beta_{k,i}}{\epsilon_k} \rightarrow 0$ when $k \rightarrow +\infty, \forall \ i$. It is easy to see that $r_k \geq \sum_{i=1}^p \beta_{k,i} m_i$ and $0 \leq \frac{h_k(x)-r_k}{\epsilon_k} \leq \sum_{i=1}^p \frac{\beta_{k,i}}{\epsilon_k}(g_i(x)\text{-}m_i)$ which goes to 0 when $k \rightarrow +\infty$ for every $x \in C$, so (b) is satisfied. The regularization functions become $F_k(x) = f(x) + \epsilon_k g(x) + \sum_{i=1}^p \beta_{k,i} g_i(x)$. In particular one may consider the functions of the kind $h_k(x) = \epsilon_k^q h(x)$, $q > 1$ with $h \colon X \rightarrow \mathbb{R}$ is any function such that $\inf_C h > -\infty$. If X is a normed space we can use also the regularization functions $F_k(x) = f(x) + \epsilon_k \|x\text{-}x_1\|^{p_1} + \sum_{i=2}^p \epsilon_k^i \|x\text{-}x_i\|^{p_i}$ where x_i is any given point in X and $p_i \in \mathbb{N}$. More generally, if we take in a general topological space $h_k(x) = \sum_{i=1}^p \beta_{k,i} \, g_{k,i}(x)$ with $g_{k,i} \colon X \rightarrow \mathbb{R}$, $\beta_{k,i} \geq 0$, $\frac{\beta_{k,i}}{\epsilon_k} \rightarrow 0$ if $k \rightarrow +\infty$, $\forall \ i = 1,2...p$ and we assume that there exist scalars $m_{k,i}$ such that $m_{k,i} \leq g_{k,i}(x), \ \forall \ x \in C$ with $\lim_{k \rightarrow +\infty} \sum_{i=1}^p \frac{\beta_{k,i}}{\epsilon_k}(g_{k,i}(x)\text{-}m_{k,i}) = 0$ for every x in C, then (b) is satisfied. In particular, one may consider the useful regularization functions $h_k(x) = \sum_{i=1}^p q_i(\epsilon_k) e^{\varphi_i(\epsilon_k)(A_i x - b_i) + \gamma_i}$ where $\varphi_i(\epsilon_k) \geq 0$, $q_i(\epsilon_k) \geq 0$, $\gamma_i \in \mathbb{R}$ for every i=1,...p and $k \in \mathbb{N}$ with $\frac{q_i(\epsilon_k)}{\epsilon_k} \rightarrow 0$ if $k \rightarrow \infty$, $b_i \in \mathbb{R}$ and $A_i \colon X \rightarrow \mathbb{R}$ is any lower semicontinuous operator. In this case the constraint set is defined by $C = \{x \in X / A_i x \text{-} b_i \leq 0, \ \forall i\}$ assumed to be nonempty. For the case $q_i(\epsilon_k) = \epsilon_k^2$,

$\varphi_i(\epsilon_k) = \epsilon_k^{-2}$, $\gamma_i = 0$ and its importance in optimization, see for instance [28] where the authors study the special regularizations $F_k(x)=\langle c,x\rangle + \epsilon_k^2 \sum_{i=1}^p e^{\epsilon_k^{-2}(A_i x - b_i)}$ for solving the linear program $\min\{\langle c,x\rangle: A_i x - b_i \leq 0, i=1...p\}$ in finite dimentional setting with A_i is a linear operator. This kind of regularizations combines the interesting properties of the interior barrier method and of the exterior penalty method. It should be pointed out that we can't always ensure higher order asymptotic developments in theorem 2.2 even under strong regular conditions on functions under consideration. For example consider the sequence of regularization functions $F_\epsilon(x)=x^2 + \epsilon x + \epsilon^2 x^2$, $x \in C = \mathbb{R}$ reaching their minimizers at the points $x_\epsilon = \frac{-\epsilon}{2}(1 + \epsilon^2)^{-1}$. If we write $F_\epsilon(x_\epsilon)=\frac{-\epsilon^2}{4}(1 + \epsilon^2)^{-1} =\min_{\mathbb{R}} x^2 + \epsilon\min_{S=\{0\}} x+\epsilon^2\min_{\mathbb{R}} x^2 + \epsilon^2\varphi(\epsilon)$ then $\varphi(\epsilon) = \frac{-1}{4}(1 + \epsilon^2)^{-1} \to -\frac{1}{4}$ if $\epsilon \to 0$. Now take another example: Let $F_\epsilon(x)=f(x)+\epsilon x^2 + \epsilon^2 x^3$, $0< \epsilon < 1$ with $f(x)=$ x-1 if $x \geqslant 1$, $f(x)=0$ if $x \in [-1, 1]$ and $f(x)= +\infty$ otherwise. Then S=argminf=$[-1, 1]$ and the minimizer of F_ϵ is attained at $x_\epsilon = 0$. If $F_\epsilon(x_\epsilon)=0 = \min_{x\geq-1} f + \varepsilon \min_{x\in S} x^2 + \epsilon^2\min_S x^3 + \epsilon^2\varphi(\epsilon)$ then $\varphi(\epsilon) = 1$.

Now we state two corollaries expressing that under suitable hypotheses, a sequence $(z_k)_k$ in C satisfying (2.1) is relatively compact. First recall that a function j: $X\to \overline{\mathbb{R}}$ is inf-compact [58] if $\{x \in X/j(x) \leq \lambda\}$ is compact for each $\lambda \in \mathbb{R}$. If for a subset A of X $\{x \in A/j(x) \leq \lambda\}$ is compact for each $\lambda \in \mathbb{R}$, we say that j is inf-compact on A. If X is a metric space and $j_k : X \to \overline{\mathbb{R}}$, $k \in \mathbb{N}$, we denote by e-$\underline{\lim}j_k(x) = \inf_{\{(x_k)_k/x_k\to x\}}\underline{\lim} j_k(x_k)$ the epi-limit inferior of the sequence $(j_k)_k$ [3] .

COROLLARY 2.5. Let F_k be the functions considered in Th.2.2. Let $(z_k)_k$ be a sequence in C such that: (a_1) $i_k =\inf_C F_k$ is finite for every $k\geq k_1$ and $\lim_{k\to+\infty} \frac{F_k(z_k)-i_k}{\epsilon_k} = 0$; (a_2) hypothesis (b) of Th.2.2 is satisfied; (a_3) $\{x \in C/g(x) \leq \lambda_0\}$ is compact for some scalar $\lambda_0 >g(s_0)$ and some $s_0 \in$ S. Then $(z_k)_k$ is relatively compact.

Proof. From the proof of Th.2.2, we see that $\overline{\lim}_k g(z_k) \leq g(s)$ for every $s \in$ S. Let $\lambda_0 >g(s_0)$ for some $s_0 \in$ S such that $L_{\lambda_0} =\{x \in C/g(x) \leq \lambda_0\}$ is compact . We have $\overline{\lim}_k g(z_k) < \lambda_0$ and $z_k \in L_{\lambda_0}$ for every k sufficiently large, so $(z_k)_{k\ k}$ is relatively compact and the conclusions of Th.2.2 hold.

COROLLARY 2.6. Let F_k be the functions considered in Th.2.2 such that hypothesis (a_1) of Corollary 2.5 is satisfied with $\overline{\lim}_k F_k(z_k) < +\infty$ and g is bounded below on C by a scalar m. Consider the following hypotheses (H_1) and (H_2) with (H_1) : There exists $\gamma \in \mathbb{R}$ such that $r_k \geq \gamma$, $\forall k \geq k_2$ and f is inf-compact on C; (H_2) : X is a vector space of finite dimension, $h_k : X \to \mathbb{R}$ are lsc and $\forall \lambda \in \mathbb{R} \exists k_\lambda \in \mathbb{N}$ such that $\forall k\geq k_\lambda$ the set $\{x \in X/h_k(x) \leq \lambda\}$ is connected (in particular this is true if h_k is convex) and the function $x \in X\to$ e-$\underline{\lim}h_k(x)$ is inf-compact. If (H_1) or (H_2) is satisfied then $(z_k)_{k\ k}$ is relatively compact; accordingly if hypothesis (b) of Th.2.2 is satisfied then the conclusions of Th.2.2 hold.

Proof. First we observe that there exist two scalars δ, β such that for all k sufficiently large $F_k(z_k) \leq \delta$ and $F_k(x)\geq f(x)+\beta+h_k(x)$ $\forall x \in$C, so $F_k(x)\geq f(x)+\beta+\gamma$ if (H_1) is verified, and then $z_k \in \{x \in C/f(x) \leq \delta - \beta - \gamma\}$ which is compact. On the other hand, it is clear that there exists a number α such that $F_k(x)\geq \alpha+h_k(x)$ for every $x \in$C and every k large

enough. Now if (H_2) is satisfied, by [109] the functions h_k are uniformly inf- compact, in particular there exist a compact K and k'$\in \mathbb{N}$ such that $\forall k \geq k'$, $\{x \in X/h_k(x) \leq \delta - \alpha\} \subseteq$K, so $z_k \in$K. The reminder follows wich completes the proof.

REMARK 2.7. Hypothesis (H_2) in Corollary 2.6 is in particular satisfied if X is a vector space of finite dimension and $(h_k)_k$ is a sequence of convex functions from X into $\mathbb{R}$ epi-converging to a proper inf-compact function h:X$\to \mathbb{R} \bigcup \{+\infty\}$ [109].

COROLLARY 2.8. Let f:C$\to \mathbb{R} \bigcup \{+\infty\}$, g_i: C$\to \mathbb{R}$, i=1,2...p, be lower semi-continuous functions and assume that f is finite at a point of a compact C of X.Then, $\lim_{\epsilon \to 0^+} \frac{\min_{x \in C}(f(x) + \epsilon g_1(x) + \sum_{i=2}^{p} a_{i\epsilon} g_i(x)) - \min_{x \in C} f(x)}{\epsilon} = \min_{x \in S} g_1(x)$ for every sequences $(a_{i\epsilon})_\epsilon$, i=2, ...p such that $a_{i\epsilon} \geq 0$ and $\lim_{\epsilon \to 0^+} \frac{a_{i\epsilon}}{\epsilon} = 0$.

Proof. The proof is an immediate consequence of theorem 2.2, remark 2.4 and the fact that $\lim_{\epsilon \to 0^+} \frac{\inf_{x \in C} h_\epsilon(x)}{\epsilon} = 0$ with $h_\epsilon(x) = \sum_{i=2}^{p} \epsilon^i g_i(x)$.

