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A STUDY ON GROWTH PROPERTIES OF n-ITERATED ENTIRE FUNCTIONS

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Abstract. We consider a new type of iteration of n entire functions and study some growth properties.

1. Introduction, Definitions and Notations

If f and g be two transcendental entire functions defined in the open complex plane $\mathbb{C}$, then Clunie [5] proved that $\lim_{r\to\infty} \frac{T(r,fog)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{T(r,fog)}{T(r,g)} = \infty$. In [15], Singh proved some comparative growth properties of $logT(r, fog)$ and $T(r, f)$ and raised the problem of investigating the comparative growth properties of $logT(r, fog)$ and $T(r, g)$. After this several authors {see [4], [10], [11] etc.} made close investigation on comparative growth of $logT(r, fog)$ and $T(r, g)$ by imposing certain restrictions on orders of f and g.

In [8] Lahiri and Banerjee introduced the concept of relative iterations of $f(z)$ with respect to $g(z)$ as follows:

$f_1(z) = f(z)$

$f_2(z) = f(g(z)) = f(g_1(z))$

$f_3(z) = f(g(f(z))) = f(g_2(z)) = f(g(f_1(z)))$

... ...

$f_n(z) = f(g(f...(f(z)$ *or* $g(z))$...according as n is odd or even)))

and so are $g_n(z)$.

Using this idea of relative iterations growth properties of iterated entire functions have been studied closely by many authors {see [1],[2],[3], [6] } to achieve great results. But in this paper we consider a more general situation by taking n entire functions and form a new type of iteration [defined below] to study the comparative growth of iterations of n entire functions that includes some previous results.

For n entire functions $f_1(z), f_2(z), ..., f_n(z)$, let

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$F_1(z) = f_1(z)$
$F_2(z) = f_2(f_1(z))$
... ...
$F_n(z) = f_n(f_{n-1}(...(f_2(f_1(z))))) = f_n(F_{n-1}(z)),\ n \geq 2.$
Following Sato [14], we write $log^{[0]}x = x$, $exp^{[0]}x = x$ and for positive integer m, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.
First we need the following definitions.

Definition 1.1. The order ρ_f and the lower order λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{logT(r,f)}{logr}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{logT(r,f)}{logr}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{log^{[2]}M(r,f)}{logr}$$

$$and \qquad \lambda_f = \liminf_{r\to\infty} \frac{log^{[2]}M(r,f)}{logr}.$$

Definition 1.2. The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function f are defined as

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{log^{[2]}T(r,f)}{logr}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{log^{[2]}T(r,f)}{logr}.$$

If f is entire then

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{log^{[3]}M(r,f)}{logr}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{log^{[3]}M(r,f)}{logr}.$$

Definition 1.3. A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function f if
(i) $\lambda_f(r)$ is nonnegative and continuous for $r \geq r_0$, say;
(ii) $\lambda_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda_f'(r-0)$ and $\lambda_f'(r+0)$ exist;
(iii) $\lim_{r\to\infty} \lambda_f(r) = \lambda_f < \infty$;
(iv) $\lim_{r\to\infty} r\lambda_f'(r)logr = 0$;

and (v) $\liminf_{r\to\infty}\frac{T(r,f)}{r^{\lambda_f(r)}}=1.$

Throughout we assume $f_1(z), f_2(z), ...f_n(z)$ be n non-constant entire functions having orders ρ_{f_k} and lower orders $\lambda_{f_k}(k=1,2,.....n)$ respectively. Also we follow Hayman [7] for standard definitions, terminologies and conventions.

2. Lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1. [1] *For two non-constant entire functions $f(z)$ and $g(z)$*

$$logM(r,f(g))\leq logM(M(r,g),f),\ \ r>0.$$

Lemma 2.2. [7] *Let $f(z)$ be an entire function. For $0\leq r<R<\infty$, we have*

$$T(r,f)\leq log^{+}M(r,f)\leq\frac{R+r}{R-r}T(R,f).$$

In particular, for large r, $T(r,f)\leq logM(r,f)\leq 3T(2r,f)$.

