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A COMMON FIXED POINT THEOREM IN COMPLETE METRIC SPACE

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Abstract. We establish a common fixed point theorem for quadruple weakly compatible mappings satisfying construction modulus in complete metric space. Presented theorem generalize and extend the results of [Parvaneh Vahid, Some common fixed point theorem in complete metric space, Int. J. Pure Appl. Math. 76 (2012), 1-8]. We also provide an example which shows that our result is a proper generalization of the existing one.

1. Introduction

Until 1968, Banach contraction principle was the main tool used to establish existence and uniqueness of fixed points. It has been used in many different fields of mathematics. One of the essential and initial result in this direction was proved by Stefan Banach in 1922, when Banach stated and proved his famous result (Banach contraction principle). The field of fixed point theory that is involving four single valued maps, began with the assumption that all of the maps are commuted. In 1998, the concept of weakly compatible pairs of mappings has been introduced by Jungck [5], that is the class of mappings such that they commute at their coincidence points. Recently literature of fixed point theory obtained coincidence point and common fixed point results for different classes of mappings on various metric spaces such as complete metric spaces, partially ordered metric spaces, cone metric spaces. For a survey of coincidence point theory, its applications, comparison of different contractive conditions and related results, we refer to [4] and references contained in it. Recently, Choudhury [2] prove a common fixed point theorem for weakly C-contractive mappings satisfying the weakly compatibility without using the notion of continuity and Vahid [6] prove a two

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self mappings of a metric space to be weakly compatible. The main purpose of this paper is to present fixed point results for two pair of maps satisfying a new contractive condition by using the concept of weakly compatible maps in a complete metric space.

2. Preliminaries

We recall the definitions of complete metric space and other results that will be needed in the sequel.

Definition 2.1. A mapping $T : X \to X$ where (X, d) is a metric space is said to be a C-contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx))$$

The concept of C-contraction was defined by S.K. Chatterjea [1] in 1972 and he has proved that if (X, d) is a complete metric space, then every C-contraction on X has a unique fixed point.

In 2009, Choudhury has introduced weak C-contraction given by the following definition.
Definition 2.2. (see [2]) A mapping $T : X \to X$, where (X, d) is a metric space is said to be weakly C-contractive (or a weak C-contraction) if for all $x, y \in X$,

$$d(Tx, Ty) \leq \tfrac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi((d(x, Ty), d(y, Tx))),$$

where $\varphi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$.

In [2], he proved that if (X, d) is a complete metric space, then every weak C-contraction on X has a unique fixed point. Recently vahid [6] proved the following result.

Theorem 2.3. Let (X,d) be a complete metric space and E be a nonempty closed subset of X. Let $T, S : E \to E$ be such that,
$d(Tx, Sy) \leq \frac{1}{2}(d(Rx, Sy) + d(Ry, Tx)) - \varphi((d(Rx, Sy), d(Ry, Tx)))$

for every pair $(x, y) \in X \times X$, where $\varphi : [0, \infty)^2 \to [0, \infty)$ is a continuous function such that $\varphi(x, y) = 0$ if and only if $x = y = 0$ and $R : E \to X$ satisfying the following hypotheses:

(i) $TE \subseteq RE$ and $SE \subseteq RE$.
(ii) The pairs (T,R) and (S,R) are weakly compatible. xms In addition, assume that R(E)

is a closed subset of X. Then, T and R and S have a unique common fixed point.

Most recently K.C. Deshmukh et al. [3] prove a Generalization of a fixed point theorem of Suzuki type in complete metric space given the following,

Theorem 2.4. Let (X, d) be a nonempty complete metric space and $T : X \to X$ be a mapping satisfying $\frac{1}{2}d(x,Tx) \leq d(x,y)$ implies

$$d(Tx,Ty) + \ p\ max[d(x,y), d(x,Tx) + d(y,Ty), d(x,Ty) + d(y,Tx)]$$

$$\leq ad(x,y) + b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)]$$

where $a \geq 0$, $b > 0$, $c > 0$, $p \geq 0$ and $a+2b+2c-2p = 1$. Then T has a unique fixed point.