2.9. General Asymptotic Developments with Particular Regularizations

It is useful to provide general asymptotic developments with particular regularizations. In the sequel we are concerned by the regularizations of the kind $F_k(x) = f(x) + \epsilon_k g(x) + \sum_{i=1}^{p} \beta_{k,i} g_i(x)$ as it is mentioned above with $\beta_{k,i} > 0$, $\frac{\beta_{k,i}}{\epsilon_k} \to 0$ for every i when k$\to +\infty$. We assume furthermore that all functions g_i: X$\to \mathbb{R}$ are lsc with $\frac{\beta_{k,i+1}}{\beta_{k,i}} \to 0$ when k$\to +\infty$, $\forall i$ and C is compact.Then $S_i = \text{argmin}(g_i, C)$ is nonempty and $g_i(x) \geqslant m_i \ \forall \ x \in$C for a scalar m_i. Our goal is to compute $\min_C F_k$. In each following computation, assume that we are within the framework of hypotheses of theorem 2.2 with functions under consideration. We have $\min_C(f + \epsilon_k g + \beta_{k,1} g_1) = \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \epsilon_k \alpha_{k,1}$; $\min_C(f + \epsilon_k g + \beta_{k,1} g_1 + \beta_{k,2} g_2) = \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C (g_1 + \frac{\beta_{k,2}}{\beta_{k,1}} g_2) + \epsilon_k \alpha_{k,2} = \min_C f + \epsilon_k \min_S g + \beta_{k,1}(\min_C g_1 + \frac{\beta_{k,2}}{\beta_{k,1}} \min_{S_1} g_2 + \frac{\beta_{k,2}}{\beta_{k,1}} \delta_{k,2}) + \epsilon_k \alpha_{k,2} = \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \beta_{k,2} \min_{S_1} g_2 + \beta_{k,2} \delta_{k,2} + \epsilon_k \alpha_{k,2}$. By the same argument we have $\min_C(f + \epsilon_k g + \beta_{k,1} g_1 + \beta_{k,2} g_2 + \beta_{k,3} g_3) = \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \beta_{k,2} \min_{S_1} g_2 + \beta_{k,3} \min_C g_3 + \beta_{k,2} \delta_{k,3} + \epsilon_k \alpha_{k,3}$ and $\min_C(f + \epsilon_k g + \beta_{k,1} g_1 + \beta_{k,2} g_2 + \beta_{k,3} g_3 + \beta_{k,4} g_4) = \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \beta_{k,2} \min_{S_1} g_2 + \beta_{k,3} \min_C g_3 + \beta_{k,4} \min_{S_3} g_4 + \beta_{k,2} \delta_{k,4} + \beta_{k,4} \delta'_{k,4} + \epsilon_k \alpha_{k,4}$. Then the obvious generalization is $\min_C(f + \epsilon_k g + \beta_{k,1} g_1 + \beta_{k,2} g_2 + ... + \beta_{k,p} g_p) = \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \beta_{k,2} \min_{S_1} g_2 + ... + \beta_{k,p} \min_{S_{p-1}} g_p + d_k^p + \epsilon_k v_{k,p}$ if p is even and $\min_C(f + \epsilon_k g + \beta_{k,1} g_1 + \beta_{k,2} g_2 + ... + \beta_{k,p} g_p) = \min_C f + \epsilon_k \min_S g + \beta_{k,1} \min_C g_1 + \beta_{k,2} \min_{S_1} g_2 + ... + \beta_{k,p} \min_C g_p + e_k^p + \epsilon_k w_{k,p}$ if p is odd with $v_{k,p}$, $w_{k,p}$, $\frac{d_k^p}{\epsilon_k}$, $\frac{e_k^p}{\epsilon_k}$ converge to 0 when k$\to +\infty$.

3. STABILITY OF VARIATIONAL ASYMPTOTIC DEVELOPMENTS BY SEQUENTIAL EPI-CONVERGENCE

In this short subsection we investigate the stability of asymptotic developments under epi-convergence, that is a variational convergence [3] which preserves the stability of the minimum of functions if a relative compactness hypothesis is satisfied. In order to state our result, we recall the definition of epi-convergence and its importance in variational analysis. Let f_n,f :X$\to \overline{\mathbb{R}}$ be a sequence of functions. We say that $(f_n)_n$ epi-converges sequentially to f

on X and we write f_n $\underrightarrow{epi-seq}$ f if: (1) $\forall x, \forall x_n \to x, f(x) \leq \underline{\lim}_n f_n(x_n)$; (2)$\forall x, \exists x_n \to x$ such that $f_n(x_n) \to f(x)$. Convergence in this sense has remarkable properties in the literature. One of the crucial properties in a general topological setting is the following stability result:

THEOREM 3.1 [3] . Assume that f_n $\underrightarrow{epi-seq}$ f and consider a sequence $(x_n)_n$ in X such that $f_n(x_n) \leq \inf_X f_n + \gamma_n, \gamma_n \to 0$ with $\gamma_n \geqslant 0$ and $\inf_X f_n \in \mathbb{R}$. Then for every converging subsequence $(x_{n_k})_k$ to an element $\overline{x}$, we have $\overline{x} \in \text{argmin}(f, X)$ and $f_{n_k}(x_{n_k}) \to \min_X f$ when k$\to +\infty$.

Now return to problem (P) with C=X and assume that the lsc proper functions f_n:X$\to \mathbb{R} \bigcup \{+\infty\}$ epi-converge sequentially to f:X$\to \mathbb{R} \bigcup \{+\infty\}$ and there exists a sequence $(x_n)_n$ in X having a subsequence $x_{n_p} = y_p \to \overline{t}$ and $\min_X f_n = f_n(x_n)$ $\forall n$. Denote by $F_k^n(x)= f_n(x)+\epsilon_k g_n(x)+h_k(x)$ the regularization functions associated to f_n with g_n:X$\to \mathbb{R}$ is lsc and suppose that h_k $\underrightarrow{epi-seq}$ h with h:X$\to \mathbb{R} \bigcup \{+\infty\}$ is proper. Assume also that there exists a sequence $(z_k^n)_{k,n}$ relatively compact such that for every k, n, $i_k^n = \inf_X F_k^n$ is finite, $\frac{F_k^n(z_k^n)-i_k^n}{\epsilon_k} \to 0, \frac{h_k(s)-r_k}{\epsilon_k} \to 0$ when k$\to +\infty$, $\forall s \in S_n = \text{argmin}(f_n, X)$, $\forall$ n. Following theorem 2.2, there exists a sequence $(\delta_k^n)_{k,n}$ of scalars converging to 0 for each fixed n when k$\to +\infty$ such that we have the following asymptotic development $A_k^n = F_k^n(z_k^n) = \min_X f_n + \epsilon_k \min_{S_n} g_n + \inf_X h_k(x) + \epsilon_k \delta_k^n$. The stability result is stated as follows:

THEOREM 3.2. There exists a subsequence $(n_k')_k$ satisfying

$(A_k^{n_k'}, \delta_k^{n_k'}, \frac{F_k^{n_k'}(z_k^{n_k'})-i_k^{n_k'}}{\epsilon_k}) \to (\min_X f+\min_X h = f(\overline{t})+h(\overline{t}),0,0)$ if k$\to +\infty$ where $\overline{t} \in \text{argmin}(f, X) \bigcap \text{argmin}(h, X)$. In particular if $z_k^n = x_k^n \in \text{argmin}(F_k^n, X)$ we have $\min_X(f_{n_k'}+\epsilon_k g_{n_k'}+h_k) \to \min_X(f+h)=\min_X f+\min_X h$ when k$\to +\infty$.

Proof. Because $d_k^p = \frac{h_k(y_p)-r_k}{\epsilon_k} \to 0$ when k$\to +\infty$ $\forall p$, the diagonalization lemma [3] shows that there exits a subsequence $(p_k)_k$ (which can be computed) such that $d_k^{p_k} \to 0$ if k$\to +\infty$, so for a given $\epsilon > 0, h_k(y_{p_k}) \leq \inf_X h_k + \epsilon_k\epsilon$ for k large enough, and by theorem 3.1 we conclude that $\inf_X h_k \to \min_X h$ and $\overline{t} \in \text{argmin}(f, X) \bigcap \text{argmin}(h, X)$. But for each fixed p, $(A_k^{n_p}, \delta_k^{n_p}, \frac{F_k^{n_p}(z_k^{n_p})-i_k^{n_p}}{\epsilon_k}) \to (\min_X f_{n_p}+\min_X h, 0,0)$ if k$\to +\infty$ and $\min_X f_{n_p} \to \min_X f$ when p$\to +\infty$; accordingly by the diagonalization lemma again, there exists a subsequence $(n_{p_k}=n_k')_k$ satisfying $(A_k^{n_k'}, \delta_k^{n_k'}, \frac{F_k^{n_k'}(z_k^{n_k'})-i_k^{n_k'}}{\epsilon_k}) \to (\min_X f+\min_X h ,0,0)$ if k$\to +\infty$. The remaider follows, which completes the proof. $\square$

REMARK 3.3. It is worth pointing out that the stability result established here holds regardless of the epi-convergence or not of the sequence ($f_{n_k'}+\epsilon_k g_{n_k'}+h_k)_k$ to f+h! and without having any information on the behavior of the sequence $(\epsilon_k g_{n_k'})_k$!.

4.THE CONVEX CASE

In this section we apply the above results to convex functions defined on a normed space and derive asymptotic developments for the Legendre-Fenchel transform.

PROPOSITION 4.1. Let X be a reflexive Banach space and f: $X \to \mathbb{R} \bigcup \{+\infty\}$, g, $h_k : X \to \mathbb{R}$ are convex proper lower semicontinuous functions. Let C be a nonempty closed convex set of X such that S=argmin(f, C) is nonempty. Assume that:

(a) $\{x \in C / \, g(x) \leq \lambda\}$ is bounded $\forall \lambda \in \mathbb{R}$;

(b) $\frac{h_k(s) - r_k}{\epsilon_k} \to 0$ if $k \to +\infty$, $\forall s \in S$.

Let $(z_k)_k$ be a sequence of C such that $\frac{F_k(z_k) - i_k}{\epsilon_k} \to 0$ when $k \to +\infty$. Then $(z_k)_k$ is weakly relatively compact and the conclusion of theorem 2.2 holds. If argmin(g, S)=$\{\bar{a}\}$ (particularly when g is strictly convex) then $(z_k)_k$ has a unique weakly cluster point $\bar{z}=\bar{a}$ and $z_k \rightharpoonup \bar{a}$ where $\rightharpoonup$ denotes the weak convergence. Moreover if the conditions $g(t_k) \to g(t)$ and $t_k \rightharpoonup t$ imply that $\|t_k - t\| \to 0$, $k \to +\infty$ one has $\|z_k - \bar{a}\| \to 0$ when $k \to +\infty$.

Proof. First, we point out that $L_\lambda^k = \{x \in C / \, F_k(x) \leq \lambda\} \subset \{x \in C / g(x) \leq \frac{\lambda - \min_C f - r_k}{\epsilon_k}\}$ for each k and $\lambda \in \mathbb{R}$, so L_λ^k is weakly compact and by a classical argument [58] , $i_k = \min_C F_k = F_k(x_k)$ for some $x_k \in C$. Now by reflexivity, convexity and Corollary 2.5, it is immediat that $(z_k)_k$ is weakly relatively compact. The remainder follows by an obvious verification. $\square$

In the theorem below, we give sufficient conditions ensuring that the minimization problem (F_k , C) is strongly well-posed in the Tikhonov sense. Given two functions p, q: $X \to \mathbb{R} \bigcup \{+\infty\}$ and consider the following hypotheses:

(H_p): $A_\lambda = \{x \in C / p(x) - \lambda \|x\| \leq 0\}$ is bounded for every $\lambda \in \mathbb{R}$.

(H_q'): $q(x_n) \to q(x)$ and $x_n \rightharpoonup x$ imply that $\|x_n - x\| \to 0$ when $n \to +\infty$.

THEOREM 4.2. Let X be a reflexive Banach space and suppose that $F_k(x) = f(x) + \epsilon_k g(x) + h_k(x)$ is strictly convex on a closed convex subset C of X. Assume that (H_p) and (H_q') hold simultaneously at least for two functions p, q (eventually identical) belonging in the set $\{f, g, h_k\}$ where f: $X \to \mathbb{R} \bigcup \{+\infty\}$, g, $h_k : X \to \mathbb{R}$ are convex proper lower semicontinuous functions. Then (F_k, C) is well-posed in the Tikhonov sense.