Lemma 2.3. [13] *Let $f(z)$ and $g(z)$ be two entire functions. Then we have*

$$T(r,f(g))\geq\frac{1}{3}logM(\frac{1}{8}M(\frac{r}{4},g)+O(1),f).$$

Lemma 2.4. [9] *Let f be an entire function. Then for $k>2$*

$$\liminf_{r\to\infty}\frac{log^{[k-1]}M(r,f)}{log^{[k-2]}T(r,f)}=1.$$

Lemma 2.5. [11] *Let f be a meromorphic function. Then for $\delta(>0)$ the function $r^{\lambda_f+\delta-\lambda_f(r)}$ is an increasing function of r.*

Lemma 2.6. [12] *Let f be an entire function of finite lower order. If there exist entire functions $a_i(i=1,2,3,...n;\ n\leq\infty)$ satisfying $T(r,a_i)=o\{T(r,f)\}$ and*

$$\sum_{i=1}^{n}\delta(a_i,f)=1$$

then

$$\lim_{r\to\infty}\frac{T(r,f)}{logM(r,f)}=\frac{1}{\pi}.$$

Lemma 2.7. *Let $f_1(z), f_2(z), ..., f_n(z)$ be such that $0 < \lambda_{f_i} \leq \rho_{f_i} < \infty$ where $i = 2, 3, 4, ..., n$. Then for arbitrary $\epsilon > 0$ and for large r*

$$log^{[n-k]}T(r, F_n) \leq (\rho_{f_{k+1}} + \epsilon)logM(r, F_k) + O(1)$$

and

$$log^{[n-k]}T(r, F_n) \geq (\lambda_{f_{k+1}} - \epsilon)logM(\frac{r}{4^{n-k}}, F_k) + O(1)$$

where $1 \leq k \leq n-1$.

Proof. We get from Lemma 2.1 and Lemma 2.2 for all large values of r and $\epsilon > 0$

$$\begin{aligned} T(r, F_n) &\leq logM(r, F_n) \\ &\leq logM(M(r, F_{n-1}), f_n) \\ &\leq M(r, F_{n-1})^{\rho_{f_n}+\epsilon} \\ or, logT(r, F_n) &\leq (\rho_{f_n} + \epsilon)logM(M(r, F_{n-2}), f_{n-1}) \\ &\leq (\rho_{f_n} + \epsilon)[M(r, F_{n-2})]^{\rho_{f_{n-1}}+\epsilon} \\ i.e.,\ log^{[2]}T(r, F_n) &\leq (\rho_{f_{n-1}} + \epsilon)logM(r, F_{n-2}) + O(1). \end{aligned}$$

Repeating the process at some stage, we have

$$log^{[n-k]}T(r, F_n) \leq (\rho_{f_{k+1}} + \epsilon)logM(r, F_k) + O(1),$$

where $1 \leq k \leq n-1$.
Again for choosing ϵ $(0 < \epsilon < min\{\lambda_{f_i}; i = 2, 3, 4, ..., n\})$ we have from Lemma 2.2 and Lemma 2.3 and for all large values of r

$$\begin{aligned} T(r, F_n) &\geq \frac{1}{3}logM(\frac{1}{8}M(\frac{r}{4}, F_{n-1}) + O(1), f_n) \\ &\geq \frac{1}{3}[\frac{1}{9}M(\frac{r}{4}, F_{n-1})]^{\lambda_{f_n}-\epsilon} \\ i.e.,\ logT(r, F_n) &\geq (\lambda_{f_n} - \epsilon)logM(\frac{r}{4}, F_{n-1}) + O(1) \\ &\geq (\lambda_{f_n} - \epsilon)T(\frac{r}{4}, F_{n-1}) + O(1) \\ &\geq (\lambda_{f_n} - \epsilon)\frac{1}{3}[\frac{1}{9}M(\frac{r}{4^2}, F_{n-2})]^{\lambda_{f_{n-1}}-\epsilon} + O(1) \\ or,\ log^{[2]}T(r, F_n) &\geq (\lambda_{f_{n-1}} - \epsilon)logM(\frac{r}{4^2}, F_{n-2}) + O(1). \end{aligned}$$

Repeating the process at some stage, we have

$$log^{[n-k]}T(r, F_n) \geq (\lambda_{f_{k+1}} - \epsilon)logM(\frac{r}{4^{n-k}}, F_k) + O(1)$$

where $1 \leq k \leq n-1$.

