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CONTROLLED OPERATOR FRAMES FOR $End_{\mathcal{A}}^{*}(\mathcal{H})$

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Abstract. Frame theory is dynamic with applications to a wide variety of areas. This paper is devoted to the introduction and the study of the concept of Controlled operator frames for Hilbert C^*-Modules. Also we discuss characterizations of controlled operator frames and we give some properties.

1. Introduction and Preliminaries

Frames were first introduced in 1952 by Duffin and Schaeffer [17] in the study of nonharmonic fourier series. Frames possess many nice properties which make them very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields. This last decade have seen tremendous activity in the development of frame theory and many generalizations of frames have come into existence [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18].

In this paper, we introduce the controlled operator frame for Hilbert C^*-modules and we give some properties.

Let I be a countable index set. In this section we briefly recall the definitions and basic properties of C^*-algebra, Hilbert C^*-modules, operator frame in Hilbert C^*-modules. For information about frames in Hilbert spaces we refer to [16]. Our reference for C^*-algebras is [3, 4]. For a C^*-algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is positive $(a \geq 0)$ if $a = a^*$ and $sp(a) \subset \mathbf{R}^+$. $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1.1. [2]. Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x\rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x\rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.
(ii) $\langle ax + y, z\rangle_{\mathcal{A}} = a\langle x, z\rangle_{\mathcal{A}} + \langle y, z\rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.
(iii) $\langle x, y\rangle_{\mathcal{A}} = \langle y, x\rangle_{\mathcal{A}}^*$ for all $x, y \in \mathcal{H}$.

Keywords: Operator frame, Controlled operator frames, C^*-algebra, Hilbert C^*-modules.

For $x \in \mathcal{H}$, we define $||x|| = ||\langle x,x\rangle_{\mathcal{A}}||^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. For every a in C^*-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x,x\rangle_{\mathcal{A}}^{\frac{1}{2}}$ for $x \in \mathcal{H}$.

Example 1.2. [19] If $\{\mathcal{H}_k\}_{k\in\mathbf{N}}$ is a countable set of Hilbert $\mathcal{A}$-modules, then one can define their direct sum $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$. On the $\mathcal{A}$-module $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$ of all sequences $x = (x_k)_{k\in\mathbb{N}} : x_k \in \mathcal{H}_k$, such that the series $\sum_{k\in\mathbb{N}}\langle x_k, x_k\rangle_{\mathcal{A}}$ is norm-convergent in the $\mathcal{C}^*$-algebra $\mathcal{A}$, we define the inner product by

$$\langle x,y\rangle := \sum_{k\in\mathbb{N}}\langle x_k, y_k\rangle_{\mathcal{A}}$$

for $x, y \in \oplus_{k\in\mathbb{N}}\mathcal{H}_k$.

Then $\oplus_{k\in\mathbb{N}}\mathcal{H}_k$ is a Hilbert $\mathcal{A}$-module.

The direct sum of a countable number of copies of a Hilbert $\mathcal{C}^*$-module $\mathcal{H}$ is denoted by $l^2(\mathcal{H})$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

We also reserve the notation $End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H})$ is abbreviated to $End^*_{\mathcal{A}}(\mathcal{H})$.

The following lemmas will be used to prove our mains results

Lemma 1.3. [1]. *If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is an $*$-homomorphism between $\mathcal{C}^*$-algebras, then φ is increasing, that is, if $a \leq b$, then $\varphi(a) \leq \varphi(b)$.*

Lemma 1.4. [1]. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$.*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*
$$\|(T^*T)^{-1}\|^{-1}I_{\mathcal{H}} \leq T^*T \leq \|T\|^2 I_{\mathcal{H}}.$$
(ii) *If T is surjective, then the adjointable map TT^* is invertible and*
$$\|(TT^*)^{-1}\|^{-1}I_{\mathcal{K}} \leq TT^* \leq \|T\|^2 I_{\mathcal{K}}.$$

Lemma 1.5. [20]. *Let $\mathcal{H}$ be Hilbert $\mathcal{A}$-module. If $T \in End^*_{\mathcal{A}}(\mathcal{H})$, then*

$$\langle Tx, Tx\rangle \leq \|T\|^2\langle x,x\rangle \qquad \forall x \in \mathcal{H}.$$

Lemma 1.6. [5]. *Let $\mathcal{H}$ and $\mathcal{K}$ two Hilbert $\mathcal{A}$-modules and $T \in End^*(\mathcal{H},\mathcal{K})$. Then the following statements are equivalent:*

(i) *T is surjective.*
(ii) *T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$ for all $x \in \mathcal{K}$.*
(iii) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^*x, T^*x\rangle \geq m'\langle x,x\rangle$ for all $x \in \mathcal{K}$.*

Definition 1.7. [8] A family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be an operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, if there exist two positives constants $A, B > 0$ such that

$$A\langle x,x\rangle \leq \sum_{i\in I}\langle T_i x, T_i x\rangle \leq B\langle x,x\rangle, \forall x\in\mathcal{H}. \tag{1.1}$$

The numbers A and B are called lower and upper bound of the operator frame, respectively.
If $A = B = \lambda$, the operator frame is λ-tight.
If $A = B = 1$, it is called a normalized tight operator frame or a Parseval operator frame.
If only upper inequality of (1.1) hold, then $\{T_i\}_{i\in I}$ is called an operator Bessel sequence for $End_{\mathcal{A}}^{*}(\mathcal{H})$.
If the sum in the middle of (1.1) is convergent in norm, the operator frame is called standard.

