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ADDENDUM TO "ON COMPLETENESS AND BICOMPLETIONS OF QUASI b-METRIC SPACES"

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Abstract. Related to the paper "On completeness and bicompletions of quasi b-metric spaces" we introduce strong quasi b-metric spaces and show that every strong quasi b-metric space admits a Cauchy bicompletion which is unique up to isometry.

1. Introduction

In the literature one find a rich theory on completeness of metric spaces, quasi metric spaces and quasi uniform spaces. Our basic reference for quasi uniform spaces and quasi metric spaces is [3].

Recall that:

Definition 1.1. [3] *A* ***quasi metric space*** *is a pair (X, d) where the mapping $d : X \times X \to [0, \infty)$ satisfies the following for all $x, y, z \in X$:*

a) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

b) $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 1.2. [4] *A* ***quasi b-metric space*** *is a pair (X, d) where $d : X \times X \to [0, \infty)$ satisfies the following for all $x, y, z \in X$ and $s \geq 1$:*

a) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

b) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Definition 1.3. *Given a quasi b-metric space (X, d) we define $d^{-1} : X \times X \to [0, \infty)$ by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$. Then (X, d^{-1}) is also a quasi b-metric space. We call d^{-1} the* ***conjugate*** *of d on X. Next, we define the function $d^* : X \times X \to [0, \infty)$ by $d^*(x, y) = max\{d(x, y), d^{-1}(x, y)\}$. Note that d^* is a b-**metric** on X. We say that the quasi b-metric space is Cauchy bicomplete if the b-metric space (X, d^*) is Cauchy complete.*

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Note that every quasi metric space is a quasi b-metric space but not conversely.

Let (X, d) be a quasi b-metric space. We say that a quasi b-metric space $(\tilde{X}, \tilde{d})$ is a **Cauchy bicompletion** of (X, d) if:

$a)$ $(\tilde{X}, \tilde{d})$ is Cauchy bicomplete;

$b)$ $X \subseteq \tilde{X}$, and $\tilde{d}|_{X\times X} = d$;

$c)$ There exists an isometry $T : X \to \tilde{X}$, such that $T(X)$ is $\tilde{d}^*$-dense in $\tilde{X}$.

Recall that a **strong b-metric space** is a pair (X, d), with a mapping $d : X \times X \to [0, \infty)$ satisfying for all $x, y, z \in X$ and $s \geq 1$:

$a)$ $d(x, y) = 0$ if and only if $x = y$;

$b)$ $d(x, y) = d(y, x)$;

$c)$ $d(x, y) \leq d(x, z) + sd(z, y)$.

So, we define a **strong quasi b-metric space** as a pair (X, d) with a mapping $d : X\times X \to [0, \infty)$ satisfying for all $x, y, z \in X$ and $s \geq 1$:

$a)$ $d(x, y) = d(y, x) = 0$ if and only if $x = y$;

$b)$ $d(x, y) \leq d(x, z) + sd(z, y)$.

Note that every strong quasi b-metric space is a quasi b-metric space but not conversely, also every strong b-metric space is a strong quasi b-metric space and not conversely.

The completion of a strong b-metric space is discussed in [2], that is,

Theorem 1.1. [2] *Let (X, d) be a strong b-metric space. There exists a Cauchy completion strong b-metric space $(\tilde{X}, \tilde{d})$ of (X, d) which is unique up to isometry.*

We now extend Theorem 1.1:

Theorem 1.2. *Let (X, d) be a strong quasi b-metric space. There exists a strong quasi b-metric space $(\tilde{X}, \tilde{d})$ of (X, d) which is Cauchy bicomplete and unique up to isometry. The strong quasi b-metric space $(\tilde{X}, \tilde{d})$ is the Cauchy bicompletion of (X, d).*

Proof: Let (X, d) be a strong quasi b-metric space and $\mathcal{C}$ be the set of all d^*-Cauchy sequences in (X, d). Define the relation $\sim$ on $\mathcal{C}$ as follows: $\{x_n\} \sim \{y_n\}$ if $\lim_n d(x_n, y_n) = 0$. Then $\sim$ is an equivalence relation in $\mathcal{C}$. Let $\tilde{X} = \mathcal{C}/\sim$ be the set of all equivalence classes for $\sim$, that is, $\tilde{X} = \{[\{x_n\}] : x_n \in \mathcal{C}\}$. Define $\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty)$ by $\tilde{d}([\{x_n\}], [\{y_n\}]) = \lim_n d(x_n, y_n)$.