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let $f_1, f_2, ..., f_n$ be entire functions having positive lower orders. Then*

(i)

$$\liminf_{r\to\infty} \frac{log^{[n-k]}T(r,F_n)}{T(r,F_k)} \le 3\rho_{f_{k+1}} 2^{\lambda_{F_k}},$$

(ii)

$$\limsup_{r\to\infty} \frac{log^{[n-k]}T(r,F_n)}{T(r,F_k)} \ge \frac{\lambda_{f_{k+1}}}{(4^{n-k})^{\lambda_{F_k}}}$$

where $1 \le k \le n-1$.

Proof. We may clearly assume $0 < \lambda_{f_i} \le \rho_{f_i} < \infty$ where $i = 1, 2, 3, ..., n-1$.
Now from Lemma 2.7 and for arbitrary $\epsilon > 0$

$$log^{[n-k]}T(r,F_n) \le (\rho_{f_{k+1}} + \epsilon) logM(r,F_k) + O(1), \quad 1 \le k \le n-1. \tag{3.1}$$

We choose $\epsilon > 0$ such that $\epsilon < min\{1, \lambda_{f_i};\ i = 2, 3, 4, ..., n\}$.
Since

$$\liminf_{r\to\infty} \frac{T(r,F_k)}{r^{\lambda_{F_k}(r)}} = 1,$$

there is a sequence of values of r tending to infinity for which

$$T(r,F_k) < (1+\epsilon) r^{\lambda_{F_k}(r)} \tag{3.2}$$

and for all large values of r

$$T(r,F_k) > (1-\epsilon) r^{\lambda_{F_k}(r)}. \tag{3.3}$$

Thus for a sequence of values of r tending to infinity, we get for any $\delta > 0$
$\frac{logM(r,F_k)}{T(r,F_k)} \le \frac{3T(2r,F_k)}{T(r,F_k)} \le \frac{3(1+\epsilon)(2r)^{\lambda_{F_k}+\delta}}{(1-\epsilon)(2r)^{\lambda_{F_k}+\delta-\lambda_{F_k}(2r)}} \cdot \frac{1}{r^{\lambda_{F_k}(r)}} \le \frac{3(1+\epsilon)}{1-\epsilon} 2^{\lambda_{F_k}+\delta}$, because $r^{\lambda_{F_k}+\delta-\lambda_{F_k}(r)}$ is an increasing function of r by Lemma 2.5.
Since $\epsilon, \delta > 0$ be arbitrary, we have

$$\liminf_{r\to\infty} \frac{logM(r,F_k)}{T(r,F_k)} \le 3.2^{\lambda_{F_k}}. \tag{3.4}$$

Therefore from (3.1) and (3.4) we get

$$\liminf_{r\to\infty} \frac{log^{[n-k]}T(r,F_n)}{T(r,F_k)} \le 3\rho_{f_{k+1}} 2^{\lambda_{F_k}}, \quad where \ \ 1 \le k \le n-1.$$

This proves (i).

Again we have from Lemma 2.2, Lemma 2.7 and (3.3)

$$\begin{aligned} log^{[n-k]}T(r,F_n) &\geq (\lambda_{f_{k+1}}-\epsilon)logM(\frac{r}{4^{n-k}},F_k)+O(1) \\ &\geq (\lambda_{f_{k+1}}-\epsilon)T(\frac{r}{4^{n-k}},F_k)+O(1) \\ &\geq (\lambda_{f_{k+1}}-\epsilon)(1-\epsilon)(1+o(1))\frac{(\frac{r}{4^{n-k}})^{\lambda_{F_k}+\delta}}{(\frac{r}{4^{n-k}})^{\lambda_{F_k}+\delta-\lambda_{F_k}(\frac{r}{4^{n-k}})}}. \end{aligned}$$

Since $r^{\lambda_{F_k}+\delta-\lambda_{F_k}(r)}$ is an increasing function of r, we have

$$log^{[n-k]}T(r,F_n) \geq (\lambda_{f_{k+1}}-\epsilon)(1-\epsilon)(1+o(1))\frac{r^{\lambda_{F_k}(r)}}{(4^{n-k})^{\lambda_{F_k}+\delta}}$$

for all large values of r.
So by (3.2) for a sequence of values of r tending to infinity

$$log^{[n-k]}T(r,F_n) \geq (\lambda_{f_{k+1}}-\epsilon)\frac{1-\epsilon}{1+\epsilon}(1+o(1))\frac{T(r,F_k)}{(4^{n-k})^{\lambda_{F_k}+\delta}}.$$