Throughout the paper, series like (1.1) are assumed to be convergent in the norm sense.

2. Controlled operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$

Let $GL^{+}(\mathcal{H})$ be the set for all positive bounded linear invertible operators on $\mathcal{H}$ with bounded inverse.

Definition 2.1. Let $C, C^{'} \in GL^{+}(H)$, a family of adjointable operators $\{T_i\}_{i\in I}$ on a Hilbert $\mathcal{A}$-module $\mathcal{H}$ over a unital C^*-algebra is said to be an $(C, C^{'})$-controlled operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, if there exists two positives constants $A, B > 0$ such that

$$A\langle x,x\rangle \leq \sum_{i\in I}\langle T_i C x, T_i C^{'} x\rangle \leq B\langle x,x\rangle, \qquad \forall x\in\mathcal{H}. \tag{2.1}$$

The numbers A and B are called lower and upper bounds of the $(C, C^{'})$-controlled operator frame , respectively.
If $A = B$, the $(C, C^{'})$-controlled operator frame is tight.
If $A = B = 1$, it is called a Parseval $(C, C^{'})$-controlled operator frame .
If only upper inequality of (2.1) hold, then $\{T_i\}_{i\in i}$ is called an $(C, C^{'})$-controlled operator Bessel sequence for $End_{\mathcal{A}}^{*}(\mathcal{H})$.

Example 2.2. Let $T \in GL^{+}(\mathcal{H})$ and $(x_i)_{i\in I}$ be a T-controlled frame from $\mathcal{H}$
Let $(\Gamma_i)_{i\in I} \in End_{\mathcal{A}}^{*}(\mathcal{H})$ such that : $\Gamma_i(x) = \langle x, x_i\rangle e_i \qquad \forall i\in I, \forall x\in\mathcal{H}$
, where $\langle e_i, e_j\rangle = \delta_{ij}1_{\mathcal{A}}$ from definition of $(x_i)_{i\in I}$, there exist $A, B > 0$ such that :

$$A\langle x,x\rangle \leq \sum_{i\in I}\langle Tx, x_i\rangle\langle x_i, x\rangle \leq B\langle x,x\rangle \qquad \forall x\in\mathcal{H}$$

$$A\langle x,x\rangle \leq \sum_{i\in I}\langle Tx, x_i\rangle\langle e_i, e_i\rangle\langle x_i, x\rangle \leq B\langle x,x\rangle \qquad \forall x\in\mathcal{H}$$

$$A\langle x,x\rangle \leq \sum_{i\in I}\langle Tx, x_i\rangle e_i, \langle x_i, x\rangle e_i\rangle e_i\rangle \leq B\langle x,x\rangle \qquad \forall x\in\mathcal{H}$$

So,

$$A\langle x,x\rangle \leq \sum_{i\in I}\langle \Gamma_i Tx, \Gamma_i x\rangle \leq B\langle x,x\rangle \qquad \forall x \in \mathcal{H}$$

Then $(\Gamma_i)_{i\in I}$ is a $(T, Id_{\mathcal{H}})$-controlled operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$

Let $\{T_i\}_{i\in I}$ be a $(C, C^{'})$-controlled operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$.

The synthesis operator $T_{CC'} : l^2(\mathcal{H}) \longrightarrow \mathcal{H}$ is given by

$$T_{CC'}(\{y_i\}_{i\in I}) = \sum_{i\in I}(CC^{'})^{\frac{1}{2}}T_i^* y_i \qquad \forall \{y_i\}_{i\in I} \in l^2(\mathcal{H})$$

is called the synthesis operator for the $(C, C^{'})$-controlled operator frame $\{T_i\}_{i\in I}$.
The adjoint operator $T^*_{CC'} : \mathcal{H} \to l^2(\{\mathcal{H}\})$ given by

$$T^*_{CC'}(x) = \{T_i(C^{'}C)^{\frac{1}{2}}x\}_{i\in I} \qquad \forall x \in \mathcal{H} \tag{2.2}$$

is called the analysis operator for the $(C, C^{'})$-controlled operator frame $\{T_i\}_{i\in I}$.
When C and $C^{'}$ commute with each other, and commute with the operator $T_i^*T_i$ for each $i \in I$, then the $(C, C^{'})$-controlled frames operator:

$S_{CC'} : U \longrightarrow U$ is defined as: $S_{CC'}x = T_{CC'}T^*_{CC'}x = \sum_{i\in I} C^{'}T_i^*T_iCx$
In the next we suppose that C and $C^{'}$ commute with each other, and commute with the operator $T_iT_i^*$ for each $i \in I$.

