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SOME FIXED POINT THEOREMS FOR GENERALIZED CONTRACTION INVOLVING RATIONAL EXPRESSIONS IN COMPLEX VALUED B-METRIC SPACES

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Abstract. The purpose of this paper is to prove the existence and uniqueness of a common fixed point for a pair of mappings satisfying generalized contraction under rational expressions having point-dependent control functions as coefficients in complex valued b-metric spaces. Our results generalize and extend the results of Azam et al., Dass et al., Dubey et al., Mukheimer, Nashine et al., Rao et al. and Sitthikul et al.

1. Introduction and Preliminaries

Complex valued metric spaces were introduced in 2011 by Azam et al. [1] and established the existence of fixed point theorems for maps satisfying the contraction condition involving rational expressions. Several fixed point results in such spaces were obtained, for example, in [2,8,9,11,12,13,14]. The concept of complex valued b-metric space as a generalization of complex valued metric space .[1] was initiated by Rao et al. [10]. Subsequently, many authors proved fixed and common fixed point results in complex valued b-metric spaces [4,5,6,7].

The aim of this paper is to give the development of fixed point theorems on complex valued b-metric space. We prove fixed and common fixed point results for generalized contraction involving rational expressions in complex valued b-metric space which generalize all the existing results. In particular, complex valued b-metric type version of very well known results of Nashine et al. [9], Sitthikul et al.[12] and others are obtained.

Keywords: Common fixed point, Complex valued b-metric spaces, Complete complex valued b-metric spaces, Cauchy sequence, Fixed point.

In what follows, we recall some notations and definitions due to Rao et al. [10], that will be needed in our subsequent discussion.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2).$$

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i),(ii) and (iii) is satisfied, also we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that

(a) If $0 \precsim z_1 \precnsim z_2$ then $|z_1| < |z_2|$.
(b) If $z_1 \precsim z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$.
(c) If $a, b \in \mathbb{R}$ and $a \leq b$ then $az \precsim bz$ for all $z \in \mathbb{C}$.

The following definition is recently introduced by Rao et al. [10].

Definition 1.1[10] Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{C}$ is called a complex valued b-metric on X if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \precsim s[d(x, z) + d(z, y)]$.

The pair (X, d) is called a complex valued b-metric space.

Example 1.2[10] Let $X = [0, 1]$. Define the mapping $d : X \times X \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$, for all $x, y \in X$. Then (X, d) is a complex valued b-metric space with $s = 2$.

Definition 1.3[10] Let (X, d) be a complex valued b-metric space.

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$.

(ii) A point $x \in X$ is called a limit point of a set A whenever for every $0 \prec r \in \mathbb{C}, B(x, r) \cap (A - \{x\}) \neq \phi$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of a set A.

(iv) A subset $A \subseteq X$ is called closed whenever each element of A belongs to A.

(v) A sub-basis for a Hausdorff topology τ on X is a family $F = \{B(x, r) : x \in X \, and \, 0 \prec r\}$.

Definition 1.4[10] Let (X, d) be a complex valued b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$converges to x and x is the limit point of $\{x_n\}$. We denote this by $lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x \, as \, n \to \infty$.

(ii) If for every $c \in \mathbb{C}$, with $0 \prec r$ there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$is said to be Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 1.5[10] Let (X, d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 1.6[10]Let (X, d) be a complex valued b-metric space and let $\{x_n\}$be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

2. Main Result

We start to this section with the following observation, that will be used in our subsequent result.

Proposition 2.1. Let (X, d) be a complex valued b-metric space and $S, T : X \to X$. Let $x_0 \in X$ and defined the sequence $\{x_n\}$ by

$x_{2n+1} = Sx_{2n}$

$x_{2n+2} = Tx_{2n+1}$, for all $n = 0, 1, 2, - - -$. (2.1)

Assume that there exists a mapping $\propto$: $X \times X \to [0, 1)$ such that $\propto (TSx, y) \leq \propto (x, y)$ and $\propto (x, STy) \leq \propto (x, y)$ for all $x, y \in X$. Then $\propto (x_{2n}, y) \leq \propto (x_0, y)$ and $\propto (x, x_{2n+1}) \leq \propto (x, x_1)$ for all $x, y \in X$ and $n = 0, 1, 2, - - -$.