Proof. Let $F_k(x) = f(x) + \epsilon_k g(x) + h_k(x)$, $\epsilon_k > 0$. Each function f, g, h_k has a continuous affine minorant, so if $p \in \{f, g, h_k\}$ then there exist $\alpha \geq 0$, $\beta \in \mathbb{R}$ such that $F_k(x) \geq -\alpha \|x\| + \beta + \gamma p(x)$ ($\gamma = \varepsilon_k$ if p=g and γ=1 otherwise). It turns out that $L_\lambda^k = \{x \in C / F_k(x) \leq \lambda\} \subseteq \{x \in C / p(x) - \frac{\alpha}{\gamma} \|x\| + \frac{\beta}{\gamma} \leq \frac{\lambda}{\gamma}\}$ which is bounded by (H_p) for every $\lambda \in \mathbb{R}$; so L_λ^k is weakly compact and (F_k, C) has a unique solution x_k. Now let $(x_n)_n$ be a mimizing sequence for (F_k, C). Since $F_k(x_n) \to F_k(x_k) \in \mathbb{R}$ when $n \to +\infty$, $(x_n)_n$ belongs to a sublevel of F_k and $(x_n)_n$ is bounded. By lower semicontinuity of F_k, every cluster point $z \in C$ of $(x_n)_n$ for the weak topology satisfies $F_k(z) = F_k(x_k)$ so $z = x_k$ and $x_n \rightharpoonup x_k$. Set $a_n = f(x_n) + \varepsilon_k g(x_n) + h_k(x_n) - f(x_k) - \varepsilon_k g(x_k) - h_k(x_k) \to 0$ if $n \to +\infty$. We have $f(x_n) - f(x_k) = a_n + \varepsilon_k(g(x_k) - g(x_n)) + h_k(x_k) - h_k(x_n)$ and $\overline{\lim}_n(f(x_n) - f(x_k)) \leq \varepsilon_k(g(x_k) - \underline{\lim}_n g(x_n)) + h_k(x_k) - \underline{\lim}_n h_k(x_n) \leq 0$; accordingly $\overline{\lim}_n f(x_n) \leq f(x_k) \leq \underline{\lim}_n f(x_n)$ and $f(x_n) \to f(x_k)$. By the same argument one has $g(x_n) \to g(x_k)$ and $h_k(x_n) \to h_k(x_k)$. From (H_q') we conclude that $\|x_n - x_k\| \to 0$ when $n \to +\infty$. $\square$

Note that a reflexive Banach space may be always renormed by a strictly convex norm $\|.\|$ such that (H'_q) is satisfied with q(x)=$\|.\|$ [104]; so one can take for instance in proposition 4.1 or in theorem 4.2, g(x)=$\|x - x_0\|^r$, r$\geq$ 1 and x_0 is any given point in X. In this case the sequence $(x_k)_k$, where x_k is the minimizer of F_k on C, converges strongly to $\text{proj}_S x_0$ in Proposition 4.1 (or in Theorem 4.2 if (b) is satisfied). Also we point out that, even though $(H_p)$and (H'_q) fail to be satisfied with p, q$\in$\{f, g\}which can be imposed by an algorithm, a large choice of the " negligible" terms h_k may guarantee the verification of $(H_{h_k})$and (H'_{h_k}). It is worth noting that the only role of hypothesis (H_p) is to ensure the boundedness of L^k_λ. It can be replaced for instance by the following hypothesis: two functions of $\{f,g,h_k\}$are bounded below and the third is weakly inf-compact; for example f, g are bounded below and h_k is weakly inf-compact.

4.3. Application to Legendre-Fenchel Transform

In the sequel we are interesting by asymptotic developments of the Legendre-Fenchel transform [58] $(f+\epsilon^k_1 g_1 + \epsilon^k_2 g_2 + ...\epsilon^k_n g_n)^*(y)$ where y is a fixed point of the topological dual X^* of a locally convex space X and f: X$\to \mathbb{R}\bigcup \{+\infty\}$ is a convex proper lower semicontinuous function, $g_k : X\to \mathbb{R}$, k= $1, 2, ...n$ are convex continuous functions. Assume that $\epsilon^k_i \to 0, \epsilon^k_i > 0$ and $\frac{\epsilon^k_{i+1}}{\epsilon^k_i} \to 0$ if k$\to +\infty$ $\forall i = 1...n$. For simplicity, take $\epsilon^k_i = \epsilon^i$ and $\epsilon \to 0$. If n=1 we have $(f+\epsilon g_1)^*(y) = \sup_{x\in X}\{\langle x,y\rangle - (f + \epsilon g_1)(x)\}$. Set $\alpha_1 = -(f+\epsilon g_1)^*(y) = \inf_{x\in X}\{f(x) - \langle x,y\rangle + \epsilon g_1(x)\}$ and assume that there exists a class of hypotheses such that $S_y = \arg\min_X(f(.) - \langle .,y\rangle)$ is nonempty and $\alpha_1 = \min_{x\in X}\{f(x) - \langle x,y\rangle + \epsilon g_1(x)\} = \min_{x\in X}(f(x) - \langle x,y\rangle)+ \epsilon \min_{x\in S_y} g_1(x) + \epsilon\varphi_1(y,\epsilon)$ where $\varphi_1(y,\epsilon) \to 0$ if $\epsilon \to 0$. But z$\in S_y$ if and only if f(z)-$\langle z,y\rangle \leq \inf_{x\in X}$f(x)-$\langle x,y\rangle = -f^*(y)$ or equivalently, f(z)+f$^*(y) \leq \langle z,y\rangle$, i.e z$\in \partial$f*(y) the subdifferential in the sense of convex analysis of f* at y [58]. It turns out that $(f+\epsilon g_1)^*(y) = f^*(y) + \epsilon(g_1 + \delta_{\partial f^*(y)})^*(0)+ \epsilon\varphi_1(y,\epsilon)$ where $\delta_{\partial f^*(y)}(z) = 0$ if z$\in \partial$f*(y) and $\delta_{\partial f^*(y)}(z) = +\infty$ otherwise. By [58, Th.6.5.8], one has $(g_1 + \delta_{\partial f^*(y)})^*(0) = (g_1^*\nabla\delta_{\partial f^*(y)}^*)(0) = g_1^*(t) + \delta_{\partial f^*(y)}^*(-t)$ for some t$\in X^*$ where the operation $\bigtriangledown$ stands for the infimal convolution, called also epi-sum in the literature (for the importance of this operation in optimization, see for instance [5, 58, 76] and references therein). Furthermore if f*is finite and $\tau(X^*, X)$ continuous at y then $\delta_{\partial f^*(y)}^*(w) = \max_{r\in\partial f^*(y)}\langle w,r\rangle = (f^*)'(y,w)$ $\forall w$ [58, Th.6.4.8]. Here $(f^*)'(y,w) = \lim_{\lambda\to 0^+} \frac{f^*(y+\lambda w)-f^*(y)}{\lambda}$ is the directional derivative and $\tau(X^*, X)$ is the Mackey topology on X^*. So $(f+\epsilon g_1)^*(y) = f^*(y)+\epsilon(g_1^*\nabla(f^*)'(y,.))(0) + \epsilon\varphi_1(y,\epsilon) = f^*(y) + \epsilon(g_1^*(t)+ (f^*)'(y,-t)) + \epsilon\varphi_1(y,\epsilon)$. For n=2, assume that under suitable hypotheses we have $\min_{x\in X}(f(x) - \langle x,y\rangle + \epsilon g_1(x) + \epsilon^2 g_2(x)) = \min_{x\in X}(f(x) - \langle x,y\rangle) + \epsilon \min_{x\in S_y=\partial f^*(y)} g_1(x) + \epsilon^2\inf_{x\in X} g_2(x)+ \epsilon\varphi_2(y,\epsilon), \varphi_2(y,\epsilon) \to 0$ when $\epsilon \to 0$. Hence, $(f+\epsilon g_1+\epsilon^2 g_2)^*(y) = f^*(y)+\epsilon(g_1+ \delta_{\partial f^*(y)})^*(0) + \epsilon^2 g_2^*(0) + \epsilon\varphi_2(y,\epsilon)$=f$^*(y)+\epsilon(g_1^*\nabla(f^*)'(y,.))(0) + \epsilon^2 g_2^*(0)+ \epsilon\varphi_2(y,\epsilon)$ if f*is finite and $\tau(X^*, X)$ continuous at y. For n=3 assume that there exist some assumptions under which one has the following equalities: -$(f+\epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3)^*(y) = \min_{x\in X}(f(x) - \langle x,y\rangle +$

$\epsilon g_1(x) + \epsilon^2(g_2(x) + \epsilon g_3(x)))) = \min_{x \in X}(f(x) - \langle x, y \rangle) + \epsilon \min_{x \in \partial f^*(y)} g_1(x) + \epsilon^2 \min_{x \in X}(g_2(x) + \epsilon g_3(x)) + \epsilon k(y, \epsilon)$

$= \min_{x \in X}(f(x) - \langle x, y \rangle) + \epsilon \min_{x \in \partial f^*(y)} g_1(x) + \epsilon^2(\min_X g_2) + \epsilon^3 \min_{x \in \partial g_2^*(0)} g_3(x) + \epsilon^3 \gamma(\epsilon) + \epsilon k(y, \epsilon)$; then we compute $(f + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3)^*(y) = f^*(y) + \epsilon(g_1 + \delta_{\partial f^*(y)})^*(0) + \epsilon^2 g_2^*(0) + \epsilon^3(g_3 + \delta_{\partial g_2^*(0)})^*(0) + \epsilon \varphi_3(y, \epsilon)$ with $\varphi_3(y, \epsilon) \to 0$ when $\epsilon \to 0$. If f*and g_2^* are respectively $\tau(X^*, X)$ continuous at y and 0 then $(f + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3)^*(y) = f^*(y) + \epsilon(g_1^* \nabla (f^*)'(y, .))(0) + \epsilon^2 g_2^*(0) + \epsilon^3(g_3^* \nabla (g_2^*)'(0, .))(0) + \epsilon \varphi_3(y, \epsilon)$. A straightforward generalization shows that under suitable hypotheses we have:

$(f + \epsilon g_1 + \epsilon^2 g_2 + ... + \epsilon^n g_n)^*(y) = f^*(y) + \epsilon(g_1 + \delta_{\partial f^*(y)})^*(0) + \epsilon^2 g_2^*(0) + \epsilon^3(g_3 + \delta_{\partial g_2^*(0)})^*(0) + \epsilon^4 g_4^*(0) + \epsilon^5(g_5 + \delta_{\partial g_4^*(0)})^*(0) + ... \epsilon^{2i} g_{2i}^*(0) + \epsilon^{2i+1}(g_{2i+1} + \delta_{\partial g_{2i}^*(0)})^*(0) + ... \epsilon \varphi_n(y, \epsilon)$

(4.1) $= f^*(y) + \epsilon(g_1^* \nabla(f^*)'(y, .))(0) + \epsilon^2 g_2^*(0) + \epsilon^3(g_3^* \nabla(g_2^*)'(0, .))(0) + ... \epsilon^{2i} g_{2i}^*(0) + \epsilon^{2i+1}(g_{2i+1}^* \nabla(g_{2i}^*)'(0, .))(0) + ... \epsilon \varphi_n(y, \epsilon)$ with $\lim_{\epsilon \to 0} \varphi_n(y, \epsilon) = 0$.

THEOREM 4.4. Let X be a locally convex space and X^* its topological dual. Let $f: X \to \mathbb{R} \bigcup \{+\infty\}$ be a convex proper lower semicontinuous function and $g_i: X \to \mathbb{R}$ is convex continuous $\forall i$. If $f(.) - \langle ., y \rangle$, $(g_{2i})_{i \geq 1}$ are weakly inf-compact and $(g_{2j+1})_{j \geq 0}$ are bounded below for every i, j satisfying 2i, 2j+1 $\leq$ n then (4.1) holds.