Proposition 2.3. *The $(C, C^{'})$-controlled frame operator $S_{CC'}$ is bounded, positive, seefladjoint and invertible.*

Proof. From definition of the $(C, C^{'})$-controlled frame operator we have:

$$\begin{aligned} A\langle x,x\rangle \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle &= \sum_{i\in I}\langle C^{'}T_i^*T_iCx, x\rangle \\ &= \langle\sum_{i\in I} C^{'}T_i^*T_iCx, x\rangle \\ &= \langle S_{CC'}x, x\rangle \end{aligned}$$

By definition $S_{CC'}$ is bounded . Then $S_{CC'}$ is a positive operator. On the other hand, we have:

$$\begin{aligned} \langle S_{CC'}x, y\rangle &= \langle\sum_{i\in I} C^{'}T_i^*T_iCx, x\rangle \\ &= \sum_{i\in I}\langle C^{'}T_i^*T_iCx, x\rangle \\ &= \sum_{i\in I}\langle x, CT_iT_i^*C^{'}y\rangle \\ &= \langle x, S_{C'C}y\rangle \end{aligned}$$

So $S_{CC'}^{*} = S_{C'C}$, but $C^{'}C = CC^{'}$, then $S_{CC'}^{*} = S_{CC'}$. From the hypothese C and $C^{'}$ commute with each other, and commute with the operator $T_iT_i^{*}$ for each $i \in I$, we have : $S_{CC'} = S_{C'C}$, so the controlled operator frame is self adjoint.
Also, we have :

$$A\langle x, x\rangle \leq \langle S_{CC'}x, x\rangle \leq B\langle x, x\rangle$$

$$\Longrightarrow A.Id_{\mathcal{H}} \leq S_{CC'} \leq B.Id_{\mathcal{H}}$$

Thus the controlled operator frame $S_{CC'}$ is invertible. □

Theorem 2.4. *Let $\{T_i\}_{i\in I} \in End_{\mathcal{A}}^{*}(\mathcal{H})$ and $C, C^{'} \in GL^{+}(\mathcal{H})$, suppose that $\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle$ converge in norm for any $x \in \mathcal{H}$. Then $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ if and only if there exist two constants positives A and B such that:*

$$A\|x\|^2 \leq \|\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle\| \leq B\|x\|^2 \tag{2.3}$$

Proof. We suppose $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. Then :

$$A\langle x, x\rangle \leq \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle \leq B\langle x, x\rangle, \qquad \forall x \in \mathcal{H}. \tag{2.4}$$

Since $0 \leq \langle x, x\rangle$ for all $x \in \mathcal{H}$, then we can take the norme on the left, middle and right terms of the above inequality (2.4), thus we have :

$$\|A\langle x, x\rangle\| \leq \|\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle\| \leq \|B\langle x, x\rangle\|$$

$$\Longrightarrow A\|x\|^2 \leq \|\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle\| \leq B\|x\|^2$$

Conversely, suppose that (2.3) holds, the $(C, C^{'})$-controlled operator frame $S_{CC'}$ is positive and self adjoint and invertible, then:

$$\begin{aligned}
\langle S_{CC'}^{\frac{1}{2}}x, S_{CC'}^{\frac{1}{2}}x\rangle &= \langle S_{CC'}x, x\rangle \\
&= \langle \sum_{i\in I} C^{'}T_i^{*}T_iCx, x\rangle \\
&= \sum_{i\in I}\langle C^{'}T_i^{*}T_iCx, x\rangle \\
&= \sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle
\end{aligned}$$

So

$$\sqrt{A}\|x\| \leq \|S_{CC'}^{\frac{1}{2}}x\| \leq \sqrt{B}\|x\| \tag{2.5}$$

According the lemma 1.6 , there exist constant $m, M > 0$ such that :

$$\begin{aligned} m\langle x,x\rangle &\leq \langle S_{CC'}^{\frac{1}{2}}x, S_{CC'}^{\frac{1}{2}}x\rangle \\ &= \sum_{i\in I}\langle T_iCx, T_iC'x\rangle \\ &\leq M\langle x,x\rangle \end{aligned}$$

Therefore $\{T_i\}_{i\in I}$ is a (C,C')-controlled operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ □

Definition 2.5. Let $C \in GL^{+}(\mathcal{H})$, the sequence $\{T_i\}_{i\in I} \in End_{\mathcal{A}}^{*}(\mathcal{H})$ is said to be a (C,C)-controlled operator frame or C^2-controlled operator frame if there exist two positives $A, B > 0$ such that :

$$A\langle x,x\rangle \leq \sum_{i\in I}\langle T_iCx, T_iCx\rangle \leq B\langle x,x\rangle, \qquad \forall x \in \mathcal{H}. \tag{2.6}$$

or equivalently

$$A\|x\|^2 \leq \|\sum_{i\in I}\langle T_iCx, T_iCx\rangle\| \leq B\|x\|^2, \qquad \forall x \in \mathcal{H}. \tag{2.7}$$