Proof. Let $x, y \in X$ and $n = 0, 1, 2, - - -$. Then we have

$$\propto (x_{2n}, y) = \propto (Tx_{2n-1}, y) = \propto (TSx_{2n-2}, y)$$
$$\leq \propto (x_{2n-2}, y)$$
$$= \propto (Tx_{2n-3}, y) = \propto (TSx_{2n-4}, y)$$
$$\leq \propto (x_{2n-4}, y) \leq - - - - \leq \propto (x_0, y).$$

Similarly, we have

$$\propto (x, x_{2n+1}) = \propto (x, Sx_{2n}) = \propto (x, STx_{2n-1})$$
$$\leq \propto (x, x_{2n-1})$$
$$= \propto (x, Sx_{2n-2}) = \propto (x, STx_{2n-3})$$
$$\leq \propto (x, x_{2n-3}) \leq - - - - \leq \propto (x, x_1).$$

Lemma 2.2[12] Let $\{x_n\}$ be a sequence in X and $h \in [0,1)$. If $a_n = |d(x_n, x_{n+1})|$ satisfies $a_n \leq h a_{n-1}$, for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

Now, we are ready to prove the following fixed point theorems for generalized contractions.

Theorem 2.3. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$. If there exist mappings $\propto, \beta, \gamma, \delta : X \times X \to [0,1)$ such that for all $x, y \in X$:

(a) $\propto(TSx, y) \leq \propto(x, y)\, and \propto(x, STy) \leq \propto(x, y)$,

$\beta(TSx, y) \leq \beta(x, y)\, and\, \beta(x, STy) \leq \beta(x, y)$,

$\gamma(TSx, y) \leq \gamma(x, y)\, and\, \gamma(x, STy) \leq \gamma(x, y)$,

$\delta(TSx, y) \leq \delta(x, y)\, and\, \delta(x, STy) \leq \delta(x, y)$;

(b) $\propto(x, y) + \beta(x, y) + 2\gamma(x, y) + 2s\delta(x, y) < 1$;

(c) $$d(Sx, Ty) \precsim \propto(x, y)d(x, y) + \frac{\beta(x,y)[1+d(x,Sx)]d(y,Ty)}{1+d(x,y)}$$

$$+\gamma(x, y)[d(x, Sx) + d(y, Ty)] + \delta(x, y)[d(x, Ty) + d(y, Sx)]. \qquad (2.2)$$

Then S and T have a unique common fixed point.

Proof. Let $x, y \in X$, from (2.2) we have

$$d(Sx, TSx) \precsim \propto(x, Sx)d(x, Sx) + \frac{\beta(x,Sx)[1+d(x,Sx)]d(Sx,TSx)}{1+d(x,Sx)}$$

$$+\gamma(x, Sx)[d(x, Sx) + d(Sx, TSx)]$$

$$+\delta(x, Sx)[d(x, TSx) + d(Sx, Sx)]$$

$$\precsim \propto(x, Sx)d(x, Sx) + \beta(x, Sx)d(Sx, TSx)$$

$$+\gamma(x, Sx)[d(x, Sx) + d(Sx, TSx)]$$

$$+s\delta(x,Sx)[d(x,Sx)+d(Sx,TSx)]$$

which implies that

$$|d(Sx,TSx)| \leq \propto (x,Sx)|d(x,Sx)| + \beta(x,Sx)|d(Sx,TSx)|$$

$$+\gamma(x,Sx)|d(x,Sx)+d(Sx,TSx)|$$

$$+s\delta(x,Sx)|d(x,Sx)+d(Sx,TSx)|. \qquad (2.3)$$

Similary, from (2.2) we have,

$$d(STy,Ty) \precsim \propto (Ty,y)d(Ty,y) + \frac{\beta(Ty,y)[1+d(Ty,STy)]d(y,Ty)}{1+d(Ty,y)}$$