Since $\epsilon, \delta > 0$ be arbitrary, it follows from the above that

$$\limsup_{r\to\infty}\frac{log^{[n-k]}T(r,F_n)}{T(r,F_k)} \geq \frac{\lambda_{f_{k+1}}}{(4^{n-k})^{\lambda_{F_k}}}.$$

This proves (ii).
Example 3.1. Let $f_1(z)=z,\ f_2(z)=z,\ f_3(z)=expz$. Then $\lambda_{f_1}=\lambda_{f_2}=0$ and $\lambda_{f_3}=1$. Now for $n=3$ *and* $k=2$, $F_n(z)=expz$ and $F_k(z)=z$, so that $T(r,F_n)=\frac{r}{\pi},\ T(r,F_k)=logr$ *and* $\lambda_{F_2}=0$.
Therefore,

$$\frac{log^{[n-k]}T(r,F_n)}{T(r,F_k)}=\frac{log\frac{r}{\pi}}{logr}.$$

So,

$$\limsup_{r\to\infty}\frac{log^{[n-k]}T(r,F_n)}{T(r,F_k)}=1.$$

Also $\frac{\lambda_{f_{k+1}}}{(4^{n-k})^{\lambda_{F_k}}}=\frac{1}{4^0}=1$.
So from this example we can say that if the function are not of positive lower orders then the theorem is also true.

Theorem 3.2. *Let* $f_1, f_2, ..., f_n$ *be such that* $\lambda_{f_k}(k=1,2,3,...,n-1)$ *are non zero finite. Also there exist entire functions* $a_j(j=1,2,3,...,n;\ n\leq\infty)$ *satisfying* $T(r,a_j)=o\{T(r,F_k)\}$ *as* $r\to\infty$ *and* $\sum_{j=1}^{n}\delta(a_j,F_k)=1$.
Then

$$\frac{\pi\lambda_{f_{k+1}}}{(4^{n-k})^{\lambda_{F_k}}} \leq \limsup_{r\to\infty}\frac{log^{[n-k]}T(r,F_n)}{T(r,F_k)} \leq \pi\rho_{f_{k+1}}$$

where $1 \leq k \leq n-1$.

Proof. If $\lambda_{f_{k+1}} = 0$ then the first inequality is obvious. Now we suppose that $\lambda_{f_{k+1}} > 0$ $(k = 1, 2, 3, ..., n-2)$.
Choose $\epsilon > 0$ such that $\epsilon < min\{1, \lambda_{f_i};\ for\ i = 2\ to\ n\ \}$. Then we have from Lemma 2.7 and for all large r

$$\frac{log^{[n-k]}T(r, F_n)}{T(r, F_k)} \geq (\lambda_{f_{k+1}} - \epsilon)\frac{logM(\frac{r}{4^{n-k}}, F_k)}{T(r, F_k)} + O(1)$$

$$\text{(3.5)} \qquad i.e., \ \frac{log^{[n-k]}T(r, F_n)}{T(r, F_k)} \geq (\lambda_{f_{k+1}} - \epsilon)\frac{logM(\frac{r}{4^{n-k}}, F_k)}{T(\frac{r}{4^{n-k}}, F_k)}.\frac{T(\frac{r}{4^{n-k}}, F_k)}{T(r, F_k)} + O(1).$$