Theorem 2.6. *Let $\{T_i\}_{i\in I} \in End_{\mathcal{A}}^{*}(\mathcal{H})$,suppose that $\sum_{i\in I}\langle T_ix, T_ix\rangle$ converge in norm for all $x \in \mathcal{H}$. Then $\{T_i\}_{i\in I}$ is an operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$ if and only if there exist two positives constants $A, B > 0$ such that*

$$A\|x\|^2 \leq \|\sum_{i\in I}\langle T_ix, T_ix\rangle\| \leq B\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

Proof. Result of the theorem 2.4 for $C = C' = Id_{\mathcal{H}}$ □

Let $C \in GL^{+}(\mathcal{H})$,and $\{T_i\}_{i\in I}$ a family of adjointable operator in H.

Proposition 2.7. *$\{T_i\}_{i\in I}$ is an operator frame for H if and only if $\{T_i\}_{i\in I}$ is a C^2-controlled operator frame for H .*

Proof. Suppose that $\{T_i\}_{i\in I}$ is a C^2-controlled operator frame with bounds A and B, then :

$$A\|x\|^2 \leq \|\sum_{i\in I}\langle T_iCx, T_iCx\rangle\| \leq B\|x\|^2, \qquad \forall x \in \mathcal{H}. \tag{2.8}$$

Now, for any $x \in \mathcal{H}$ we have :

$$\begin{aligned} A\|x\|^2 &= A\|CC^{-1}x\|^2 \\ &\leq A\|C\|^2\|C^{-1}x\|^2 \\ &\leq \|C\|^2\|\sum_{i\in I}\langle T_iCC^{-1}x, T_iCC^{-1}x\rangle\| = \|C\|^2\|\sum_{i\in I}\langle T_ix, T_ix\rangle\| \end{aligned}$$

Hence :

$$A\|C\|^{-2}\|x\|^2 \leq \|\sum_{i\in I}\langle T_ix, T_ix\rangle\| \tag{2.9}$$

Again for any $x \in \mathcal{H}$:

$$\|\sum_{i\in I}\langle T_ix, T_ix\rangle\| = \|\sum_{i\in I}\langle T_iCC^{-1}x, T_iCC^{-1}x\rangle\| \leq B\|C^{-1}x\|^2 \leq B\|C^{-1}\|^2\|x\|^2 \tag{2.10}$$

By (2.9), (2.10) and theorem 2.6 we have $\{T_i\}_{i\in I}$ is an operator frame with bounds $A\|C\|^{-2}$ and $B\|C^{-1}\|^2$.

Conversely, let $\{T_i\}_{i\in I}$ be an operator frame with bounds A and B, then :

$$A\langle x, x\rangle \leq \sum_{i\in I}\langle T_ix, T_ix\rangle \leq B\langle x, x\rangle \qquad \forall x \in \mathcal{H} \tag{2.11}$$

so for all $x \in \mathcal{H}$ we have : $Cx \in \mathcal{H}$ and :

$$\sum_{i\in I}\langle T_iCx, T_iCx\rangle \leq B\langle Cx, Cx\rangle \leq B\|C\|^2\langle x, x\rangle \tag{2.12}$$

Also for any $x \in \mathcal{H}$ we have :

$$A\langle x, x\rangle = A\langle C^{-1}Cx, C^{-1}Cx\rangle \tag{2.13}$$

$$\leq A\|C^{-1}\|^2\langle Cx, Cx\rangle \tag{2.14}$$

$$\leq \|C^{-1}\|^2\sum_{i\in I}\langle T_iCx, T_iCx\rangle \tag{2.15}$$

from (2.12) and (2.13) we have :

$$A\|C^{-1}\|^{-2}\langle x, x\rangle \leq \sum_{i\in I}\langle T_iCx, T_iCx\rangle \leq B\|C\|^2\langle x, x\rangle. \tag{2.16}$$

then $\{T_i\}_{i\in I}$ is a C^2-controlled operator frame with bounds $A\|C^{-1}\|^{-2}$ and$B\|C\|^2$. □

Let $C, C^{'} \in GL^+(\mathcal{H})$, then $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled operator frame

Proposition 2.8. *Let $\{T_i\}_{i\in I}$ be an operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$. with frame operator S_T . Then $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled operator frame .*