Definition 2.5. Let T and S be two self mappings of a metric space (X, d). T and S are said to be weakly compatible if for all $x \in X$ the equality $Tx = Sx$ implies $TSx = STx$.

3. Main Result

In this section we prove a common fixed point theorem for two pairs of weakly compatible mappings in complete metric spaces using a contractive modulus. This is the generalization of [Theorem 4] of Vahid [6].

Theorem 3.1. *Let (X, d) be a complete metric space and let E be a nonempty closed subset of X. Let $T, S : E \to E$ be such that,*

$$d(Tx,Sy) \leq \frac{1}{2}(d(Rx,Sy) + d(Py,Tx)) - \varphi((d(Rx,Sy), d(Py,Tx))). \tag{3.1}$$

For every pair $(x,y) \to X \times X$, where $\varphi : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\varphi(x,y) = 0$ if and only if $x = y = 0$ and $R, P : E \to X$ satisfying the following hypotheses:
(i) $T(E) \subseteq R(E)$ and $S(E) \subseteq P(E)$,
(ii) The pairs (T, P) and (S, R) are weakly compatible and we assume that if $R(E)$ and $P(E)$ are closed subsets of X. Then T, R, S and P have a unique common fixed point.

Proof. Let $x_0 \in E$ be arbitrary. Using (i) there exist tow sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $y_0 = Tx_0 = Rx_1, y_1 = Sx_1 = Px_2, y_2 = Tx_2 = Rx_3, ..., y_{2n} = Tx_{2n} = Rx_{2n+1}, y_{2n+1} = Sx_{2n+1} = Px_{2n+2}, \dots$
We complete the proof in three steps.
Step I. We will prove that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Then from (3.1), we suppose that $n = 2k$ then,

$$\begin{aligned}
d(y_{2k+1}, y_{2k}) =& d(Tx_{2k}, Sx_{2k+1}) \\
\leq& \frac{1}{2}(d(Rx_{2k}, Sx_{2k+1}) + d(Px_{2k+1}, Tx_{2k})) \\
&- \varphi(d(Rx_{2k}, Sx_{2k+1}), d(Px_{2k+1}, Tx_{2k})) \\
=& \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k})) \\
&- \varphi(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k})) \\
\leq& \frac{1}{2}d(y_{2k-1}, y_{2k+1}) \\
\leq& \frac{1}{2}(d(y_{2k-1}, y_{2k}) + d(y_{2k}, y_{2k+1}))
\end{aligned} \tag{3.2}$$

Hence, $d(y_{2k+1}, y_{2k}) \leq d(y_{2k}, y_{2k-1})$.

If $n = 2k + 1$, similarly we can say that

$$d(y_{2k+2}, y_{2k+1}) \leq d(y_{2k+1}, y_{2k})$$

Therefore $d(y_{n+1}, y_n)$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Suppose that, $lim_{n\to\infty} d(y_{n+1}, y_n) = r$. From the above argument we have,

$$\begin{aligned}
d(y_{n+1}, y_n) \leq& \frac{1}{2}d(y_{n-1}, y_{n+1}) \\
\leq& \frac{1}{2}(d(y_{n-1}, y_n) + d(y_n, y_{n+1})).
\end{aligned} \tag{3.3}$$

Taking $n \to \infty$, we have

$$r \leq lim_{n\to\infty} \frac{1}{2} d(y_{n-1}, y_{n+1}) \leq r.$$

Therefore, $lim_{n\to\infty} d(y_{n-1}, y_{n+1}) = 2r$.

We have proved in (3.2)

$$\begin{aligned}
d(y_{2k+1}, y_{2k}) =& d(Tx_{2k}, Sx_{2k+1})) \\
\leq& \frac{1}{2}(d(y_{2k-1}, y_{2k+1}) + d(y_{2k}, y_{2k})) - \varphi(d(y_{2k-1}, y_{2k+1}), d(y_{2k}, y_{2k})).
\end{aligned} \tag{3.4}$$

Now, if $k \to \infty$ and using the continuity of φ we obtain

$$r \leq \frac{1}{2}2r - \varphi(2r, 0),$$

and consequently, $\varphi(2r, 0) = 0$. This gives that

$$r = lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \tag{3.5}$$

by our assumption about φ.