Also from (3.2) and (3.3) we get for a sequence of values of $r \to \infty$ and $\delta > 0$

$$\begin{aligned}
\frac{T(\frac{r}{4^{n-k}}, F_k)}{T(r, F_k)} &> \frac{1-\epsilon}{1+\epsilon}.\frac{(\frac{r}{4^{n-k}})^{\lambda_{F_k}(\frac{r}{4^{n-k}})}}{r^{\lambda_{F_k}(r)}} \\
&\geq \frac{1-\epsilon}{1+\epsilon}.\frac{(\frac{r}{4^{n-k}})^{\lambda_{F_k}+\delta}}{(\frac{r}{4^{n-k}})^{\lambda_{F_k}+\delta-\lambda_{F_k}(\frac{r}{4^{n-k}})}}.\frac{1}{r^{\lambda_{F_k}(r)}} \\
&\geq \frac{1-\epsilon}{1+\epsilon}.\frac{1}{(4^{n-k})^{\lambda_{F_k}+\delta}},
\end{aligned}$$

because $r^{\lambda_{F_k}+\delta-\lambda_{F_k}(r)}$ is an increasing function of r.
Since $\epsilon, \delta > 0$ be arbitrary, so using Lemma 2.6, we have from (3.5)

$$\limsup_{r\to\infty} \frac{log^{[n-k]}T(r, F_n)}{T(r, F_k)} \geq \frac{\pi\lambda_{f_{k+1}}}{(4^{n-k})^{\lambda_{F_k}}}.$$

If $\rho_{f_{k+1}} = \infty$, the second inequality is obvious. So we may suppose $\rho_{f_{k+1}} < \infty$.
Now from Lemma 2.7 we get

$$\begin{aligned}
log^{[n-k]}T(r, F_n) &\leq (\rho_{f_{k+1}} + \epsilon)logM(r, F_k) + O(1) \\
or, \ \frac{log^{[n-k]}T(r, F_n)}{T(r, F_k)} &\leq (\rho_{f_{k+1}} + \epsilon)\frac{logM(r, F_k)}{T(r, F_k)} + O(1) \\
&\leq (\rho_{f_{k+1}} + \epsilon)\frac{1}{\frac{T(r,F_k)}{logM(r,F_k)}} + O(1).
\end{aligned}$$

Now from Lemma 2.6 we have

$$\limsup_{r\to\infty} \frac{log^{[n-k]}T(r, F_n)}{T(r, F_k)} \leq \pi(\rho_{f_{k+1}} + \epsilon).$$

Since $\epsilon > 0$ is arbitrary, so we write

$$\limsup_{r\to\infty} \frac{log^{[n-k]}T(r, F_n)}{T(r, F_k)} \leq \pi\rho_{f_{k+1}}.$$

This proves the theorem.

Theorem 3.3. *Let $f_1, f_2, ..., f_n$ be such that $0 < \lambda_{f_i} \leq \rho_{f_i} < \infty$, where $i = 2, 3, ..., n$. Then for $m = 0, 1, 2, ...$*

$$\frac{\overline{\lambda}_{F_k}}{\rho_{F_k}} \leq \liminf_{r\to\infty} \frac{log^{[n-k+2]}T(r, F_n)}{logT(r, F_k^{(m)})} \leq \limsup_{r\to\infty} \frac{log^{[n-k+2]}T(r, F_n)}{logT(r, F_k^{(m)})} \leq \frac{\overline{\rho}_{F_k}}{\lambda_{F_k}}$$

where $F_k^{(m)}$ denotes the m^{th} derivative of F_k and $1 \leq k \leq n-1$.

Proof. For given ϵ $(0 < \epsilon < min\{\lambda_{f_i},\ i = 2\ to\ m\})$ we get from Lemma 2.2 and Lemma 2.7 for all large values of r

$$log^{[n-k]}T(r, F_n) \geq (\lambda_{f_{k+1}} - \epsilon)logM(\frac{r}{4^{n-k}}, F_k) + O(1).$$

$$So,\ log^{[n-k+2]}T(r, F_n) \geq log^{[2]}T(\frac{r}{4^{n-k}}, F_k) + O(1).$$

So for all large values of r

$$\frac{log^{[n-k+2]}T(r, F_n)}{logT(r, F_k^{(m)})} \geq \frac{log^{[2]}T(\frac{r}{4^{n-k}}, F_k) + O(1)}{logT(r, F_k^{(m)})}$$

$$(3.6)\qquad i.e.,\ \frac{log^{[n-k+2]}T(r, F_n)}{logT(r, F_k^{(m)})} \geq \frac{log^{[2]}T(\frac{r}{4^{n-k}}, F_k)}{log\frac{r}{4^{n-k}}}.\frac{log\frac{r}{4^{n-k}}}{logT(r, F_k^{(m)})} + o(1).$$