Proof. Let $\{T_i\}_{i\in I}$ be an operator frame with bounds A, B, then we have :

$$A\|x\|^2 \leq \|\sum_{i\in I}\langle T_ix, T_ix\rangle\| \leq B\|x\|^2, \qquad \forall x \in \mathcal{H}.$$

$$\Longrightarrow A\|x\|^2 \leq \|\langle S_Tx, x\rangle\| \leq B\|x\|^2, \qquad \forall x \in \mathcal{H}. \tag{2.17}$$

but :

$$\|\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle\| = \|\langle S_{CC^{'}}x, x\rangle\|$$

and ,

$$\|\sum_{i\in I}\langle T_iCx, T_iCx\rangle\| = \|C\|\|C^{'}\|\|\sum_{i\in I}\langle T_ix, T_ix\rangle\| \tag{2.18}$$

$$= \|C\|\|C^{'}\|\|\langle S_Tx, x\rangle\| \tag{2.19}$$

then we have :

$$A\|C\|\|C^{'}\|\|x\|^2 \leq \|\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle\| \leq B\|C\|\|C^{'}\|\|x\|^2$$

From, we conclude that $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled operator frame with bounds $A\|C\|\|C^{'}\|$ and $B\|C\|\|C^{'}\|$. □

Theorem 2.9. *Let $C, C^{'} \in GL^+(\mathcal{H})$ and $\{T_i\}_{i\in I}$ be an operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$. Then the sequence $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled Bessel operator frame with bound B if and only if the operator $T_{CC'} : l^2(\mathcal{H}) \to \mathcal{H}$ given by*

$$T_{CC'}(\{y_i\}_{i\in I}) = \sum_{i\in I}(CC^{'})^{\frac{1}{2}}T_i^*y_i \qquad \forall\{y_i\}_{i\in I} \in l^2(\mathcal{H})$$

is well defined and bounded operator with $\|T_{CC'}\| \leq \sqrt{B}$

Proof. Let $\{T_i\}_{i\in I}$ be a $(C, C^{'})$-controlled Bessel operator frame with bounds A and B, then :

$$\|\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle\| \leq B\|x\|^2 \qquad \forall x \in \mathcal{H}$$

In the other hand, for any sequence $\{y_i\}_{i\in I} \in l^2(\mathcal{H})$ we have :

$$\begin{aligned}\|T_{CC'}(\{y_i\}_{i\in I})\|^2 &= \sup_{x\in\mathcal{H},\|x\|=1}\|\langle T_{CC'}(\{y_i\}_{i\in I}), x\rangle\|^2 \\ &= \sup_{x\in\mathcal{H},\|x\|=1}\|\langle\sum_{i\in I}(CC^{'})^{\frac{1}{2}}T_i^*y_i, x\rangle\|^2 \\ &\leq \sup_{x\in\mathcal{H},\|x\|=1}\|\sum_{i\in I}\langle y_i, y_i\rangle\| \sup_{x\in\mathcal{H},\|x\|=1}\|\sum_{i\in I}\langle T_i(CC^{'})^{\frac{1}{2}}x, T_i(CC^{'})^{\frac{1}{2}}x\rangle\| \\ &= \sup_{x\in\mathcal{H},\|x\|=1}\|\sum_{i\in I}\langle y_i, y_i\rangle\| \sup_{x\in\mathcal{H},\|x\|=1}\|\sum_{i\in I}\langle T_iCx, T_iC^{'}x\rangle\| \\ &\leq B\|\{y_i\}_{i\in I}\|^2\end{aligned}$$

then

$$\|T_{CC'}\| \leq \sqrt{B}$$

So the operator $T_{CC'}$ is well defined, bounded and $\|T_{CC'}\| \leq \sqrt{B}$
The converse is clear . □

Proposition 2.10. *Let $\{T_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ be two $(C, C^{'})$-controlled Bessel operator frame for $End^*_{\mathcal{A}}(\mathcal{H})$ with bounds B_1 and B_2 respectively.*
The operator $L_{CC'} : \mathcal{H} \longrightarrow \mathcal{H}$ given by :

$$L_{CC'}(x) = \sum_{i\in I}C^{'}\Gamma_i^*T_iCx \tag{2.20}$$

is well defined , bounded and adjointable with $\|L_{CC'}\| \leq \sqrt{B_1B_2}$.
and,

$$L_{CC'}^{*}(f) = \sum_{i\in I} CT_i^{*}\Gamma_i^{*}C^{'}f \tag{2.21}$$

Proof. For all $x \in \mathcal{H}$ we have :

$$\begin{aligned}
\|\sum_{i\in I} C^{'}\Gamma_i^{*}T_iCx\| &= \sup_{y\in\mathcal{H},\|y\|=1} \|\langle\sum_{i\in I} C^{'}\Gamma_i^{*}T_iCx, y\rangle\|^2 \\
&= \sup_{y\in\mathcal{H},\|y\|=1} \|\sum_{i\in K}\langle T_iCx, \Gamma_iC^{'}y\rangle\|^2 \\
&\leq \sup_{y\in\mathcal{H},\|y\|=1} \|\sum_{i\in I}\langle T_iCx, T_iCx\rangle\|\|\sum_{i\in K}\langle\Gamma_iC^{'}y, \Gamma_iC^{'}y\rangle\| \\
&\leq \|\sum_{i\in I}\langle T_iCx, T_iCx\rangle\|F \\
&\leq B_1B_2\|x\|^2
\end{aligned}$$