$$+\gamma(Ty,y)[d(Ty,STy)+d(y,Ty)]$$

$$+\delta(Ty,y)[d(Ty,Ty)+d(y,STy)]$$

which implies that

$$|d(STy,Ty)| \leq \propto (Ty,y)|d(Ty,y)| + \beta(Ty,y)|1+d(Ty,STy)||\frac{d(y,Ty)}{1+d(y,Ty)}|$$

$$+\gamma(Ty,y)|d(Ty,STy)+d(y,Ty)| + s\delta(Ty,y)|d(y,Ty)+d(Ty,STy)|$$

$$|d(STy,Ty)| \leq \propto (Ty,y)|d(Ty,y)| + \beta(Ty,y)|d(STy,Ty)|$$

$$+\gamma(Ty,y)|d(STy,Ty)+d(Ty,y)|$$

$$+s\delta(Ty,y)|d(STy,Ty)+d(Ty,y)|. \qquad (2.4)$$

Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by (2.1).

We show that $\{x_n\}$ is a Cauchy sequence. From Propostion 2.1 and inequalities (2.3), (2.4) and for all $K = 0, 1, 2, -----$, we obtain

$$|d(x_{2K+1}, x_{2K})| = |d(STx_{2K-1,}Tx_{2K-1})|$$

$$\leq \propto (Tx_{2K-1}, x_{2K-1})|d(Tx_{2K-1}, x_{2K-1})|$$

$$+\frac{\beta(Tx_{2K-1},x_{2K-1})|1+d(Tx_{2K-1},STx_{2K-1})||d(x_{2K-1},Tx_{2K-1})|}{|1+d(Tx_{2K-1},x_{2K-1})|}$$

$$+\gamma(Tx_{2K-1}, x_{2K-1})|d(Tx_{2K-1}, STx_{2K-1}) + d(x_{2K-1}, Tx_{2K-1})|$$

$$+\delta(Tx_{2K-1}, x_{2K-1})|d(Tx_{2K-1}, Tx_{2K-1}) + d(x_{2K-1}, STx_{2K-1})|$$

$$=\propto (x_{2K}, x_{2K-1})|d(x_{2K}, x_{2K-1})|$$

$$+\frac{\beta(x_{2K},x_{2K-1})|1+d(x_{2K},x_{2K+1})||d(x_{2K-1},x_{2K})|}{|1+d(x_{2K},x_{2K-1})|}$$

$$+\gamma(x_{2K}, x_{2K-1})|d(x_{2K}, x_{2K+1}) + d(x_{2K-1}, x_{2K})|$$

$$+\delta(x_{2K}, x_{2K-1})|d(x_{2K}, x_{2K}) + d(x_{2K-1}, x_{2K+1})|$$

$$|d(x_{2K+1}, x_{2K})| \leq \propto (x_{2K}, x_{2K-1})|d(x_{2K}, x_{2K-1})|$$

$$+\beta(x_{2K}, x_{2K-1})|d(x_{2K+1}, x_{2K})|$$

$$+\gamma(x_{2K}, x_{2K-1})|d(x_{2K+1}, x_{2K}) + d(x_{2K}, x_{2K-1})|$$

$$+s\delta(x_{2K}, x_{2K-1})|d(x_{2K+1}, x_{2K}) + d(x_{2K}, x_{2K-1})|$$

$$\leq \propto (x_0, x_{2K-1})|d(x_{2K}, x_{2K-1})|$$

$$+\beta(x_0, x_{2K-1})|d(x_{2K+1}, x_{2K})|$$

$$+\gamma(x_0, x_{2K-1})|d(x_{2K+1}, x_{2K}) + d(x_{2K}, x_{2K-1})|$$

$$+s\delta(x_0, x_{2K-1})|d(x_{2K+1}, x_{2K}) + d(x_{2K}, x_{2K-1})|$$

$$\leq \propto (x_0, x_1)|d(x_{2K}, x_{2K-1})| + \beta(x_0, x_1)|d(x_{2K+1}, x_{2K})|$$