Proof. Let $m_i \in \mathbb{R}$ such that $g_i(x) \geqslant m_i$ for every i and $x \in X$. We have $F_\epsilon(x) = f(x) - \langle x, y \rangle + \sum_{i=1}^n \epsilon^i g_i(x) \geqslant f(x) - \langle x, y \rangle + \sum_{i=1}^n \epsilon^i m_i \geqslant f(x) - \langle x, y \rangle - 1$ for every $\epsilon \leq \epsilon_0$, so F_ϵ is weakly inf-compact and reaches its minimum at a point x_ϵ. Let $a \in Domf$; we have $f(x_\epsilon) - \langle x_\epsilon, y \rangle + \sum_{i=1}^n \epsilon^i m_i \leq f(a) - \langle a, y \rangle + \sum_{i=1}^n \epsilon^i g_i(a)$ and $f(x_\epsilon) - \langle x_\epsilon, y \rangle \leq f(a) - \langle a, y \rangle + \sum_{i=1}^n \epsilon^i(g_i(a) - m_i) \leq f(a) - \langle a, y \rangle + 1$ for every ϵ sufficiently small. Then $(x_\epsilon)_\epsilon$ is weakly relatively compact, and by theorem 2.2 one has $\min_X F_k = \min_{x \in X}(f(x) - \langle x, y \rangle) + \epsilon \min_{x \in S_y} g_1(x) + \inf_{x \in X} \sum_{i=2}^n \epsilon^i g_i(x) + \epsilon \varphi(\epsilon)$. Now by the same argument $\inf_{x \in X} \sum_{i=2}^n \epsilon^i g_i(x) = \epsilon^2 \inf_{x \in X}(g_2(x) + \epsilon g_3(x) + ... \epsilon^{n-2} g_n(x)) = \epsilon^2 \min_{x \in X}(g_2(x) + \epsilon g_3(x) + ... \epsilon^{n-2} g_n(x)) = \epsilon^2 \min_{x \in X} g_2(x) + \epsilon^3 \min_{x \in \arg\min(g_2, X)} g_3(x) + \epsilon^4 \{\inf_{x \in X} \sum_{i=4}^n \epsilon^{i-4} g_i(x)\} + \epsilon^3 \varphi_1(\epsilon)$, and step by step we derive formula (4.1) by [58, Th.6.4.8 , Th.6.5.8] and by the fact that the weak inf -compactness of $f(.) - \langle ., y \rangle$ and g_{2i} implies that f* and g_{2i}^* are respectively $\tau(X^*, X)$ continuous at y and 0 [58,Th.6.3.9 and its Corollary]. □

COROLLARY 4.5. Let X be a reflexive Banach space and X^* its topological dual. Let $f: X \to \mathbb{R} \bigcup \{+\infty\}$ be a convex proper lower semicontinuous function and $g_i(x) = \|x\|^i$ for every i=1,2...n. If f* is $\|\|_{X^*}$ −continuous at y then (4.1) holds with $g_{2i}^*(0) = 0$ for every $i \geq 1$ such that 2i$\leq$n .

Proof. It is clear that g_i i=1,2...n satisfy all hypotheses in theorem 4.4 and the Mackey topology $\tau(X^*, X)$ on X^* is exactly the norme $\|\|_{X^*}$ topology, so the $\|\|_{X^*}$ − continuity of f* at y is equivalent [58] to the weak inf-compactness of $f(.) - \langle ., y \rangle$. □

5. NEW GENERALIZED REGULARIZATIONS FOR SADDLE FUNCTIONS AND ASYMPTOTIC DEVELOPMENTS

In what follows we are concerned by new generalized regularizations of saddle functions and their associated asymptotic developments. Consider two general topological Hausdorff spaces X,Y and f:X×Y→ $\overline{\mathbb{R}}$, g:X×Y→ $\mathbb{R}$, h_ϵ:X×Y→ $\mathbb{R}$ be three functions with $\epsilon > 0$. Each function f, g is assumed to be lower semicontinuous (lsc) at the first variable and upper semicontinuous (usc) at the second variable. Denote by $h_\epsilon^1 = \sup_{y\in Y} \inf_{x\in X} h_\epsilon(x,y)$ and $h_\epsilon^2 = \inf_{x\in X} \sup_{y\in Y} h_\epsilon(x,y)$ which are supposed finite for every $\epsilon > 0$ sufficiently small. Assume that the set S={(a,b)∈X×Y/(a,b) is a saddle point of f } is nonempty. Set $F_\epsilon(x,y) = f(x,y) + a_\epsilon g(x,y) + h_\epsilon(x,y)$ with $a_\epsilon > 0$, $a_\epsilon \to 0$ if $\epsilon \to 0$. If $h_\epsilon = 0$ and g(x,y)= $a_i \|x\|^p - b_i \|y\|^q$ with a_i, b_i are positive real numbers and p,q∈ $\mathbb{N}^*$ then F_ϵ reduces to the classical Tikhonov regularization.

Using a similar technique considered in the proof of theorem 2.2 with more difficult and sophisticated arguments we can state the following result:

THEOREM. 5.1. Let $(x_\epsilon,y_\epsilon)_\epsilon$ be a relatively compact sequence such that $\alpha_\epsilon = \sup_y F_\varepsilon(x_\epsilon,y)$, $\beta_\epsilon = \inf_x F_\varepsilon(x,y_\epsilon)$, $\gamma_\epsilon(t)= \sup_y h_\varepsilon(t,y)$, $\delta_\varepsilon(z) = \inf_x h_\varepsilon(x,z)$ are finite for every ϵ sufficiently small and every (t,z)∈X×Y. Assume that the following condition holds:

(5.1) $\lim_{\varepsilon\to 0} \frac{\alpha_\epsilon - \beta_\varepsilon}{a_\epsilon} = \lim_{\varepsilon\to 0} \frac{\gamma_\epsilon(t) - \delta_\varepsilon(z)}{a_\epsilon} = 0 \ \forall(t,z) \in X \times Y$

Then:

(i) any cluster point $(\overline{x},\overline{y})$ of $(x_\epsilon,y_\epsilon)_\epsilon$ is a saddle point of f on X×Y and is a saddle point of g on S. Furthermore for every $\alpha \in \mathbb{R}$, there exists a sequence $(\delta_\epsilon^\alpha, \theta_\epsilon^{1,\alpha}, \theta_\epsilon^{2,\alpha}) \to 0_{\mathbb{R}^3}$ if $\epsilon \to 0$ depending on the scheme under consideration such that $F_\epsilon(x_\epsilon,y_\epsilon)=f(\overline{x},\overline{y}) + a_\epsilon g(\overline{x},\overline{y}) + \alpha h_\epsilon^1 + (1-\alpha)h_\epsilon^2 + a_\epsilon \delta_\epsilon^\alpha$ and the sequence $(g(x_\epsilon,\overline{y}), g(\overline{x},y_\epsilon), \frac{f(x_\epsilon,\overline{y}) - f(\overline{x},\overline{y})}{a_\epsilon}, \frac{f(\overline{x},y_\epsilon) - f(\overline{x},\overline{y})}{a_\epsilon}, \frac{h_\epsilon^2 - h_\epsilon^1}{a_\epsilon})_\varepsilon$ converges to $(g(\overline{x},\overline{y}),g(\overline{x},\overline{y}),0,0,0)$ if $\epsilon \to 0$;

(ii) $F_\epsilon^i = f(\overline{x},\overline{y}) + a_\epsilon g(\overline{x},\overline{y}) + \alpha h_\epsilon^1 + (1-\alpha)h_\epsilon^2 + a_\epsilon \theta^{i,\alpha}$ and $\lim_{\epsilon\to 0} \frac{F_\epsilon^2 - F_\epsilon^1}{a_\epsilon} = 0$ where $F_\epsilon^1 = \sup_{y\in Y} \inf_{x\in X} F_\epsilon(x,y)$ and $F_\epsilon^2 = \inf_{x\in X} \sup_{y\in Y} F_\epsilon(x,y)$.

REMARK 5.2. The first limit in (5.1) is straightforward satisfied if (x_ϵ,y_ϵ) is a saddle point of F_ϵ. Also we observe that there exists a wide class of functions h_ϵ :X×Y → $\mathbb{R}$ satisfying the second limit in (5.1). Take for instance the functions of the kind $h_\epsilon(x,y) = \sum_{i=1}^n b_\epsilon^i g_i(x,y)$ where $m_i(y) \leq g_i(x,y) \leq M_i(x) \ \forall(x,y)$ ∈X×Y. Here $m_i(y)$, $M_i(x)$ are real numbers and $b_\epsilon^i \geq 0$ satisfying $\lim_{\epsilon\to 0} \frac{b_\epsilon^i}{a_\epsilon} = 0 \ \forall i$. Then $0 \leq \frac{\sup_{y\in Y} h_\epsilon(a,y) - \inf_{x\in X} h_\epsilon(x,b)}{a_\epsilon} \leq \frac{\sum_{i=1}^n b_\epsilon^i \cdot (M_i(a) - m_i(b))}{a\epsilon}$ and $\lim_{\epsilon\to 0} \frac{\sum_{i=1}^n b_\epsilon^i \cdot (M_i(a) - m_i(b))}{a\epsilon} = 0 \ \forall(a,b)$ ∈X×Y. In Particular one can consider the classical functions used in many schemes of saddle point approximation methods $g_i(x,y) = \alpha_i \|x - x_i\|^{p_i} - \beta_i \|y - y_i\|^{q_i}$ where x_i, y_i are given points in the normed spaces X, Y , $p_i, q_i \in \mathbb{N}^*$ and $\alpha_i, \beta_i > 0$ (For instance, see [88, 94] and references therein). More generally one may take $h_\epsilon(x,y) = \sum_{i=1}^n b_\epsilon^i g_{\epsilon i}(x,y)$ where there exist two real functions $m_{\epsilon i}(y)$ and $M_{\epsilon i}(x)$ such that for every (x,y) we have $m_{\epsilon i}(y) \leq g_{\epsilon i}(x,y) \leq M_{\epsilon i}(x)$ and $\lim_{\epsilon\to 0} \frac{\sum_{i=1}^n b_\epsilon^i \cdot (M_{\epsilon i}(a) - m_{\epsilon i}(b))}{a\epsilon} = 0 \ \forall(a,b)$ ∈X×Y then $\lim_{\epsilon\to 0} \frac{\sup_{y\in Y} h_\epsilon(a,y) - \inf_{x\in X} h_\epsilon(x,b)}{a_\epsilon} = 0$. For example, see the regularization function considered in theorem 6.9.

COROLLARY 5.3. Let X, Y be two convex compacts of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Assume that $F_\epsilon : X \times Y \to \mathbb{R}$ is finite, convex-concave and continuous with $h_\epsilon(x,y) = \sum_{i=1}^p b_\epsilon^i g_i(x,y)$ where $g_i : X \times Y \to \mathbb{R}$, i=1...p are such that $m_i(y) \leq g_i(x,y) \leq M_i(x)$ $\forall (x,y) \in X \times Y$. $m_i(y)$, $M_i(x)$ are real numbers and $a_\epsilon > 0, b_\epsilon^i \geq 0$ satisfying $\lim_{\epsilon \to 0} \frac{b_\epsilon^i}{a_\epsilon} = 0$ $\forall i$, $\lim_{\epsilon \to 0} a_\epsilon = 0$. Then $\lim_{\epsilon \to 0} \frac{\min_{x \in X} \max_{y \in Y} (f(x,y) + a_\epsilon g(x,y) + \sum_{i=1}^p b_\epsilon^i g_i(x,y)) - \min_{x \in X} \max_{y \in Y}(f(x,y))}{a_\epsilon} = \min_{x \in A} \max_{y \in B}(g(x,y))$ where $S = A \times B$.

Proof. By [96], F_ϵ has a saddle point (x_ϵ, y_ϵ) and the sequence $((x_\epsilon, y_\epsilon))_\epsilon$ is relatively compact. The limits in (5.1) are obviously satisfied, so the conclusions of theorem 5.1 hold. But for every $(x,y) \in X \times Y$ we have $\sum_{i=1}^p b_\epsilon^i m_i(y)) \leq h_\epsilon^1 \leq h_\epsilon^2 \leq \sum_{i=1}^p b_\epsilon^i M_i(x))$ then $\lim_{\epsilon \to 0} \frac{h_\epsilon^1}{a_\epsilon} = \lim_{\epsilon \to 0} \frac{h_\epsilon^2}{a_\epsilon} = 0$ which completes the proof.