then , $\|L_{CC'}\| \leq \sqrt{B_1B_2}$ Morover, we see that :

$$\begin{aligned}
\langle L_{CC'}x, y\rangle &= \langle\sum_{i\in I} C^{'}\Gamma_i^{*}T_iCx, y\rangle = \sum_{i\in I}\langle C^{'}\Gamma_i^{*}T_iCx, y\rangle \\
&= \sum_{i\in I}\langle x, CT_i^{*}\Gamma_iC^{'}y\rangle = \langle x, \sum_{i\in I} CT_i^{*}\Gamma_iC^{'}y\rangle
\end{aligned}$$

Thus:

$$L_{CC'}^{*}x = \sum_{i\in I} CT_i^{*}\Gamma_iC^{'}x \tag{2.22}$$

□

Theorem 2.11. *Let* $\{T_i\}_{i\in I}$ *be a* $(C, C^{'})$*-controlled operator frame for* $End_{\mathcal{A}}^{*}(\mathcal{H})$ *and* $\{\Gamma_i\}_{i\in I}$ *be a* $(C, C^{'})$*-controlled Bessel operator frame for* $End_{\mathcal{A}}^{*}(\mathcal{H})$*. Assume that* C *and* $C^{'}$ *commute with each other and commute with* $\Gamma_i^{*}\Gamma_i$*. If the operator* $L_{CC'}$ *defined in* (2.20) *is surjective then* $\{\Gamma_i\}_{i\in I}$ *is also a* $(C, C^{'})$*-controlled operator frame for* $End_{\mathcal{A}}^{*}(\mathcal{H})$

Proof. $\{T_i\}_{i\in I}$ is a $(C, C^{'})$-controlled operator frame for $End_{\mathcal{A}}^{*}(\mathcal{H})$, then by theorem 2.9 the operator $T_{CC'} : l^2(\mathcal{H}) \to \mathcal{H}$ given by

$$T_{CC'}(\{y_i\}_{i\in I}) = \sum_{i\in I}(CC^{'})^{\frac{1}{2}}T_i^{*}y_i \qquad \forall\{y_i\}_{i\in I} \in l^2(\mathcal{H})$$

is well defined , bounded and adjointable
his operator adjoint is: $T_{CC'}^{*} : \mathcal{H} \longrightarrow l^2(\mathcal{H})$ given by

$$T_{CC'}^{*}(x) = (T_i(C^{'}C)^{\frac{1}{2}}x)_{i\in I} \qquad \forall x \in \mathcal{H}$$

Since $\{\Gamma_i\}_{i\in I}$ is also a $(C,C^{'})$-controlled Bessel operator frame, by theorem 2.9, the operator $P_{CC'}: l^2(\mathcal{H}) \to \mathcal{H}$ given by

$$P_{CC'}(\{y_i\}_{i\in I}) = \sum_{i\in I}(CC^{'})^{\frac{1}{2}}\Gamma_i^* y_i \qquad \forall \{y_i\}_{i\in I} \in l^2(\mathcal{H})$$

is well defined and bounded operator, his operator adjoint is: $P_{CC'}^* : \mathcal{H} \longrightarrow l^2(\mathcal{H})$ given by

$$P_{CC'}^*(x) = (\Gamma_i(C^{'}C)^{\frac{1}{2}}x)_{i\in I} \qquad \forall x \in \mathcal{H}$$

Hence, for any $x \in \mathcal{H}$, the operator defined in (2.21) can be written as

$$L_{CC'}(x) = \sum_{i\in I} C^{'}\Gamma_i^* T_i C x = P_{CC'}T_{CC'}^* x$$

Since $L_{CC'}$ is surjective, then for any $x \in \mathcal{H}$, there exist $y \in \mathcal{H}$ such that:
$x = L_{CC'}(y) = P_{CC'}T_{CC'}^*(y)$ and $T_{CC'}^*(y) \in l^2(\mathcal{H})$.
This implies that $P_{CC'}$ is surjective.
As a result of lemma 1.6, we have $P_{CC'}^*$ is bounded below, that is there exist $0 \leq m$ such that :

$$\begin{aligned}
& m\langle x,x\rangle \leq \langle P_{CC'}^* x, P_{CC'}^* x\rangle \\
\Longrightarrow \quad & m\langle x,x\rangle \leq \langle P_{CC'}P_{CC'}^* x, x\rangle \\
\Longrightarrow \quad & m\langle x,x\rangle \leq \langle \sum_{i\in I}(C^{'}C)^{\frac{1}{2}}\Gamma_i^*\Gamma_i(C^{'}C)^{\frac{1}{2}}x, x\rangle \\
\Longrightarrow \quad & m\langle x,x\rangle \leq \sum_{i\in I}\langle \Gamma_i C x, \Gamma_i C^{'} x\rangle
\end{aligned}$$