$$+\gamma(x_0, x_1)|d(x_{2K+1}, x_{2K}) + d(x_{2K}, x_{2K-1})|$$

$$+s\delta(x_0, x_1)|d(x_{2K+1}, x_{2K}) + d(x_{2K}, x_{2K-1})|$$

which yeilds that

$$|d(x_{2K+1}, x_{2K})| \leq \frac{\propto(x_0,x_1)+\gamma(x_0,x_1)+s\delta(x_0,x_1)}{1-\beta(x_0,x_1)-\gamma(x_0,x_1)-s\delta(x_0,x_1)}|d(x_{2K}, x_{2K-1})|.$$

Similarly, one can obtain

$$|d(x_{2K+2}, x_{2K+1})| \leq \frac{\propto(x_0,x_1)+\gamma(x_0,x_1)+s\delta(x_0,x_1)}{1-\beta(x_0,x_1)-\gamma(x_0,x_1)-s\delta(x_0,x_1)}|d(x_{2K+1}, x_{2K})|.$$

Let $\mu = \frac{\propto(x_0,x_1)+\gamma(x_0,x_1)+s\delta(x_0,x_1)}{1-\beta(x_0,x_1)-\gamma(x_0,x_1)-s\delta(x_0,x_1)} < 1$.

Since $\propto (x_0, x_1) + \beta(x_0, x_1) + 2\gamma(x_0, x_1) + 2s\delta(x_0, x_1) < 1$, thus we have

$$|d(x_{n+1}.x_{n+2})| \leq \mu|d(x_n, x_{n+1})| \leq - - - \leq \mu^{n+1}|d(x_0, x_1)|. \qquad (2.5)$$

Thus for any $m > n, m, n \in \mathbb{N}$,

$$|d(x_n, x_m)| \leq s|d(x_n, x_{n+1})| + s|d(x_{n+1}, x_m)|$$

$$\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^2|d(x_{n+2}, x_m)|$$

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$$\leq s|d(x_n, x_{n+1})| + s^2|d(x_{n+1}, x_{n+2})| + s^3|d(x_{n+2}, x_{n+3})|$$

$$+ - - - - + s^{m-n-1}|d(x_{m-2}, x_{m-1})| + s^{m-n}|d(x_{m-1}, x_m)|.$$

By using (2.5), we get

$$|d(x_n, x_m)| \leq s\mu^n|d(x_0, x_1)| + s^2\mu^{n+1}|d(x_0, x_1)| + s^3\mu^{n+2}|d(x_0, x_1)|$$

$$+ - - - - + s^{m-n-1}\mu^{m-2}|d(x_0, x_1)| + s^{m-1}\mu^{m-1}|d(x_0, x_1)|$$

$$= \sum_{i=1}^{m-n} s^i\mu^{i+n-1}|d(x_0, x_1)|.$$

Therefore,

$$|d(x_n, x_m)| \leq \sum_{i=1}^{m-n} s^{i+n-1}\mu^{i+n-1}|d(x_0, x_1)|$$

$$= \sum_{t=n}^{m-1} s^t\mu^t|d(x_0, x_1)| \leq \sum_{t=n}^{\infty}(s\mu)^t|d(x_0, x_1)|$$

$$= \tfrac{(s\mu)^n}{1-s\mu}|d(x_0, x_1)|$$

and so,

$$|d(x_n, x_m)| \leq \tfrac{(s\mu)^n}{1-s\mu}|d(x_0, x_1)| \to 0 \text{ as } m, n \to \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Let on contrary $u \neq Su$, then there exists $z \in X$ such that