For all large values of r and arbitrary $\epsilon > 0$ we have

$$(3.7)\qquad logT(r, F_k^{(m)}) < (\rho_{F_k} + \epsilon)\ logr.$$

Since $\epsilon > 0$ is arbitrary, so from (3.6) and (3.7) we have

$$(3.8)\qquad \liminf_{r\to\infty} \frac{log^{[n-k+2]}T(r, F_n)}{logT(r, F_k^{(m)})} \geq \frac{\overline{\lambda}_{F_k}}{\rho_{F_k}}.$$

Again from Lemma 2.7, for all large values of r

$$(3.9)\qquad \frac{log^{[n-k+2]}T(r, F_n)}{logT(r, F_k^{(m)})} \leq \frac{log^{[3]}M(r, F_k)}{logT(r, F_k^{(m)})} + o(1).$$

Also for all large values of r and arbitrary ϵ $(0 < \epsilon < \lambda_{F_k},\ k = 1\ to\ n)$ we have

$$(3.10)\qquad logT(r, F_k^{(m)}) > (\lambda_{F_k} - \epsilon)\ logr.$$

Since $\epsilon > 0$ is arbitrary so from (3.9) and (3.10) we have,

$$\limsup_{r\to\infty} \frac{log^{[n-k+2]}T(r,F_n)}{logT(r,F_k^{(m)})} \leq \frac{\overline{\rho}_{F_k}}{\lambda_{F_k}}. \tag{3.11}$$

Thus the results follows from (3.8) and (3.11).

Theorem 3.4. *Let $f_1, f_2, ..., f_n$ be such that $0 < \lambda_{f_i} \leq \rho_{f_i} < \infty$, where $i = 2, 3, ..., n$. Then*

$$\frac{\lambda_{F_k}}{\rho_{F_k}} \leq \liminf_{r\to\infty} \frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)} \leq 1 \leq \limsup_{r\to\infty} \frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)} \leq \frac{\rho_{F_k}}{\lambda_{F_k}}$$

where $1 \leq k \leq n-1$.

Proof. For given ϵ $(0 < \epsilon < min\{\lambda_{f_i},\ i = 2\ to\ n-1\})$, we get from Lemma 2.7 for all large r

$$\frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)} \leq \frac{log^{[2]}M(r,F_k)}{logT(r,F_k)} + o(1). \tag{3.12}$$

i.e., $\liminf_{r\to\infty} \frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)} \leq \liminf_{r\to\infty} \frac{log^{[2]}M(r,F_k)}{logT(r,F_k)}$

$$or,\ \liminf_{r\to\infty} \frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)} \leq 1(\ by\ Lemma\ 2.4). \tag{3.13}$$

Also,

$$\begin{aligned}\frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)} &\geq \frac{log^{[2]}M(\frac{r}{4^{n-k}},F_k)}{logT(r,F_k)} + o(1)\\ &= \frac{log^{[2]}M(\frac{r}{4^{n-k}},F_k)}{log\frac{r}{4^{n-k}}}.\frac{log\frac{r}{4^{n-k}}}{logT(r,F_k)} + o(1).\end{aligned}$$

Also for all large values of r and for arbitrary $\epsilon_1 > 0$, we have

$$logT(r,F_k) < (\rho_{F_k} + \epsilon_1)\ logr.$$

Since $\epsilon_1 > 0$ is arbitrary, so we have from above

$$\liminf_{r\to\infty} \frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)} \geq \frac{\lambda_{F_k}}{\rho_{F_k}}. \tag{3.14}$$

Also from (3.12), we get for all large values of r

$$\begin{aligned}\frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)} &\leq \frac{log^{[2]}M(r,F_k)}{logr}.\frac{logr}{logT(r,F_k)} + o(1)\\ &\leq \frac{log^{[2]}M(r,F_k)}{logr}.\frac{1}{\frac{logT(r,F_k)}{logr}} + o(1).\end{aligned}$$

Therefore,

$$\limsup_{r\to\infty}\frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)}\leq\frac{\rho_{F_k}}{\lambda_{F_k}}. \tag{3.15}$$

Again from Lemma 2.7

$$\log^{[n-k+1]}T(r,F_n)\geq log^{[2]}M(\frac{r}{4^{n-k}},F_k)+O(1). \tag{3.16}$$