Hence $\{\Gamma_i\}_{i\in I}$ is also a a $(C,C^{'})$-controlled operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ □

Theorem 2.12. *Let $(\mathcal{H},\mathcal{A},\langle.,.\rangle_{\mathcal{A}})$ and $(\mathcal{H},\mathcal{B},\langle.,.\rangle_{\mathcal{B}})$ be two Hilbert $\mathcal{C}^*$-modules and let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a $*$-homomorphism and θ be a adjointable map on $\mathcal{H}$ such that $\langle \theta x, \theta y\rangle_{\mathcal{B}} = \varphi(\langle x,y\rangle_{\mathcal{A}})$ for all $x,y \in \mathcal{H}$. Also, suppose that $\{T_i\}_{i\in I}$ be a $(C,C^{'})$-controlled operator frame for $End_{\mathcal{A}}^*(\mathcal{H})$ with frame operator $S_{\mathcal{A}}$ and lower and upper operator frame bounds A, B respectively. If θ is invertible, then $\{\theta T_i\}_{i\in I}$ is a $(C,C^{'})$-controlled operator frame for $End_{\mathcal{B}}^*(\mathcal{H})$ with frame operator $S_{\mathcal{B}}$ and lower and upper operator frame bounds $\|(\theta^*\theta)^{-1}\|^{-1}A$ and $\|\theta\|^2 B$ respectively, and $\langle S_{\mathcal{B}}x, y\rangle_{\mathcal{B}} = \varphi(\langle S_{\mathcal{A}}x, y\rangle_{\mathcal{A}})$.*

Proof. By the definition of $(C,C^{'})$-controlled operator frame, we have :

$$A\langle x,x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle T_i C x, T_i C^{'} x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}, \qquad \forall x \in \mathcal{H}.$$

by lemma 1.3, we have:

$$\varphi(A\langle x,x\rangle_{\mathcal{A}}) \leq \varphi(\sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}) \leq \varphi(B\langle x,x\rangle_{\mathcal{A}})$$

$$\Longrightarrow \qquad A\varphi(\langle x,x\rangle_{\mathcal{A}}) \leq \sum_{i\in I}\varphi(\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}) \leq B\varphi(\langle x,x\rangle_{\mathcal{A}})$$

$$\Longrightarrow \qquad A\langle \theta x,\theta x\rangle_{\mathcal{B}} \leq \sum_{i\in I}\langle \theta T_iCx,\theta T_iC^{'}x\rangle_{\mathcal{A}} \leq B\langle \theta x,\theta x\rangle_{\mathcal{B}}$$

By lemma 1.4, we have :

$$\|(\theta^*\theta)^{-1}\|^{-1}\langle x,x\rangle_{\mathcal{B}} \leq \langle \theta x,\theta x\rangle_{\mathcal{B}} \leq \|\theta\|^2\langle x,x\rangle_{\mathcal{B}}$$

then

$$\|(\theta^*\theta)^{-1}\|^{-1}A\langle x,x\rangle_{\mathcal{B}} \leq \sum_{i\in I}\langle \theta T_iCx,\theta T_iC^{'}x\rangle_{\mathcal{A}} \leq \|\theta\|^2 B\langle x,x\rangle_{\mathcal{B}}$$

then $\{\theta T_i\}_{i\in I}$ is a $(C,C^{'})$-controlled operator frame for $End_{\mathcal{B}}^{*}(\mathcal{H})$ with frame operator $S_{\mathcal{B}}$ and lower and upper operator frame bounds $\|(\theta^*\theta)^{-1}\|^{-1}A$ and $\|\theta\|^2B$ respectively
On the other hand, we have :

$$\begin{aligned}
\varphi(\langle S_{\mathcal{A}}x,y\rangle_{\mathcal{A}}) &= \varphi(\langle\sum_{i\in I}C^{'}T_i^*T_iCx,y\rangle_{\mathcal{A}}) \\
&= \sum_{i\in I}\varphi(\langle C^{'}T_i^*T_iCx,y\rangle_{\mathcal{A}}) \\
&= \sum_{i\in I}\varphi(\langle T_iCx,T_iC^{'}y\rangle_{\mathcal{A}}) \\
&= \sum_{i\in I}\langle \theta T_iCx,\theta T_iC^{'}y\rangle_{\mathcal{A}}) \\
&= \langle\sum_{i\in I}C^{'}(\theta T_i)^*\theta T_iCx,y\rangle_{\mathcal{A}}) \\
&= \langle S_{\mathcal{B}}x,y\rangle_{\mathcal{B}}
\end{aligned}$$