Step II. Now we prove that, $\{y_n\}$ is Cauchy.

If $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$, it is sufficient to show that the sub- sequence $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ such that $n(k)$ is the least index for which $n(k) > m(k) > k$ and $d(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon$.

It is clear that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon \tag{3.6}$$

From triangle inequality, we get

$$\begin{aligned} \varepsilon \leq & d(y_{2m(k)}, y_{2n(k)}) \\ \leq & d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(_{y2n(k)-1}, y_{2n(k)}) \\ \leq & d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)})). \end{aligned} \tag{3.7}$$

Letting $k \to \infty$ and using (3.5) we conclude that

$$lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon \tag{3.8}$$

Now, we have

$$|d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2m(k)}, y_{2n(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1}) \tag{3.9}$$

and

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d(y_{2m(k)}, y_{2m(k)-1}) \tag{3.10}$$

and

$$|d(y_{2n(k)}, y_{2m(k)-2}) - d(y_{2n(k)}, y_{2m(k)-1})| \leq d(y_{2m(k)-2}, y_{2m(k)-1}). \tag{3.11}$$

Using (3.5), (3.8), (3.9), (3.10) and (3.11) we get

$$\begin{aligned} lim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)}) = & llim_{k\to\infty} d(y_{2m(k)-1}, y_{2n(k)-1}) \\ = & lim_{k\to\infty} d(y_{2m(k)-2}, y_{2n(k)}) \\ = & \varepsilon. \end{aligned} \tag{3.12}$$

Then, from (3.1) we have

$$
\begin{aligned}
d(y_{2m(k)-1}, y_{2n(k)}) =& d(Tx_{2n(k)}, Sx_{2m(k)-1}) \\
\leq& \frac{1}{2}(d(Rx_{2n(k)}, Sx_{2m(k)-1}) + d(Px_{2m(k)-1}, Tx_{2n(k)})) \\
& - \varphi(d(Rx_{2n(k)}, Sx_{2m(k)-1}), d(Px_{2m(k)-1}, Tx_{2n(k)})) \\
=& \frac{1}{2}(d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2n(k)})) \\
& - \varphi(d(y_{2n(k)-1}, y_{2m(k)-1}), d(y_{2m(k)-2}, y_{2n(k)})) \\
\leq& \frac{1}{2}(d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2m(k)}, y_{2m(k)+1})).
\end{aligned} \tag{3.13}
$$

Letting $k \to \infty$ in the above inequality, from (3.12) and the continuity of φ, we have

$$\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) - \varphi(\varepsilon, \varepsilon)$$

and from the last inequality $\varphi(\varepsilon, \varepsilon) = 0$. By our assumption about φ, we have $\varepsilon = 0$ which is a contradiction. At last we show that,

Step III. T, S, R and P have a common fixed point. Since (X, d) is complete and $\{y_n\}$ is Cauchy, there exists $z \in X$ such that $lim_{n\to\infty} y_n = z$. Since E is closed and $\{y_n\} \subseteq E$, we have $z \in E$. By assumption $R(E)$ and P(E) are closed, so there exists $u \in E$ such that $z = Ru = Pu$. For all $n \in N$,

$$
\begin{aligned}
d(Tu, y_{2n+1}) =& d(Tu, Sx_{2n+1}) \\
\leq& \frac{1}{2}(d(Ru, Sx_{2n+1}) + d(Px_{2n+1}, Tu)) \\
& - \varphi(d(Ru, Sx_{2n+1}), d(Px_{2n+1}, Tu)) \\
=& \frac{1}{2}(d(z, y_{2n+1}) + d(y_{2n}, Tu)) \\
& - \varphi(d(Ru, Sx_{2n+1}), d(Px_{2n+1}, Tu))).
\end{aligned} \tag{3.14}
$$

Letting $n \to \infty$,

$$d(Tu, z) \leq \frac{1}{2}(d(z, z) + d(z, Tu)) - \varphi(d(Pu, z), d(z, Tu))$$

and hence

$$\varphi(0, d(z, Tu)) \leq \ -\frac{1}{2}(d(Tu, z)) \leq 0,$$

which implies $d(z, Tu) = 0$. Therefore $Tu = z$.