$$|d(u, Su)| = |z| > 0. \qquad (2.6)$$

So by using the triangular inequality and (2.2), we get

$$z = d(u, Su)$$

$$\precsim sd(u, x_{2n+2}) + sd(x_{2n+2}, Su)$$

$$= sd(u, x_{2n+2}) + sd(Tx_{2n+1}, Su)$$

$$\precsim sd(u, x_{2n+2}) + s \propto (u, x_{2n+1})d(u, x_{2n+1})$$

$$+ \tfrac{s\beta(u, x_{2n+1})[1+d(u,Su)]d(x_{2n+1}, Tx_{2n+1})}{1+d(u, x_{2n+1})}.$$

$$+s\gamma(u,x_{2n+1})[d(u,Su)+d(x_{2n+1},Tx_{2n+1})]$$

$$+s\delta(u,x_{2n+1})[d(u,Tx_{2n+1})+d(x_{2n+1},Su)]$$

$$=sd(u,x_{2n+2})+s\propto(u,x_{2n+1})d(u,x_{2n+1})$$

$$+\frac{s\beta(u,x_{2n+1})[1+d(u,Su)]d(x_{2n+1},x_{2n+2})}{1+d(u,x_{2n+1})}$$

$$+s\gamma(u,x_{2n+1})[d(u,Su)+d(x_{2n+1},x_{2n+2})]$$

$$+s\delta(u,x_{2n+1})[d(u,x_{2n+2})+d(x_{2n+1},Su)]$$

$$\precsim sd(u,x_{2n+2})+s\propto(u,x_1)d(u,x_{2n+1})$$

$$+\frac{s\beta(u,x_1)[1+d(u,Su)]d(x_{2n+1},x_{2n+2})}{1+d(u,x_{2n+1})}$$

$$+s\gamma(u,x_1)[d(u,Su)+d(x_{2n+1},x_{2n+2})]$$

$$+s\delta(u,x_1)[d(u,x_{2n+2})+d(x_{2n+1},Su)].$$

This implies that

$$|z|\leq|d(u,Su)|\leq s|d(u,x_{2n+2})|+s\propto(u,x_1)|d(u,x_{2n+1})|$$

$$+\frac{s\beta(u,x_1)|[1+d(u,Su)]||d(x_{2n+1},x_{2n+2})|}{|1+d(u,x_{2n+1})|}$$

$$+s\gamma(u,x_1)|z+d(x_{2n+1},x_{2n+2})|$$

$$+s\delta(u,x_1)|d(u,x_{2n+2})+z|.$$

Letting $n\to\infty$, it follows that

$|z| \leq s[\gamma(u, x_1) + \delta(u, x_1)]|z|$

$\leq s[\propto (u, x_1) + \beta(u, x_1) + 2\gamma(u, x_1) + 2\delta(u, x_1)]|z|$

$< |z|,$

a contradiction and so $|z| = 0$, that is $u = Su$. It follows similarly that $u = Tu$.

We now show that S and T have unique common fixed point. For this, assume that $u^{\star}$in X is another common fixed point of S and T. Then

$d(u, u^{\star}) = d(Su, Tu^{\star})$

$\precsim \propto (u, u^{\star})d(u, u^{\star}) + \frac{\beta(u,u^{\star})[1+d(u,Su)]d(u^{\star},Tu^{\star})}{1+d(u,u^{\star})}$

$+\gamma(u, u^{\star})[d(u, Su) + d(u^{\star}, Tu^{\star})]$

$+\delta(u, u^{\star})[d(u, Tu^{\star}) + d(u^{\star}, Su)]$

$\precsim [\propto (u, u^{\star}) + 2\delta(u, u^{\star})]d(u, u^{\star}).$

Therefore, we have

$|d(u, u^{\star})| \leq [\propto (u, u^{\star}) + 2\delta(u, u^{\star})]|d(u, u^{\star})|.$

Since $\propto (u, u^{\star}) + 2\delta(u, u^{\star}) < 1$, we have $|d(u, u^{\star})| = 0$. Thus $u = u^{\star}$, which proves the uniqueness of common fixed point in X. This completes the proof of the Theorem. By putting $S = T$ in Theorem 2.3, we deduce the following corollary.