5.4. Conjugacy of Bivariate Functions and Asymptotic Developments

Theorem 5.1 can provide an interesting tool for application to the conjugacy of bivariate functions as follows: Assume that X,Y are normed spaces with their topological dual X* and Y* respectively. Fix (x^*, y^*) in X*×Y* and set $K(x^*, y^*) = \sup_y \inf_x (\langle x^*, x \rangle + \langle y^*, y \rangle + g(x,y))$ (for instance see [96] for the importance of this function in saddle functions theory and conjugacy), $K_\epsilon(x^*, y^*) = \sup_y \inf_x (f(x,y) + \epsilon g_1(x,y) + \epsilon^2 g_2(x,y) + ... \epsilon^n g_n(x,y))$, $\epsilon > 0$ where g, g_i: $X \times Y \to \mathbb{R}$ are given functions, i=1,2...n and $f(x,y) = \langle x^*, x \rangle + \langle y^*, y \rangle + g(x,y)$. Denote by S_f and $S_{g_{2k}}$ respectively the sets of saddle points of f and g_{2k} on X×Y assumed to be nonempty. Under suitable hypotheses as in the previous theorem applied many times to f, g_i and to the regularized scheme under consideration with $h_\epsilon(x,y) = \epsilon^2 (g_2(x,y) + \epsilon g_3(x,y) + \epsilon^2 g_4(x,y) + \epsilon^3 g_5(x,y) + ... \epsilon^{n-2} g_n(x,y)) = \epsilon^2 H_\epsilon(x,y)$, one can derive easily the following formula: If n=2p+1 then $K_\epsilon(x^*, y^*) = f(x_0, y_0) + \epsilon g_1(x_0, y_0) + \epsilon^2 g_2(x_2, y_2) + \epsilon^3 g_3(x_2, y_2) + ... + \epsilon^{2k} g_{2k}(x_{2k}, y_{2k}) + \epsilon^{2k+1} g_{2k+1}(x_{2k}, y_{2k}) + ... \epsilon^{2p} g_{2p}(x_{2p}, y_{2p}) + \epsilon^{2p+1} g_{2p+1}(x_{2p}, y_{2p}) + \sum_{i=1}^{2p+1} \epsilon^i \gamma_\epsilon^i$ for some γ_ϵ^i converging to 0 if $\epsilon \to 0$. If n=2p and $\alpha \in \mathbb{R}$ then $K_\epsilon(x^*, y^*) = f(x_0, y_0) + \epsilon g_1(x_0, y_0) + \epsilon^2 g_2(x_2, y_2) + \epsilon^3 g_3(x_2, y_2) + ... + \epsilon^{2k} g_{2k}(x_{2k}, y_{2k}) + \epsilon^{2k+1} g_{2k+1}(x_{2k}, y_{2k}) + .. + \epsilon^{2p-2} g_{2p-2}(x_{2p-2}, y_{2p-2}) + \epsilon^{2p-1} g_{2p-1}(x_{2p-2}, y_{2p-2}) + \epsilon^{2p}(\alpha g_{2p}^1 + (1-\alpha) g_{2p}^2) + \sum_{i=1}^{2p} \epsilon^i r_\epsilon^i$ with $r_\epsilon^i \to 0$ if $\epsilon \to 0$. Here (x_0, y_0) is a saddle point of f on X×Y and is also a saddle point of g_1 on S_f. $(x_{2k}, y_{2k}) \in S_{g_{2k}}$ and is a saddle point of g_{2k+1} on $S_{g_{2k}}$.

6. WELL-POSEDNESS OF GENERALIZED REGULARIZATIONS FOR BIVARIATE FUNCTIONS

In the sequel we investigate well-posedness of generalized regularizations of saddle functions. Let X,Y be two reflexive Banach spaces renormed by strictly convex norms $\|.\|_X$, $\|.\|_Y$ making them E-spaces [3, 34] and f:X×Y→ $\overline{\mathbb{R}}$, g:X×Y → $\mathbb{R}$, h_ϵ:X×Y → $\mathbb{R}$ be three functions weakly lsc at the first variable for each fixed y and weakly usc at the second variable for each fixed x. Consider $F_\epsilon(x,y) = f(x,y) + \epsilon g(x,y) + h_\epsilon(x,y)$ and assume that the following hypotheses are satisfied for a function F:X×Y→ $\overline{\mathbb{R}}$, (H_1) : $\forall x \in X$, $\exists y \in Y$ such that F(x,y)>-∞, and (H_2) : $\forall y \in Y$, $\exists x \in X$ such that F(x,y) <+∞. Set g(x)= $\sup_{y \in Y} F(x,y)$, h(y)= $\inf_{x \in X} F(x,y)$. It is clear that $\forall x \in X$ g(x)>-∞, $\forall y \in Y$ h(y)<+∞ and the function

w(x,y)=g(x)-h(y) is well defined on X×Y with w≥ 0. If $(\overline{x}, \overline{y})$ is a saddle point of F on X×Y, then $\max_y \min_x F(x,y) = \min_x F(x,\overline{y}) = \max_y F(\overline{x},y) = \min_x \max_y F(x,y) = g(\overline{x}) = h(\overline{y}) = F(\overline{x}, \overline{y})$ is finite.

DEFINITION 6.1 [26] . A sequence $(x_n, y_n)_n$ in X×Y is called minimaximizing sequence of F if w(x_n,y_n) → 0 when n→ +∞.

The last definition is equivalent to the existence of $\epsilon_n > 0, \epsilon_n \to 0$ such that $F(x_n, y) \leq \epsilon_n + F(x, y_n) \ \forall (x, y) \in$ X×Y. Note that a function does not always possess a minimaximizing sequence [26] .

THEOREM 6.2 [26] . The following are equivalent: (a) F has a minimaximizing sequence on X×Y; (b) $\inf_x \sup_y F(x,y) = \sup_y \inf_x F(x,y)$;(c) $\inf_{(x,y)} w(x,y) = 0$.

DEFINITION 6.3 [26] . We say that F has well- posed saddle problem on X×Y or briefly (F, X×Y) is well- posed if F has a unique saddle point $\overline{z} = (\overline{x}, \overline{y})$ on X×Y and every minimaximizing sequence of F converges to $\overline{z}$ in the norm topology.

REMARK 6.4. For a stronger notion of well- posedness of saddle functions see for instance [75] .

In what follows we state sufficient conditions ensuring that $(F_\epsilon, $ X×Y) is well- posed.

THEOREM 6.5. Assume that f(x,y) sasisfies (H_1) and (H_2) and the following hypotheses are verified: (i) $F_\epsilon(., y)$ is strictly convex and lsc $\forall y \in$ Y; (ii)$F_\epsilon(x, .)$ is strictly concave and usc $\forall x \in$ X; (iii) $\exists y_0 \in Y$ such that for every $\lambda \in \mathbb{R}$, $A_\lambda = \{$x$\in X/F_\epsilon(x, y_0) \leq \lambda\}$ is bounded; (iv) $\exists \ x_0 \in$ X such that for every $\lambda \in \mathbb{R}$, $B_\lambda = \{$y$\in Y/F_\epsilon(x_0, y) \geq \lambda\}$ is bounded; v) $\exists$(a,b)$\in$X×Y such that f(a,b) is finite. Then: (a) $\inf_x \sup_y F_\epsilon$(x,y)=$\sup_y \inf_x F_\epsilon$(x,y); (b) F_ϵ has a unique saddle point $(\overline{x}_\epsilon, \overline{y}_\epsilon)$ and $F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$ is finite; (c) every minimaximizing sequence $(x_n, y_n)_n$ of F_ϵ converges weakly to $(\overline{x}_\epsilon, \overline{y}_\epsilon)$ if n→ +∞ and $F_\epsilon(x_n, \overline{y}_\epsilon), F_\epsilon(\overline{x}_\epsilon, y_n), F_\epsilon(x_n, y_n)$ converge to $F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$ when n→ +∞. Moreover if there exist two functions p, q$\in \{f, g, h_\epsilon\}$ (eventually identical) such that $p(x_n, \overline{y}_\epsilon) \to p(\overline{x}_\epsilon, \overline{y}_\epsilon)$ and $x_n \rightharpoonup \overline{x}_\epsilon$ ($\rightharpoonup$ denotes the weak convergence) imply that $\lim_n \|x_n - \overline{x}_\epsilon\| = 0$; and $q(\overline{x}_\epsilon, y_n) \to q(\overline{x}_\epsilon, \overline{y}_\epsilon)$ and $y_n \rightharpoonup \overline{y}_\epsilon$ imply that $\lim_n \|y_n - \overline{y}_\epsilon\| = 0$; then $(F_\epsilon,$X×Y) is well- posed.

Proof. (a). By (i), (iii), (v) and [8] one has $\inf_x \sup_y F_\epsilon$(x,y)=$\sup_y \inf_x F_\epsilon$(x,y). (b). Set φ_ϵ(x)=$\sup_y F_\epsilon$(x,y) which is convex lsc and φ_ϵ(x)$\geq F_\epsilon(x, y_0)$, so $\{$x$\in X/\varphi_\epsilon$(x)$\leq \lambda\}$ is bounded for every λ by (iii) and $\min_x \varphi_\epsilon$(x)=$\varphi_\epsilon(\overline{x}_\epsilon)$ [58] for some $\overline{x}_\epsilon$. Using (ii),(iv) and [58] , a symmetric argument shows that $\max_y \psi_\epsilon(y) = \psi_\epsilon(\overline{y}_\epsilon)$ for some $\overline{y}_\epsilon$ where $\psi_\epsilon(y) = \inf_x F_\epsilon$(x,y); then by (a) $\varphi_\epsilon(\overline{x}_\epsilon) = \psi_\epsilon(\overline{y}_\epsilon)$ i.e $\sup_y F_\epsilon(\overline{x}_\epsilon,$y)=$\inf_x F_\epsilon$(x,$\overline{y}_\epsilon$)=$F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$ which is finite because f(x,y) sasisfies (H_1) and (H_2); consequently $(\overline{x}_\epsilon, \overline{y}_\epsilon)$ is a saddle point of F_ϵ on X×Y. The uniqueness is immediate from strict convexity of $F_\epsilon(.,$y) $\forall y$ and strict concavity of $F_\epsilon(x, .)\forall x$. (c). First we observe by Theorem 6.2 and (a) that the set of minimaximizing sequences of F_ϵ is nonempty. Now let $(x_n, y_n)_n$ be a minimaximizing sequence of F_ϵ . We have $F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon) \leq F_\epsilon(x_n, \overline{y}_\epsilon) \leq F_\epsilon(x_0, y_n) + \epsilon_n, \epsilon_n > 0, \epsilon_n \to 0$ and $(y_n)_n$is bounded by (iv). In the same way $F_\epsilon(x_n, y_0) \leq F_\epsilon(\overline{x}_\epsilon, y_n) + \epsilon_n \leq F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon) + \epsilon_n$ and $(x_n)_n$is bounded by (iii). By weak relative compactness of $(x_n, y_n)_n$, semicontinuity and uniqueness of the saddle point $(\overline{x}_\epsilon, \overline{y}_\epsilon)$,

it is a routine to check that $x_n \rightharpoonup \overline{x}_\epsilon$ and $y_n \rightharpoonup \overline{y}_\epsilon$. On the other hand, there exist three scalars m, M, α such that m$\leq F_\epsilon(x_n, \overline{y}_\epsilon) \leq F_\epsilon(x_0, y_n) + \epsilon_n \leq$M and m$\leq F_\epsilon(x_n, \overline{y}_\epsilon) \leq F_\epsilon(\overline{x}_\epsilon, y_n) + \epsilon_n \leq \alpha$ for every n, so $F_\epsilon(x_n, \overline{y}_\epsilon)$, $F_\epsilon(\overline{x}_\epsilon, y_n)$ are bounded ; and by a classical argument they have a unique cluster point $F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$ to which they converge. But $F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon) - \epsilon_n \leq F_\epsilon(x_n, y_n) \leq$ $F_\epsilon(\overline{x}_\epsilon, y_n) + \epsilon_n$, then $F_\epsilon(x_n, y_n) \rightarrow F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon)$ when n$\rightarrow +\infty$. Since $F_\epsilon(x_n, \overline{y}_\epsilon) \rightarrow F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon) = \min_x F_\epsilon($x, $\overline{y}_\epsilon)$, $F_\epsilon(\overline{x}_\epsilon, y_n) \rightarrow F_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon) = \max_y F_\epsilon(\overline{x}_\epsilon,$ y), $(x_n)_n$ and $(y_n)_n$ are respectively minimizing and maximizing sequences for the two last extremum problems; and as in the proof of Th.4.2 we show that the sequence ($f(x_n, \overline{y}_\epsilon), g(x_n, \overline{y}_\epsilon), h_\epsilon(x_n, \overline{y}_\epsilon))_n$ converges to $(f(\overline{x}_\epsilon, \overline{y}_\epsilon), g(\overline{x}_\epsilon, \overline{y}_\epsilon), h_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon))$. A symmetric argument shows that