Similarly $Su = z$. So $Tu = Su = Ru = Pu = z$. Since the pairs (P, T) and (R, S) are weakly compatible, we have $Tz = Sz = Rz = Pz$.

Now we have

$$d(Tz, y_{2n+1}) = d(Tz, Sx_{2n+1}) \leq \frac{1}{2}(d(Rz, Sx_{2n+1}) + d(Px_{2n+1}, Tz)) - \varphi(d(Rz, Sx_{2n+1}), d(Px_{2n+1}, Tz)) = \frac{1}{2}(d(Rz, y_{2n+1}) + d(y_{2n}, Tz)) - \varphi(d(Rz, y_{2n+1}), d(y_{2n}, Tz)). \tag{3.15}$$

Letting $n \to \infty$, since $Tz = Sz = Rz = Pz$, we obtain

$$d(Tz, z) = \frac{1}{2}(d(Tz, z) + d(z, Tz)) - \varphi(d(Tz, z), d(z, Tz)). \tag{3.16}$$

Hence, $\varphi(d(Tz, z), d(z, Tz)) = 0$ and so $d(Tz, z) = 0$. Therefore $Tz = z$ and from $Tz = Sz = Rz = Pz$ we conclude that $Tz = Sz = Rz = Pz = z$.
Uniqueness of the common fixed point follows immediately from (3.1).
Remark 3.2. If we take P as identity map on X in Theorem 3.1, T = S = R and E = X, then from above theorem we obtain [Theorem 4] Vahid [6] result . □

Now we provide an example to validate our result.

4. Illustrative example

Let $X = R$ be endowed with the Euclidean metric and let $E = \{0, \frac{1}{3}, \frac{1}{2}, 1\}$.

Let $T, S : E \to E$ be defined by $T0 = T\frac{1}{3} = T\frac{1}{2} = 0$, $T1 = \frac{1}{2}$ and $Sx = 0$, for all $x \in E$.

Let $R, P : E \to X$ and $\varphi : [0, \infty)^2 \to [0, \infty)$ be defined by $R0 = 0$, $R\frac{1}{3} = \frac{1}{3}$, $R\frac{1}{2} = \frac{1}{2}$, $R1 = 1$ and $P(0) = P(1) = 0$, $P\frac{1}{3}$, $P\frac{1}{2} = 0$ and $\varphi(t, s) = \frac{t+s}{12}$.
We have from Theorem 3.1,
$d(Tx, Sy) \leq \frac{1}{2}(d(Rx, Sy) + d(Py, Tx)) - \varphi((d(Rx, Sy), d(Py, Tx)))$
$\varphi(x, y) = 0$, if $x = y = 0$ and only if
for $x = 0$,
$0 \leq \frac{1}{2}[0 + 0 - \varphi(0, 0)] \Rightarrow \frac{1}{12}$.
for $x = 1$,
$\frac{1}{2} \leq \frac{1}{2}[1 + \frac{1}{2} - \varphi(1, \frac{1}{2})] \Rightarrow \frac{33}{48}$.
for $x = \frac{1}{3}$,
$0 \leq \frac{1}{2}[\frac{1}{3} + \frac{1}{3} - \varphi(\frac{1}{3}, \frac{1}{3})] \Rightarrow \frac{11}{36}\}$.
for $x = \frac{1}{2}$,
$0 \leq \frac{1}{2}[\frac{1}{2} + \frac{1}{2} - \varphi(\frac{1}{2}, \frac{1}{2})] \Rightarrow \frac{5}{24}\}$.
By calculation we see that all conditions of Theorem 3.1 hold. Hence T, S, R and P have a unique common fixed point $(x = 0)$ by Theorem 3.1.

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