Corollary 2.4. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exist mappings $\propto, \beta, \gamma, \delta : X \times X \to [0, 1)$ such that for all $x, y \in X$:

(a) $\propto (Tx, y) \leq \propto (x, y)\, and\, \propto (x, Ty) \leq \propto (x, y),$

$\beta(Tx, y) \leq \beta(x, y)\, and\, \beta(x, Ty) \leq \beta(x, y),$

$\gamma(Tx, y) \leq \gamma(x, y)\, and\, \gamma(x, Ty) \leq \gamma(x, y),$

$\delta(Tx, y) \leq \delta(x, y)\, and\, \delta(x, Ty) \leq \delta(x, y);$

(b) $\propto (x, y) + \beta(x, y) + 2\gamma(x, y) + 2s\delta(x, y) < 1;$

(c) $d(Tx, Ty) \precsim \propto (x, y)d(x, y) + \frac{\beta(x,y)[1+d(x,Tx)]d(y,Ty)}{1+d(x,y)}$

$$+\gamma(x, y)[d(x, Tx) + d(y, Ty)]$$

$$+\delta(x, y)[d(x, Ty) + d(y, Tx)]. \qquad (2.7)$$

Then T has a unique fixed point.

The following theorem is closely related to Corollary 2.4 with $\gamma = \delta = 0$.

Theorem 2.5. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $T : X \to X$. If there exist mappings $\propto, \beta : X \times X \to [0, 1)$ such that for all $x, y \in X$:

(a) $\propto (Tx, y) \leq \propto (x, y)\, and\, \propto (x, Ty) \leq \propto (x, y),$

$\beta(Tx, y) \leq \beta(x, y)\, and\, \beta(x, Ty) \leq \beta(x, y);$

(b) $s\{\propto (x, y) + \beta(x, y)\} < 1;$

(c) $d(Tx, Ty) \precsim \propto (x, y)d(x, y) + \beta(x, y)\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}. \qquad (2.8)$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined by $x_{n+1} = Tx_n$, where $n = 0, 1, 2, - - - - - - - -.$ (2.9)

We show that $\{x_n\}$ is a Cauchy sequence. From (2.8) we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$$

$$\precsim \propto (x_n, x_{n+1})d(x_n, x_{n+1})$$

$$+\beta(x_n, x_{n+1})\frac{d(x_{n+1}, Tx_{n+1})[1 + d(x_n, Tx_n)]}{1 + d(x_n, x_{n+1})}$$

$$= \propto (x_n, x_{n+1})d(x_n, x_{n+1})$$

$$+\beta(x_n, x_{n+1})\frac{d(x_{n+1}, x_{n+2})[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n+1})}$$

$$= \propto (x_n, x_{n+1})d(x_n, x_{n+1}) + \beta(x_n, x_{n+1})d(x_{n+1}, x_{n+2}).$$

It follows from (a) that

$$d(x_{n+1}, x_{n+2}) \precsim \propto (x_n, x_{n+1})d(x_n, x_{n+1}) + \beta(x_n, x_{n+1})d(x_{n+1}, x_{n+2})$$

$$\precsim \propto (x_0, x_{n+1})d(x_n, x_{n+1}) + \beta(x_0, x_{n+1})d(x_{n+1}, x_{n+2})$$

$$\precsim \propto (x_0, x_0)d(x_n, x_{n+1}) + \beta(x_0, x_0)d(x_{n+1}, x_{n+2}).$$

Therefore

$$|d(x_{n+1}, x_{n+2})| \leq \propto (x_0, x_0)|d(x_n, x_{n+1})| + \beta(x_0, x_0)|d(x_{n+1}, x_{n+2})|$$

and hence

$$|d(x_{n+1}, x_{n+2})| \leq \frac{\propto (x_0, x_0)}{1 - \beta(x_0, x_0)}|d(x_n, x_{n+1})|, \text{ for all } n = 0, 1, 2, - - - - .$$