$(f(\overline{x}_\epsilon, y_n), g(\overline{x}_\epsilon, y_n), h_\epsilon(\overline{x}_\epsilon, y_n)) \rightarrow (f(\overline{x}_\epsilon, \overline{y}_\epsilon), g(\overline{x}_\epsilon, \overline{y}_\epsilon), h_\epsilon(\overline{x}_\epsilon, \overline{y}_\epsilon))$ when n $\rightarrow +\infty$; so $(x_n, y_n)_n$ converges in the norm topology to $(\overline{x}_\epsilon, \overline{y}_\epsilon)$ by hypothesis which completes the proof of the theorem. $\square$

As an immediate consequence of the previous theorem and the fact that $(X, \|.\|_X)$, $(Y, \|.\|_Y)$ are E-spaces, we have the following corollaries:

COROLLARY 6.6. Assume that f:X$\times$Y$\rightarrow \mathbb{R}$ is convex-concave lsc at the first variable for each fixed y and usc at the second variable for each fixed x and there exist (x_0, y_0) in X$\times$Y and scalars m, M such that f$(x, y_0) \geq$m, f$(x_0, y) \leq$M for every $(x, y) \in$X$\times$Y. Set $F_\epsilon(x, y) =$ f$(x, y) + \sum_{i=1}^p \varphi_\epsilon^i(\|x - x_i\|^{p_i}) - \sum_{j=1}^q \psi_\epsilon^j(\|y - y_j\|^{q_j})$ where ϵ is a positive parameter, p,q,p_i,q_j $\in \mathbb{N}^*$ and x_i, y_j, i=1,...p, j=1,...q are given points in X and Y respectively, $\varphi_\epsilon^i, \psi_\epsilon^j : [0, +\infty[\rightarrow \mathbb{R}$ are continuous functions at 0, convex and strictly increasing such that for every $\lambda \in \mathbb{R}$ the sets $\{x \in X / \sum_{i=1}^p \varphi_\epsilon^i(\|x - x_i\|^{p_i}) \leq \lambda\}$, $\{y \in Y / \sum_{j=1}^q \psi_\epsilon^j(\|y - y_j\|^{q_j}) \leq \lambda\}$ are bounded, then $(F_\epsilon, X \times Y)$ is well- posed.

COROLLARY 6.7. Assume that f:X$\times Y \rightarrow \mathbb{R}$ is convex-concave lsc at the first variable for each fixed y and usc at the second variable for each fixed x. Set $F_\epsilon(x, y) =$ f$(x, y) + \sum_{i=1}^p a_\epsilon^i \|x - x_i\|^{p_i} - \sum_{j=1}^q b_\epsilon^j \|y - y_j\|^{q_j}$ where $\epsilon, a_\epsilon^i, b_\epsilon^j \in \mathbb{R}_+^*$, p,q,$p_i$,$q_j \in \mathbb{N}^*$ and x_i, y_j, i=1,...p, j=1,...q are given points in X and Y respectively. If there exist (x_0, y_0) in X$\times$Y and scalars m, M such that f$(x, y_0) \geq$m, f$(x_0, y) \leq$M for every $(x, y) \in$X$\times$Y, then $(F_\epsilon, X \times Y)$ is well- posed.

COROLLARY 6.8. Assume that f:X$\times Y \rightarrow \mathbb{R}$ is convex-concave lsc at the first variable for each fixed y and usc at the second variable for each fixed x. Set
$F_\epsilon(x, y) =$ f$(x, y) + \sum_{i=1}^p \alpha_i \epsilon e^{\frac{1}{\epsilon}(\|x - x_i\|^{p_i} + c_i) + \varpi_i} - \sum_{j=1}^q \theta_j \epsilon e^{\frac{1}{\epsilon}(\|y - y_j\|^{q_j} + d_j) + \delta_j}$ where $\epsilon, \alpha_i, \theta_j \in \mathbb{R}_+^*$, $c_i, d_j, \varpi_i, \delta_j \in \mathbb{R}$, p,q,$p_i$,$q_j \in \mathbb{N}^*$ and x_i, y_j, i=1,...p, j=1,...q are given points in X and Y respectively. If there exist (x_0, y_0) in X$\times$Y and scalars m, M such that f$(x, y_0) \geq$m, f$(x_0, y) \leq$M for every $(x, y) \in$X$\times$Y, then $(F_\epsilon, X \times Y)$ is well- posed.

Finally we end our investigation by the following theorem in finite dimensional setting which combines the results of theorem 5.1 and the ones of theorem 6.5:

THEOREM 6.9. Let f:$\mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$, f_i:$\mathbb{R}^m \rightarrow \mathbb{R}$, g_j:$\mathbb{R}^n \rightarrow \mathbb{R}$,i=1...p, j=1...q be real-valued functions such that f is continuous convex-concave and f_i,g_j are convex. Assume that the sets $X = \{x \in \mathbb{R}^m / f_i(x) \leq 0, i = 1...p\}$, $Y = \{y \in \mathbb{R}^n / g_j(y) \leq 0, j = 1, ...q\}$ are

nonempty and bounded. Set $F_\epsilon(x,y) = f(x,y) + a_\epsilon(\sum_{k=1}^r \alpha_k \|\text{x-x}_k\|^{p_k} - \sum_{k=1}^s \beta_k \|\text{y-y}_k\|^{q_k}) + \sum_{i=1}^p a_{i\epsilon} r_{i\epsilon} e^{\frac{1}{r_{i\epsilon}}(f_i(x))+h_i(x)} -$

$\sum_{j=1}^q b_{j\epsilon} t_{j\epsilon} e^{\frac{1}{t_{j\epsilon}}(g_j(y))+k_j(y)}$ where $a_\epsilon, \alpha_k, \beta_k, a_{i\epsilon}, b_{j\epsilon}, r_{i\epsilon}, t_{j\epsilon}$ are real positive numbers such that $a_\epsilon \to 0$ if $\epsilon \to 0$, h_i, k_j are convex continuous functions defined on X, Y respectively such that for every $(\text{x,y}) \in X \times Y$, $h_i(x) \le \varpi_i$, $k_j(y) \le \delta_j$ and $\lim_{\epsilon \to 0} \frac{\sum_{i=1}^p a_{i\epsilon} r_{i\epsilon} e^{\varpi_i} + \sum_{j=1}^q b_{j\epsilon} t_{j\epsilon} e^{\delta_j}}{a_\epsilon} = 0$. $r, s, p, q, p_k, q_k \in \mathbb{N}^*$ and x_k, y_k are given points in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. If $z = (z_i)_i \in \mathbb{R}^w, w \in \{m,n\}$ we denote $\|z\| = (\sum_{i=1}^w z_i^2)^{\frac{1}{2}}$ the strictly convex norm of $\mathbb{R}^w$. Then we have the following results:

1) $(F_\epsilon, X \times Y)$ is well-posed;

2) The conclusions of theorem 5.1 hold.

For its proof we need the following lemma:

LEMMA 6.10 [96, Corollary 37.6.2, p.397]. Let C and D be nonempty closed bounded convex sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and K be a continuous finite convex-concave function on C×D. Then K has a saddle point with respect to C×D.

Proof of theorem 6.9. 1).The proof of this point is an immediate consequence of lemma 6.10 (applied to K=F_ϵ and C×D=X×Y) and various arguments used in the proof of theorem 6.5 and the fact that the norm $\|z\| = (\sum_{i=1}^w z_i^2)^{\frac{1}{2}}$ is strictly convex, X, Y are convex compact sets and the weak convergence reduces to the norm convergence in finite dimensional setting.

2). If $(\text{x}_\epsilon, \text{y}_\epsilon)$ is the unique saddle point of F_ϵ on X×Y, the sequence $(\text{x}_\epsilon, \text{y}_\epsilon)_\epsilon$ is relatively compact and the first limit in (5.1) is straightforward satisfied. On the other hand it is easy to see that

$$-\sum_{j=1}^q b_{j\epsilon} t_{j\epsilon} e^{\delta_j} \le h_\epsilon(x,y) \le \sum_{i=1}^p a_{i\epsilon} r_{i\epsilon} e^{\varpi_i}$$

for every $(\text{x,y}) \in X \times Y$, so

$$0 \le \frac{\sup_{y \in Y} h_\epsilon(x,y) - \inf_{x \in X} h_\epsilon(x,y)}{a_\epsilon} \le \frac{\sum_{i=1}^p a_{i\epsilon} r_{i\epsilon} e^{\varpi_i} + \sum_{j=1}^q b_{j\epsilon} t_{j\epsilon} e^{\delta_j}}{a_\epsilon}$$

which goes to 0 if $\epsilon \to 0$ and the second limit in (5.1) is also satisfied which completes the proof. □

7. STABILITY OF VARIATIONAL ASYMPTOTIC DEVELOPMENTS OF BIVARIATE FUNCTIONS BY EPI/HYPO-CONVERGENCE

Let us consider two Hausdorff tpological spaces (X, τ) and (Y, σ) and a sequence $F_n : X \times Y \to \overline{\mathbb{R}}$ of bivariate functions. First recall [4, 6, 7, 9] that the hypo/epi-limit inferior of the sequence $(F_n)_n$ denoted by $h_\sigma/e_\tau - li F_n$ is the function defined by $(h_\sigma/e_\tau - li F_n)(x,y) = \inf_{V \in N_\sigma(y)} \sup_{U \in N_\tau(x)} [\underline{\lim}_n (\inf_{u \in U} \sup_{v \in V} F_n(u,v))]$. The epi/hypo-limit superior of $(F_n)_n$ denoted by $e_\tau/h_\sigma - ls F_n$ is defined by $(e_\tau/h_\sigma - ls F_n)(x,y) = \sup_{U \in N_\tau(x)} \inf_{V \in N_\sigma(y)} [\overline{\lim}_n (\sup_{v \in V} \inf_{u \in U} F_n(u,v))]$. Here $N_\tau(x)$ is the class of all $\tau-$neighbourhoods of x. A bivariate function F: $X \times Y \to \overline{\mathbb{R}}$ is said to be an epi/hypo-limit of $(F_n)_n$ if $e_\tau/h_\sigma - ls F_n \le F \le h_\sigma/e_\tau - li F_n$ and we write

F=e_τ/h_σ−limF_n. The limit F in this sense is not in general unique and this convergence is in some sense minimal to obtain convergence of saddle points and saddle values (see for instance [9] and references therein). For other type of convergences and their importance in saddle point theory see for instance [6, 7, 24, 25, 31, 44]. When (X, τ) and (Y, σ) are metrizable, it is possible to give an easy representation of epi/hypo-limits in sequential terms [6, 9] that turns out to be very useful in practice. In this case F=e_τ/h_σ−limF_n if and only if $\forall (x,y) \in X \times Y$: (i) $\forall y_n \xrightarrow{\sigma} y, \exists\ x_n \xrightarrow{\tau} x$ such that $\overline{\lim}_n F_n(x_n, y_n) \leq F(x,y)$;(ii) $\forall x_n \xrightarrow{\tau} x$, $\exists y_n \xrightarrow{\sigma} y$ such that F(x,y)$\leq \underline{\lim}_n F_n(x_n, y_n)$. The sequence $(F_n)_n$ is said to epi/hypo-converge in the extended sense to F [6, 9] if $\underline{cl}_x\ (e_\tau/h_\sigma - lsF_n) \leq F \leq \overline{cl}^y\ (h_\sigma/e_\tau - liF_n)$ where $\underline{cl}_x$ and $\overline{cl}^y$ are respectively the extended lower closure with respect to x for fixed y and the extended upper closure with respect to y for fixed x [9].