Then $\tilde{d}$ is well defined and it is a strong quasi b-metric on $\tilde{X}$, with the same parameter s.

Easily, X can be regarded as a subspace of $\tilde{X}$, just define $T : X \to \tilde{X}$ by $Tx = [\{x\}]$ for each $x \in X$ and also T is an isometry.

Let $x = [\{x_n\}] \in \tilde{X}$. Given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for any $n, m > N$, $d(x_n, x_m) < \epsilon$. In particular, $d(x_n, x_{N+1}) < \epsilon$ and $d(x_{N+1}, x_n) < \epsilon$ for $n > N$. Hence, $\tilde{d}(x, x_{N+1}) \leq \epsilon$ and $\tilde{d}(x_{N+1}, x) \leq \epsilon$. So, $\tilde{d}^*(x, x_{N+1}) < \epsilon$, thus X is $\tilde{d}^*$-dense in $\tilde{X}$, where $\tilde{d}^* : \tilde{X}\times\tilde{X} \to [0, \infty)$ is given by $\tilde{d}^*(x, y) = max\{\tilde{d}(x, y), \tilde{d^{-1}}(x, y)\}$, for all $x, y \in \tilde{X}$. Finally, $(\tilde{X}, \tilde{d})$ is Cauchy bicomplete and it is unique up to isometry. □

We recall:

Theorem 1.3. [1] *Let (X, d) be a quasi b-metric space. There exists a Cauchy bicompletion $(\tilde{X}, \tilde{d})$ of (X, d) which is unique up to isometry.*

It is important to note that in Theorem 1.3, $\tilde{d}$ may not be well-defined. Now consider the set X, where X is the set of rational numbers equipped with the distance function $d : X \times X \to [0, \infty)$ defined by $d(x, y) = (x - y)^2$, for all $x, y \in X$. We easily see that (X, d) is a b-metric space with the parameter $s = 2$. The Cauchy completion of X is just $\tilde{X}$, the set of real numbers with the distance function $\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty)$ defined by $\tilde{d}(x, y) = (x - y)^2$. Note that (X, d) is not a strong b-metric space, take $x = 1, z = 2$ and $y = 3$. Then $d(x, y) > d(x, z) + sd(z, y)$ with $s = 2$. So, Theorem 1.1 does not apply in this instance but Theorem 1.3 does. Generally, there are problems with this construction as noted in [2], namely: that the function $\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty)$ may not be well-defined. This also correct our statement concerning the well defineness of $\tilde{d}$ in [1].

PROBLEM: Let (X, d) be a quasi b-metric space. Construct a Cauchy bicompletion $(\tilde{X}, \tilde{d})$ of (X, d) such that $\tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty)$ is well-defined.

Finally, let (X, d) be a quasi b-metric space. It is well-known [1] that (X, d) is quasi-metrizable, since the set $U_d = \{(x, y) : d(x, y) < \frac{1}{n}; n \geq 1\}$ is a base for the quasi-uniformity $\mathcal{U}_d$ and is countable. So, let ρ be the quasi-metric on X such that $\tau(d) = \tau(\rho)$, where $\tau(d)$ denotes the topology on X induced by d. Note that the bicompletion $(\tilde{X}, \tilde{\mathcal{U}_d})$ of the quasi-uniform space $(X, \mathcal{U}_d)$ is unique up to quasi uniform isomorphism [3].

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