Since $s\{\propto (x_0, x_0) + \beta(x_0, x_0)\} < 1$ and $s \geq 1$, we get $\propto (x_0, x_0) + \beta(x_0, x_0) < 1$. Therefore with $\mu = \frac{\propto(x_0, x_0)}{1 - \beta(x_0, x_0)} < 1$,we have

$$|d(x_{n+1}, x_{n+2})| \leq \mu|d(x_n, x_{n+1})|, \text{ for all } n = 0, 1, 2, - - - - .$$

By similar argument as Theorem 2.3, we have $\{x_n\}$ is a Cauchy sequence in (X,d). By the completeness of X, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Next, we show that u is a fixed point of T. Then

$$d(u,Tu) \precsim d(u,Tx_n) + d(Tx_n,Tu)$$

$$\precsim d(u,x_{n+1}) + \propto (x_n,u)d(x_n,u)$$

$$+\beta(x_n,u)\frac{d(u,Tu)[1+d(x_n,Tx_n)]}{1+d(x_n,u)}$$

$$\precsim d(u,x_{n+1}) + \propto (x_0,u)d(x_n,u)$$

$$+\beta(x_0,u)\frac{d(u,Tu)[1+d(x_n,x_{n+1})]}{1+d(x_n,u)}.$$

Letting $n \to \infty$, it follows that

$$|d(u,Tu)| \leq \beta(x_0,u)|d(u,Tu)|,$$

a contradiction and so $|d(u,Tu)| = 0$ that is $u = Tu$.

Finally, we show the uniqueness. Suppose that there is $u^\star \in X$ such that $u^\star = Tu^\star$. Then

$$d(u,u^\star) = d(Tu,Tu^\star)$$

$$\precsim \propto (u,u^\star)d(u,u^\star) + \beta(u,u^\star)\frac{d(u^\star,Tu^\star)[1+d(u,Tu)]}{1+d(u,u^\star)}$$

$$= \propto (u,u^\star)d(u,u^\star)$$

and so $d(u,u^\star) = 0$, since $\propto (u,u^\star) < 1$. This implies that $u^\star = u$, completing the proof of the theorem.

3. Deduced results

We deduce the main result of [13] as follows.

Theorem 3.1. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$. If there exist mappings $\propto, \beta : X \to [0, 1)$ such that for all $x, y \in X$:

(a) $\propto(Sx) \leq \propto(x)\, and\, \beta(Sx) \leq \beta(x),$

$\propto(Tx) \leq \propto(x)\, and\, \beta(Tx) \leq \beta(x);$

(b) $s\{\propto(x) + \beta(x)\} < 1;$

(c) $d(Sx, Ty) \precsim \propto(x)d(x, y) + \frac{\beta(x)d(x,Sx)d(y,Ty)}{1+d(x,y)}.$ (3.1)

Then $S\, and\, T$ have a unique common fixed point.

Proof. Define $\lambda, \mu : X \times X \to [0, 1)$ by

$\lambda(x, y) = \propto(x)$ and $\mu(x, y) = \beta(x)$, for all $x, y \in X$. (3.2)

Then for all $x, y \in X$,

(a) $\lambda(TSx, y) = \propto(TSx) \leq \propto(Sx) \leq \propto(x) = \lambda(x, y)$ and

$\lambda(x, STy) = \propto(x) = \lambda(x, y),$

$\mu(TSx, y) = \beta(TSx) \leq \beta(Sx) \leq \beta(x) = \mu(x, y)$ and

$\mu(x, STy) = \beta(x) = \mu(x, y);$

(b) $s\{\lambda(x, y) + \mu(x, y)\} = s\{\propto(x) + \beta(x)\} < 1;$

(c) $d(Sx, Ty) \precsim \propto(x)d(x, y) + \frac{\beta(x)d(x,Sx)d(y,Ty)}{1+d(x,y)}$

$= \lambda(x, y)d(x, y) + \mu(x, y)\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}.$

Then by Corollary 2.6[12] (Complex valued b-metric space version), $S\,and\,T$ have a unique common fixed point.

The following Corollary is obtained from our Theorem 2.3.