From (3.3) we obtain for all large values of $r,\delta>0$ and $0<\epsilon<1$ and using Lemma 2.2 we get

$$\begin{aligned}logM(\frac{r}{4^{n-k}},F_k)&>(1-\epsilon)(\frac{r}{4^{n-k}})^{\lambda_{F_k}(\frac{r}{4^{n-k}})}\\&=\frac{(1-\epsilon)(\frac{r}{4^{n-k}})^{\lambda_{F_k}+\delta}}{(\frac{r}{4^{n-k}})^{\lambda_{F_k}+\delta-\lambda_{F_k}(\frac{r}{4^{n-k}})}}\\&\geq\frac{(1-\epsilon)}{(4^{n-k})^{\lambda_{F_k}+\delta}}.r^{\lambda_{F_k}(r)}\end{aligned}$$

because $r^{\lambda_{F_k}+\delta-\lambda_{F_k}(r)}$ is an increasing function of r.

So by (3.2) we get for a sequence of values of r tending to infinity,

$$logM(\frac{r}{4^{n-k}},F_k)\geq\frac{1-\epsilon}{1+\epsilon}\frac{1}{(4^{n-k})^{\lambda_{F_k}+\delta}}.T(r,F_k)$$

$$i.e,\;\; log^{[2]}M(\frac{r}{4^{n-k}},F_k)\geq logT(r,F_k)+O(1). \tag{3.17}$$

Now from (3.16) and (3.17) we obtain,

$$\frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)}\geq\frac{log^{[2]}M(\frac{r}{4^{n-k}},F_k)}{logT(r,F_k)}+o(1)$$

$$So,\;\; \limsup_{r\to\infty}\frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k)}\geq 1. \tag{3.18}$$

So the theorem follows from (3.13), (3.14), (3.15) and (3.18).

Theorem 3.5. *Let $f_1,f_2,...,f_n$ be such that $0<\lambda_{f_i}\leq\rho_{f_i}<\infty$, where $i=2,3,...,n$. Then for $m=0,1,2,...$*

$$\frac{\lambda_{F_k}}{\rho_{F_k}}\leq\liminf_{r\to\infty}\frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k^{(m)})}\leq\limsup_{r\to\infty}\frac{log^{[n-k+1]}T(r,F_n)}{logT(r,F_k^{(m)})}\leq\frac{\rho_{F_k}}{\lambda_{F_k}}$$

where $F_k^{(m)}$ denotes the m^{th} derivative of F_k and $1\leq k\leq n-1$.

Proof. For given ϵ $(0 < \epsilon < min\{\lambda_{f_i},\ i = 2\ to\ m\})$ we get from Lemma 2.2 and Lemma 2.7 and also for all large values of r

$$log^{[n-k]}T(r, F_n) \geq (\lambda_{f_{k+1}} - \epsilon)logM(\frac{r}{4^{n-k}}, F_k) + O(1).$$

$$So,\ log^{[n-k+1]}T(r, F_n) \geq logT(\frac{r}{4^{n-k}}, F_k) + O(1).$$

So for all large values of r

$$\frac{log^{[n-k+1]}T(r, F_n)}{logT(r, F_k^{(m)})} \geq \frac{logT(\frac{r}{4^{n-k}}, F_k) + O(1)}{logT(r, F_k^{(m)})}$$

$$i.e.,\ \frac{log^{[n-k+1]}T(r, F_n)}{logT(r, F_k^{(m)})} \geq \frac{logT(\frac{r}{4^{n-k}}, F_k)}{log\frac{r}{4^{n-k}}}.\frac{log\frac{r}{4^{n-k}}}{logT(r, F_k^{(m)})} + o(1). \tag{3.19}$$

For all large values of r and arbitrary $\epsilon > 0$ we have

$$logT(r, F_k^{(m)}) < (\rho_{F_k} + \epsilon)\ logr. \tag{3.20}$$

Since $\epsilon > 0$ is arbitrary, so from (3.19) and (3.20) we have

$$\liminf_{r\to\infty} \frac{log^{[n-k+1]}T(r, F_n)}{logT(r, F_k^{(m)})} \geq \frac{\lambda_{F_k}}{\rho_{F_k}}. \tag{3.21}$$