This section concerns the stability properties of the asymptotic developments

(7.1) $F_k^n(x_k^n,y_k^n)=f_n(x_n, y_n) + a_k g_n(x_n, y_n) + \alpha h_k^1 + (1 - \alpha)h_k^2 + a_k \delta_{k,n}^\alpha$

where $(f_n)_n$ and $(h_k)_k$ are two sequences of bivariate functions epi/hypo-converging in the extended sense to f and h respectively and (x_n, y_n) is a saddle point of f_n. Here $F_k^n(x, y) = f_n(x, y) + a_k g_n(x, y) + h_k(x, y)\ \forall (x, y) \in X \times Y$ with $a_k > 0$, $a_k \to 0$ and g_n, h_k: $X \times Y \to \mathbb{R}$ are given functions. Our goal is to study the limit of $F_k^n(x_k^n,y_k^n)$ when k,n $\to +\infty$. To this end we state the following result:

LEMMA 7.1 [71]. Assume that the functions F_n:X×Y→ $\overline{\mathbb{R}}$ epi/hypo-converge in the extended sense to a function F: X×Y→ $\overline{\mathbb{R}}$ and there exist a sequence $(x_n, y_n)_n$ of points in X×Y and a subsequence $(n_k)_k$ such that $F_{n_k}(x_k, y) \leq \epsilon_k + F_{n_k}(x, y_k)\ \forall x, y$ where $\epsilon_k \geqslant 0$ and $\epsilon_k \to 0$ when k→ $+\infty$. If the sequence $(x_n, y_n)_n$ is relatively compact, there exist cluster points $\overline{x}, \overline{y}$ of $(x_n)_n, (y_n)_n$ respectively such that $(\overline{x}, \overline{y})$ is a saddle point of:

(i) $\underline{cl}_x$F(x,y) and $F_{n_k}(x_k, y_k) \to \underline{cl}_xF(\overline{x}, \overline{y})$ when k→ $+\infty$ if $\underline{cl}_x$F(x,y) is usc at y for each fixed x;

(ii) $\overline{cl}^y$F(x,y) and $F_{n_k}(x_k, y_k) \to \overline{cl}^yF(\overline{x}, \overline{y})$ when k→ $+\infty$ if $\overline{cl}^y$F(x,y) is lsc at x for each fixed y;

(iii) F if F(x,y) is lsc at x for each fixed y and usc at y for each fixed x with $F_{n_k}(x_k, y_k) \to$ F$(\overline{x}, \overline{y})$ if k→ $+\infty$;

(iv) if $(x_n, y_n)_n$ converges to a point (x',y'), any epi/hypo-limit G of $(F_n)_n$ in the extended sense, regardless of its semicontinuity, has (x',y') as a saddle point and $F_{n_k}(x_k, y_k) \to$ G(x', y').

REMARK 7.2. If $(x_n, y_n)_n$ converges, the above lemma generalizes theorem 2.6 in [9, p.93] where the author supposes that (X, τ) and (Y, σ) are metrizable spaces and $\epsilon_k = 0$. It should be pointed out that the proof of our lemma is very different from the one presented in the last reference. If $\sup_y F_{n_k}(x_k, y)$, $\inf_x F_{n_k}$(x,y$_k$) are finite for every k≥k_0, hypothesis

$F_{n_k}(x_k, y) \leq \epsilon_k + F_{n_k}(x, y_k) \, \forall x, y$ is equivalent to $\sup_y F_{n_k}(x_k, y) - \inf_x F_{n_k}(x, y_k) \to 0$ when $k \to +\infty$.

THEOREM 7.3. Let $(f_n)_n$: $X \times Y \to \overline{\mathbb{R}}$, $(h_k)_k$: $X \times Y \to \mathbb{R}$ be two sequences of functions epi/hypo-converging in the extended sense respectively to f and h which are assumed to be lower semicontinuous at x, upper semicontinuous at y. Assume that the variational asymptotic development (7.1) holds with (x_k^n, y_k^n) and (x_n, y_n) are saddle points of F_k^n and f_n respectively and $(x_n, y_n)_n$ is supposed to be relatively compact. Suppose furthermore that $\sup_{y \in Y} h_k(t, y) - \inf_{x \in X} h_k(x, z) \to 0$ if $k \to +\infty$, $\forall (t, z) \in S_{f_n} = \{(a,b) \in X \times Y / (a,b)$ is a saddle point of $f_n\}, \forall n$. Then there exists a subsequence $(n_k)_k$ such that

$\lim_k F_k^{n_k}(x_k^{n_k}, y_k^{n_k}) = f(\overline{x}, \overline{y}) + h(\overline{z}, \overline{t})$ where $(\overline{x}, \overline{y})$ and $(\overline{z}, \overline{t})$ are two saddle points of f and h respectively.

Proof. First, fix an integer $n \in \mathbb{N}$ and set $\sup_{y \in Y} h_k(x_n, y) - \inf_{x \in X} h_k(x, y_n) = \omega_k^n$. Then $h_k(x_n, y) \leqq \omega_k^n + h_k(x, y_n)$ for all $(x,y) \in X \times Y$ and $h_k^1 \leq h_k^2 \leq h_k(x_n, y_n) + \omega_k^n \leq 2\omega_k^n + h_k^1$; so by lemma 7.1, $\lim_k h_k^1 = \lim_k h_k^2 = h(x_n, y_n)$ and (x_n, y_n) is a saddle point of h; consequently $\lim_k F_k^n(x_k^n, y_k^n) = f_n(x_n, y_n) + h(x_n, y_n)$, again by this lemma $(x_n, y_n)_n$ has cluster points $(\overline{x}, \overline{y}), (\overline{z}, \overline{t})$ which are saddle points of f and h respectively such that $f_n(x_n, y_n) \to f(\overline{x}, \overline{y})$ and $h(x_n, y_n) \to h(\overline{z}, \overline{t})$ when $n \to +\infty$. Diagonalization lemma [3] permits then to conclude. $\square$

REMARK 7.4. In (7.1) and theorem 7.3 we don't need any information neither on the behavior of the sequence $(g_n)_n$ nor on its mode of convergence.

REFERENCES

[1]N.Abu-Tabak, Stabilité dynamique des systèmes électriques multimachines : modélisation, commande, observation et simulation. Thèse, Ecole Centrale de Lyon (2008).

[2]Y.I.Albert and I.Ryazantseva, Nonlinear Ill-posed problems of monotone type, Springer, Dordrecht (2006).

[3]H.Attouch,Variational convergence for functions and operators, Applicable Maths. Series, Pitman, London (1984).

[4]H. Attouch, D. Azé and R.J.-B. Wets, Convergence of convex-concave saddle functions; continuity properties of the Legendre- Fenchel transform with applications to convex programming and mechanics. Ann. Inst. Henri Poincaré 5 (1988), 537-572.

[5]H. Attouch and H. Brézis, Duality for the sum of convex functions in general Banach spaces, Aspects of mathematics and its applications, J.B.Arroso edition North holland (1986),125-133.

[6]H.Attouch and R. Wets, A convergence theory for saddle functions,Trans. Amer. Math. Soc.280 (1983),1-44.

[7]H.Attouch and R. Wets, A convergence for bivariate functions aimed at the convergence of saddle values, in Mathematical theory of Optimization, eds J. Cecconi and T.Zolezzi, Springer- Verlag. Lecture Notes in Mathematics 979 (1981).

[8]J.-P. Aubin and I. Ekeland, Applied nonlinear Analysis. J. Wiley Inter-sciences (1984), New York.

[9]D. Azé, Convergences variationnelles et dualité: Applications en calcul des variations et en programmation mathématique. Thèse d'Etat, Université de Perpignan (1986).

[10] A. Bakushinsky and A.Goncharsky, Ill-posed Problems: Theory and applications (1994), Published by Dordrecht Kluwer.

[11]G.Beer, Lipschitz regularization and the convergence of convex functions.Num.Fun.Anal.Optim.15 (1994), 31-46.

[12]J.Beer and R. Lucchetti,The epidistance topology: continuity and stability results with applications to convex optimization problems. Math.Oper.Res.Vol.17 (3) (1992), 715-726.

[13]G.Beer and R.Lucchetti, Convex optimization and the epidistance topology.Trans.AMS 32(1991),759-813.

[14]G.Beer and R. Lucchetti, Solvability for constrained problems.Quaderno 3/1991. Dipart Matem.Univ. Milano (1991).

[15]A.Ben-Tal and A.Nemirovski, Robust convex optimization, Math.Oper. Res., 23 (1998), 769-805.

[16]M.Bertero, Regularization methods for linear inverse problems. Lecture Notes in Math.1225, Springer (1986), 52-112.

[17]D. Bertsimas, D. B. Brown and C. Caramanis, Theory and Applications of Robust Optimization.SIAM Review Vol. 53 (3) (2011), 464-501.

[18]J. Bonnet, M.Gilbert, F. Perrot, J-S. Dewallers and P. Bret, Un demi-siècle d'aéronautique en France-Les avions militaires, tome 1 (2007).

[19]H.Brezis, Analyse fonctionnelle.Théorie et applications. Masson (1983).

[20]H.Brezis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North Holland (1973).

[21]P.Brianzi and M.Frontini, On the regularized inversion of the Laplace transform, Inverse Problems 7 (1991), 355-368.

[22]F.E. Browder, Existence and approximation of solutions of nonlinear variational inequalities, Proc.Nat.Acad.Sci.U.S.A.,56(1966), 1080-1086.

[23]G.Bruckner and J.Cheng, Tikhonov regularization for an integral equation of the first kind with logarithmic kernel, J. Inverse and Ill- posed Problems (2000), 21-33.

[24]E. Cavazutti, Alcune caratterizzazioni della $\Gamma-$Convergenza multipla, Annali Mat. Pura Applicata,32(1982), 69-112.

[25]E. Cavazutti, $\Gamma-$Convergenza multipla, Convergenza di punti di sella e di max-min, Boll.Un.Mat. Ital. (6) 1-B (1982), 251-274.

[26]E.Cavazzuti and J. Morgan, Well- posed saddle point problems. Lecture Notes in Pure and Appl. Math.86., M. Dekker (1983), 61-76.

[27]M.P.Charbonier, Sur l'état actuel de la balistique extérieure. Site, http:// www.mathunion.org /ICM/ICM1924.2/Main/icm1924.2.0571 .0594.ocr.pdf.

[28]R.Cominetti and J.San Martin, Asymptotic Analysis of the Exponential Penalty Trajectory in Linear Programming, Mathematical Programming, Vol.67 (1994), 169-187.

[29]R.Courant and D.Hilbert, Methods of mathematical physics, Vol.II. Interscience (1962).

[30]M.G.Crandall and P.L.Lions, Viscosity solutions of Hamilton-Jacobi equations,Trans.Amer.Math.Soc. 277(1983),1-42.

[31]E.DeGiorgi, Convergence problems for functionals and operators,Procee.on recent methods in Nonlinear Analysis, eds.E.De Giorgi, E.Magenes and U.Mosco,Pitagora Editrice, Bologna (1980).

[32]D.Demailly, Etude du comportement non-linéaire dans le domaine frequentiel. Application à la dynamique rotor, Thèse, Ecole Centrale de Lyon (2003).

[33]J.Diestel, Geometry of Banach spaces. Selected topics. Lecture Notes in Math. 485, Springer (1975).

[34]A.L.Dontchev and T.Zolezzi, Well-posed optimization problems. Lecture Notes in Math., 1543, Springer-Verlag (1993).

[35]J.J. Dorrzapf, Un survol des lois de la balistique.Société Européenne de la balistique lésionnelle.Site:http://www.eswb-sebl.org/ lois_balistique_ _page01.htm.

[36]I.Ekeland and R.Temam, Analyse convexe et problèmes variationnels, Dunod and Gauthier-Villars (1974).

[37]Y. Elkettani and D. Mentagui, Robustesse de l'estimation spatiale par krigeage simple et par régression PLS. Revue MODULAD, (31) (2004), 32-50.

[38] H. W. Engl, K. Kunisch and A. Neubauer, Convergence rates for Tikhonov regularization of nonlinear ill-posed problems, Inverse Problems Vol. 5 (4) (1989), 523-540.

[39]Y. M.Ermoliev, I.N.Vladimir and R.J-B. Wets: The minimization of semicontinuous functions: Mollifier Subgradients. SIAM J. Control. and Optimization. Vol. 33. Number 1(1995), 149-167.