□

3. Tensor product

Theorem 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert C^*-modules over unitary C^*-algebra $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $C,C^{'}\in GL^+(\mathcal{H})$ and $C_1,C_2\in GL^+(\mathcal{K})$, also, let $\{T_i\}_{i\in I}$ a $(C,C^{'})$-controlled operator frame for $End_{\mathcal{B}}^{*}(\mathcal{H})$ and $\{\Gamma_i\}_{i\in I}$ a $(C_1,C_2$-controlled operator frame for $End_{\mathcal{B}}^{*}(\mathcal{K})$ with frames bounds (A,B) and (C,D) respectively, then $\{T_i\otimes\Gamma_i\}$ is a $(C\otimes C_1),C^{'}\otimes C_2)$-controlled operator frames for $End_{\mathcal{A}\otimes\mathcal{B}}^{*}(\mathcal{H}\otimes\mathcal{K})$ with bounds AC and BD with operator frame $S=S_T\otimes S_\Gamma$*

Proof. By the definition of $(C,C^{'})$-controlled operator frame $\{T_i\}_{i\in I}$ and (C_1,C_2)-controlled operator frame $\{\Gamma_i\}_{i\in I}$ we have

$$A\langle x,x\rangle_{\mathcal{A}}\leq\sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}\leq B\langle x,x\rangle_{\mathcal{A}},\qquad\forall x\in\mathcal{H}$$

and

$$C\langle y,y\rangle_{\mathcal{B}}\leq\sum_{i\in I}\langle\Gamma_iC_1y,\Gamma_iC_2y\rangle_{\mathcal{B}}\leq D\langle y,y\rangle_{\mathcal{B}},\qquad\forall y\in\mathcal{K}$$

therefore

$$A\langle x,x\rangle_{\mathcal{A}}\otimes C\langle y,y\rangle_{\mathcal{B}}\leq\sum_{i\in I}\langle T_iCx,T_iC^{'}x\rangle_{\mathcal{A}}\otimes\sum_{i\in I}\langle\Gamma_iC_1y,\Gamma_iC_2y\rangle_{\mathcal{B}}\leq B\langle x,x\rangle_{\mathcal{A}}\otimes D\langle y,y\rangle_{\mathcal{B}}$$

then we have :

$$(AB)(\langle x\otimes y,x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}})\leq\sum_{i\in I}\langle T_iCx\otimes\Gamma_iC_1y,T_iC^{'}x\otimes\Gamma_iC_2y\rangle_{\mathcal{A}\otimes\mathcal{B}})\leq(CD)(\langle x\otimes y,x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}})$$

$$\begin{aligned}&(A\otimes B)(\langle x\otimes y,x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}})\\&\leq\sum_{i\in I}\langle(T_iC\otimes\Gamma_iC_1)(x\otimes y),(T_iC^{'}\otimes\Gamma_iC_2)(x\otimes y)\rangle_{\mathcal{A}\otimes\mathcal{B}})\\&\leq(CD)(\langle x\otimes y,x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}})\end{aligned}$$

then

$$\begin{aligned}&(AB)(\langle x\otimes y,x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}})\\&\leq\sum_{i\in I}\langle(T_i\otimes\Gamma_i)(C\otimes C_1)(x\otimes y),(T_i\otimes\Gamma_i)(C^{'}\otimes C_2)(x\otimes y)\rangle_{\mathcal{A}\otimes\mathcal{B}})\\&\leq(CD)(\langle x\otimes y,x\otimes y\rangle_{\mathcal{A}\otimes\mathcal{B}})\end{aligned}$$

the last inequality is satisfied for every finite elements in $\mathcal{H}\otimes\mathcal{K}$ and then it is satisfied for all $z\in\mathcal{H}\otimes\mathcal{K}$. It show that $\{T_i\otimes\Gamma_i\}$ is a $(C\otimes C_1),C^{'}\otimes C_2)$-controlled operator frames for $End^{*}_{\mathcal{A}\otimes\mathcal{B}}(\mathcal{H}\otimes\mathcal{K})$ with bounds $A\otimes C$ and $B\otimes D$ In other hand we have :

$$\begin{aligned}(S_T\otimes S_\Gamma)(x\otimes y)&=(S_Tx)\otimes(S_\Gamma y)\\&=(\sum_{i\in I}C^{'}T_i^{*}T_iCx)\otimes(\sum_{i\in I}C_2\Gamma_i^{*}\Gamma_iC_1y)\\&=\sum_{i\in I}(C^{'}T_i^{*}T_iCx\otimes C_2\Gamma_i^{*}\Gamma_iC_1y)\\&=\sum_{i\in I}(C^{'}T_i^{*}T_iC\otimes C_2\Gamma_i^{*}\Gamma_iC_1)(x\otimes y)\\&=\sum_{i\in I}(C^{'}\otimes C_2)(T_i\otimes\Gamma_i)^{*}(T_i\otimes\Gamma_i)(C\otimes C_1)(x\otimes y)\\&=S_{T\otimes\Gamma}(x\otimes y)\end{aligned}$$

Now, by the uniqueness of frame operator, we have : $S_T \otimes S_\Gamma = S_{T\otimes\Gamma}$ □

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