Again from Lemma 2.7, for all large values of r

$$\frac{log^{[n-k+1]}T(r, F_n)}{logT(r, F_k^{(m)})} \leq \frac{log^{[2]}M(r, F_k)}{logT(r, F_k^{(m)})} + o(1). \tag{3.22}$$

Also for all large values of r and arbitrary ϵ $(0 < \epsilon < \lambda_{F_k},\ k = 1\ to\ n)$ we have

$$logT(r, F_k^{(m)}) > (\lambda_{F_k} - \epsilon)\ logr. \tag{3.23}$$

Since $\epsilon > 0$ is arbitrary so from (3.22) and (3.23) we have,

$$\limsup_{r\to\infty} \frac{log^{[n-k+1]}T(r, F_n)}{logT(r, F_k^{(m)})} \leq \frac{\rho_{F_k}}{\lambda_{F_k}}. \tag{3.24}$$

Thus the results follows from (3.21) and (3.24).

Theorem 3.6. *Let $f_1, f_2, ..., f_n$ be such that $0 < \lambda_{f_i} \leq \rho_{f_i} < \infty$. Then for $m = 0, 1, 2, 3, ...$*

$$\lim_{r\to\infty} \frac{log^{[n-k+1]}T(r, F_n)}{T(r, F_k^{(m)})} = 0$$

for all $n(\geq 2)$ and $1 \leq k \leq n - 1$.

Proof. From Definition 1.1 and Lemma 2.7 for all sufficiently large values of r and $\epsilon > 0$

$$log^{[n-k]}T(r, F_n) \leq (\rho_{f_{k+1}} + \epsilon) logM(r, F_k) + O(1),$$
$$logM(r, F_k) < r^{\rho_{F_k}+\epsilon}$$

and

$$T(r, F_k^{(m)}) > r^{(\lambda_{F_k}-\epsilon)}.$$

So,

$$\frac{log^{[n-k+1]}T(r, F_n)}{T(r, F_k^{(m)})} \leq \frac{log^{[2]}M(r, F_k)}{r^{(\lambda_{F_k}-\epsilon)}} + o(1)$$
$$\leq \frac{(\rho_{F_k} + \epsilon)\ logr}{r^{(\lambda_{F_k}-\epsilon)}} + o(1).$$

Therefore,

$$\lim_{r\to\infty} \frac{log^{[n-k+1]}T(r, F_n)}{T(r, F_k^{(m)})} = 0.$$

Theorem 3.7. *Let $f_1, f_2, ..., f_n$ be such that $0 < \lambda_{f_i} \leq \rho_{f_i} < \infty$ and also $\rho_{F_i} < \infty$ $(i = 1$ to $n-1)$. Then for $m = 0, 1, 2, 3, ...$*

$$\lim_{r\to\infty} \frac{log^{[n-k]}T(r, F_n)}{T(e^r, F_k^{(m)})} = 0$$

for all $n (\geq 2)$ and $1 \leq k \leq n-1$.

Proof. From Definition 1.1 and Lemma 2.7 for all sufficiently large values of r and ϵ $(0 < \epsilon < \lambda_{F_i};\ i = 1$ to $n-1)$

$$log^{[n-k]}T(r, F_n) \leq (\rho_{f_{k+1}} + \epsilon) logM(r, F_k) + O(1),$$
$$logM(r, F_k) < r^{\rho_{F_k}+\epsilon}$$

and

$$T(e^r, F_k^{(m)}) > e^{r^{(\lambda_{F_k}-\epsilon)}}.$$

So,

$$\frac{log^{[n-k]}T(r, F_n)}{T(e^r, F_k^{(m)})} \leq \frac{(\rho_{f_{k+1}} + \epsilon) logM(r, F_k)}{T(e^r, F_k^{(m)})} + o(1)$$
$$\leq \frac{(\rho_{f_{k+1}} + \epsilon).r^{\rho_{F_k}+\epsilon}}{e^{r^{(\lambda_{F_k}-\epsilon)}}} + o(1).$$

Therefore,

$$\lim_{r\to\infty} \frac{log^{[n-k]}T(r, F_n)}{T(e^r, F_k^{(m)})} = 0.$$

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