[40]J.Gagne, Exploitation de l'effet gyroscopique pour la stabilisation active de l'épicarde.Vers une chirurgie des coronaires à coeur battant. Thèse, Université de Strasbourg (2011).

[41]M. Gaydu and M. H. Geoffroy, Tikhonov regularization of metrically regular inclusions; Positivity 13 (2009), 385-398.

[42]E.Giusti, Minimal surfaces and functions of bounded variations, Birkhäuser (1984).

[43]D. Goeleven and D. Mentagui, Well-posed hemivariational inequalities, Numer. Funct.Anal. Optim., 16 (1995), 909-921.

[44]G.Greco, Saddle topology and minimax theorems, Tech. Report. Univ.Trento (1984).

[45]C.W Groetsch, The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind, Pitman, London (1984).

[46]M.Guskov, Dynamique non-lineaire des systèmes multi-rotors. Etudes numériques et experimentales, Thèse, Ecole Centrale de Lyon (2007).

[47]J.Hadamard, Lectures on Cauchy's problem in linear partial differential equations.Dover,(1953).

[48]J.Hadamard, Sur les problèmes aux dérivées partielles et leur signification physique.Bull.Univ.Princeto
13 (1902), 49-52.

[49]F.R.Hampel, Robust statistics: A brief introduction and overview. Technical Report
94, Seminar für Statistik, Eidgenössische Tchnische Hochschule, (2001).

[50]A. Haraux and F. Murat, Perturbations singulières et problèmes de contrôle optimal:
deux cas biens posés, C.R.Acad.Sc.Paris, 297, (1983), 21-24.

[51]B.Hofmann, Regularization for Applied Inverse and Ill-posed Problems, A Numerical
Approach (Leipzig, Teuber), (1986).

[52]P.J.Huber, Robust estimation of a local parameter. Annals of Mathematical Statistics,
35 (1964), 73-101.

[53]T. A. Johansen, On Tikhonov regularization. Bias and variance in nonlinear System
identification.Automatica, Vol.33 (3) (1997), 441-446.

[54]N.Kemmochi, Nonlinear operators of monotone type in reflexive Banach spaces and
nonlinear perturbations, Hiroshima Math.J.,4 (1974), 229-263.

[55]G.Klein, On spline functions and statistical regularization of ill-posed problems, J.
Comput. App. Math. Vol. 5, Number 4 (1979), 259-263.

[56]R.V. Kohn and L.Nirenberg, Degenerate elliptic parabolic equations of second or-
der,Comm. Pure appl.math., 20,(1967),797-892.

[57]P.Kouvelis and G.Yu, Robust discrete optimization and its applications, Kluwer Aca-
demic Publishers, Norwell, MA (1997).

[58]J. Laurent, Approximation et optimisation, Hermann (1972).

[59]M. M. Lavrentev, V. G. Romanov, and S. P. Shishatski, Ill-posed problems of mathe-
matical physics and analysis. (Translated from the Russian) Translations of Mathematical
Monographs, 64. American Mathematical Society, Providence, RI (1986).

[60]B.Lemaire,Well-posedness, conditioning and regularization of minimization, inclusion
and fixed-point problems, In.Proceedings of the 4th International Conference on Mathemat-
ical Methods in operations Research and 6th Workshop on Well-posedness and stability of
optimization Problems(Sozopol,1997). Pliska Stud. Math.Bulgar.,12(1998),71-84.

[61]B.Lemaire, About the convergence of the proximal method, Advances in Optimization,
Proceedings Lambrecht 1991, Lecture notes in economics and mathematical systems 382,
Springer-Verlag (1992), 39-51.

[62]B.Lemaire and C.O.A. Salem, From fixed point regularization to constraint decom-
position in variational inequalities, Optimization (Namur,1998), 250-259, Lecture Notes in
Econom. and Math.Systems, 481,Springer, Berlin (2000).

[63]J.L.Lions, Contrôle des systèmes distribués singuliers, Gauthier-Villars (1983).

[64]J.L.Lions, Perturbations singulières dans les problèmes aux limites et en contrôle op-
timal, Lecture Notes in Math. 323, Springer-Verlag(1973).

[65]J.L.Lions, Sur le contrôle optimal des systèmes gouvernés par des équations aux dérivées
partielles, Dunod et Gauthier-Villars, Paris (1968).

[66]J.L.Lions and G. Stampacchia,Variational inequalities, Comm.Pure.Appl.Math. 20(1967),493-519.

[67]O.L.Liskovets, Regularization of mixed incorrect variational inequalities of monotonic type, Soviet.J.Numer. Anal.Math.Model.6(1991),107-119.

[68]O.L.Mangasarian, Iterative solution of linear programs, SIAM J.Num.Anal.,18 (4) (1981), 606-614.

[69]B. Martinet, Algorithmes pour la résolution des problèmes d'optimisation et minimax. Thèse d'Etat, Université de Grenoble (1972).

[70]L.Mc Linden and R. Bergstrom, Preservation of convergence of convex sets and functions in finite dimensions, Trans. Amer. Math. Soc., 268(1981), 127-142.

[71]D. Mentagui, Well-posedness of saddle functions and epi/hypo-convergence. Tech. report (2013), 1-13.

[72]D.Mentagui, Analyse de recession et résultats de stabilité d'une convergence variationnelle. Application à la théorie de la dualité en programmation mathématique. ESAIM: Control, Optimisation and Calculus of Variations, Vol. 9 (2003), 297-315.

[73]D.Mentagui, Bonne position au sens de Levitin-Polyak dans le cadre de la minimisation des fonctionnelles intégrales. PIM. Nouvelle serie, tome 68 (82) (2000), 108-116.

[74]D. Mentagui, Stability results of a class of well-posed optimization problems. Optim. 36 (1996), 119-138.

[75]D. Mentagui, Poblèmes d'optimisation bien posés et convergences variationnelles: Théorie et applications dans le cadre de l'optimisation non différentiable,Thèse d'Etat, Facultés Universitaires Notre -Dame de la Paix, Namur, Belgique (1996).

[76]D. Mentagui, Caractérisation de la stabilité d'un problème de minimisation associé à une fonction de perturbation particulière. Publication de l'Institut Mathématique, Nouvelle série, 60(74) (1996), 65-74.

[77]D. Mentagui, Stabilité de l'épi-convergence en dimension finie.PIM. Nouvelle serie, tome 59 (73) (1996), 161-168.

[78]D. Mentagui, Inf-convolution polaire, stabilité de l'épi-convergence et estimation de la rapidité de convergence d'une suite de compacts, thèse de 3ème Cycle, Faculté des Sciences de Rabat, Maroc (1988).

[79]D. Mentagui, Rapport interne exposé au séminaire d'analyse convexe. Faculté des Sciences de Rabat, Maroc (1986).

[80]D.Mentagui and K. Elhajioui, Slice convergence: Stabilité et optimisation dans les espaces non réflexifs. ESAIM: Control, Optimization and Calculus of Variations, Vol. 10(2004), 505-525.

[81]D.Mentagui and K. Elhajioui, Convergence des fonctions convexes et approximations inf-convolutives généralisées . PIM. Nouvelle serie, tome 72(86) (2000),123-136.

[82]L.Modica, Gradient theory of phase transitions and minimal interface criteria, Arch.Rat.Mech.Anal.98 (1987),123-142.

[83]N.Moïsséev, Problèmes mathématiques d'analyse des systèmes, Traduction française Edition Mir, Moscou, (1985).

[84]G.Mónica - Fernández Barberis, Robustness Analysis: A Powerful Tool in the Multiple Criteria Decision Making Field EWG-MCDA Newsletter (Forum), Series 3, Number 13, Springer (2006).

[85]U. Mosco, Convergence of convex sets and of solutions of variational inequalities, Adv. Math., 3 (1969), 512-585.

[86] K.Mouallif, Convergences variationnelles et méthodes perturbées pour les problèmes d'optimisation et de point selle. Ph.D thesis, Université de Liège, Belgique (1989).

[87]A. Moudafi, Tikhonov fxed-point regularization, Optimization (Namur,1998), 320-328, Lecture Notes in Econom.and Math.Systems,481, Springer, Berlin (2000).

[88]A.Moudafi, Contribution à l'étude de problèmes variationnels et optimisation, Thèse d'Etat, Faculté des Sciences Ben M'sik, Casablanca, Maroc (1995).

[89]J. Mulvey, R.Vanderbei and S. Zenios, Robust optimization of large-scale systems, Oper. Res. 43 (1995), 264-281.

[90]S.Postigo, Les bases de l'hélicoptère radiocommandé. Paru dans la magasine RCM.Site: http://thunderdarkdevil.modelisme.com

/debuterenhelico.html. By Thunderdarkdevil. PDF Version by O. Dellicour (2008).

[91]J.Revalski, Generic well-posedness in some classes of optimization problems. Acta Univ.Carolin. Math.Phys.28(1987),117-125.

[92]D.Ríos-Insúa, F. Ruggeri, Robust Bayesian Analysis. Springer- Verlag. New York. USA (2000).

[93]R.T. Rockafellar, Augmented Lagrangien and application of the proximal point algorithm in convex programming, Maths of operations research Vol.1 (2) (1976), 97-116.

[94]R.T.Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control and optimization (14) (1976), 877-898.

[95]R.T.Rockafellar, Monotone operators associated with saddle functions and minimax problems, Nonlinear functional analysis, Part I, F, E.Browder ed. symposia in Pure Math., Vol.18, Amer.Math.Soc., Providence (1970), 397-407.

[96]R.T.Rockafellar, Convex analysis. Princeton University Press, Princeton (1970).

[97]B.Roy, Le concept de conclusions robustes en recherche opérationnelle. In: IFORS 1996, The 14th Trienal Conference on Operational Research, p.8, Vancouver, British Columbia, Canada (1996).

[98]R.Temam, Problèmes mathématiques en plasticité, Gauthier-Villars(1983).

[99]A.M. Thompson and J. Kay, On some Bayesian choices of regularization parameter in image restoration, Inverse Problems Vol. 9 (1993), 749-761.

[100]A. Tikhonov, Solution of incorrectly formulated problems and the regularization method, Soviet Math. Dokl.4 (1963),1035-1038.

[101]A. Tikhonov and V. Arsenin, Méthodes de résolution de problèmes mal posés, Traduction française, Ed.MIR (1976).

[102] A.N.Tikhonov, A.S.Leonov and A.G.Yagola, Nonlinear ill-posed Problems, Vols. 1, 2, Chapman and Hall, London (1998).

[103]P.Tossing, The perturbed Tikhonov's algorithm and some of its applications, RAIRO modél.Math.Anal.Numér.,28(2) (1994), 189-221.

[104]S.L.Troyanski, On locally uniformly convex and differentiable norms in certain non separable Banach spaces. Studia. Math., 37, (1971), 173-180.

[105]Ph.Vincke, About Robustness Analysis. EWG-MCDA, Series 3, Number 8 (2003), 7-9.

[106]Ph.Vincke, Robust Solutions and Methods in Decision Aid. Journal of Multicriteria Decision Analysis, 8 (1999),181-187.

[107]Ph.Vincke, Robust and neutral methods for aggregating preferences into an outranking relation. European Journal of Operational Research, Vol.111(2) (1999), 405-412.

[108] G.Wahba, Practical approximate solutions to linear operator equations when data are noisy, SIAM. J. Numer. Anal. Vol. 14(1977), 651-677.

[109]R.J.-B.Wets, A formula for the level sets of epi-limits and some applications. Lecture Notes in Math. 979, Springer (1983), 256-268.

[110]M.Yamanoto, On ill-posedness and Tikhonov regularization for a multidimensional inverse hyperbolic problem, J. Math. Kyoto-Univ. Vol. 36, Number 4 (1996), 825-856.

[111]I.Zaïtsev, La Défense antimissile américaine de A à Z. Site:http://www15. fr.rian.ru/analysis/20070302/61492054.html.

[112]T.Zolezzi, Well posedness and stability analysis in optimization.North Holland Math.Studies 129 (1986), 305-320.

[113]T.Zolezzi, On equiwellset minimum problems. Appl.Math.Optim.4(1978),209-223.