Corollary 3.2. Let (X, d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and let $S, T : X \to X$. If there exist mappings $\propto, \beta, \gamma, \delta : X \to [0, 1)$ such that for all $x, y \in X$:

(a) $\propto(TSx) \leq \propto(x), \beta(TSx) \leq \beta(x), \gamma(TSx) \leq \gamma(x)$ and

$\delta(TSx) \leq \delta(x)$;

(b) $\propto(x) + \beta(x) + 2\gamma(x) + 2s\delta(x) < 1$;

(c) $d(Sx, Ty) \precsim \propto(x)d(x, y) + \frac{\beta(x)[1+d(x,Sx)]d(y,Ty)}{1+d(x,y)}$

$$+\gamma(x)[d(x, Sx) + d(y, Ty)]$$

$$+\delta(x)[d(x, Ty) + d(y, Sx)]. \qquad (3.3)$$

Then $S\,and\,T$ have a unique common fixed point.

Proof. Define $\propto, \beta, \gamma, \delta : X \times X \to [0, 1)$ by

$$\propto(x, y) = \propto(x), \beta(x, y) = \beta(x), \gamma(x, y) = \gamma(x) \text{ and } \delta(x, y) = \delta(x), \text{for all } x, y \in X. \qquad (3.4)$$

Then for all $x, y \in X$,

(a) $\propto(TSx, y) = \propto(TSx) \leq \propto(x) = \propto(x, y)$ and $\propto(x, STy) = \propto(x) = \propto(x, y)$,

$\beta(TSx, y) = \beta(TSx) \leq \beta(x) = \beta(x, y)$ and $\beta(x, STy) = \beta(x) = \beta(x, y)$,

$\gamma(TSx, y) = \gamma(TSx) \leq \gamma(x) = \gamma(x, y)$ and$\gamma(x, STy) = \gamma(x) = \gamma(x, y)$,

$\delta(TSx, y) = \delta(TSx) \leq \delta(x) = \delta(x, y)$ and $\delta(x, STy) = \delta(x) = \delta(x, y)$;

(b) $\propto(x, y) + \beta(x, y) + 2\gamma(x, y) + 2s\delta(x, y) = \propto(x) + \beta(x) + 2\gamma(x) + 2s\delta(x) < 1$;

(c) $d(Sx, Ty) \precsim \propto(x)d(x, y) + \frac{\beta(x)[1+d(x,Sx)]d(y,Ty)}{1+d(x,y)}$

$$+\gamma(x)[d(x, Sx) + d(y, Ty)]$$

$$+\delta(x)[d(x, Ty) + d(y, Sx)]$$

$$= \propto(x, y)d(x, y) + \frac{\beta(x,y)[1+d(x,Sx)]d(y,Ty)}{1+d(x,y)}$$

$$+\gamma(x, y)[d(x, Sx) + d(y, Ty)] + \delta(x, y)[d(x, Ty) + d(y, Sx)]. \qquad (3.5)$$

By Theorem 2.3, $S\, and\, T$ have a unique common fixed point.

Letting $\propto(.) = \propto, \beta(.) = \beta, \gamma(.) = \gamma$ and $\delta(.) = \delta$ in Corollary 3.2 gives the following result proved by Nashine and Fisher [9] (Complex valued b-metric space version).

Corollary 3.3. If $S\, and\, T$ are self-mappings defined on a complex valued b-metric space (X, d) with the coefficient $s \geq 1$ satisfying the condition

$$d(Sx, Ty) \precsim \propto d(x, y) + \frac{\beta[1+d(x,Sx)]d(y,Ty)}{1+d(x,y)}$$

$$+\gamma[d(x, Sx) + d(y, Ty)]$$

$$+\delta[d(x, Ty) + d(y, Sx)] \qquad (3.6)$$

for all $x, y \in X$, where $\propto, \beta, \gamma$ and δ are nonnegative reals with $\propto + \beta + 2\gamma + 2s\delta < 1$. Then $S\, and\, T$ have a unique common fixed point.

Competing interests

The authors declare that they have no competing interests.

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