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ANALYTIC RESULTS OF SEMIGROUP OF LINEAR OPERATOR WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. This paper is generated by ω-order preserving partial contraction mapping(ω-OCP_n) (a semigroup of linear operator) as the infinitesimal generator of a C_0-semigroup in order to obtain analytic results by studying the existence and regularity properties of the Stokes equation. We also investigated parabolic and elliptic problems with dynamic boundary conditions.

1. Introduction

Stokes equation describe the flow of moderate speed of a vicious incompressible fluid within a domain Ω in $\mathbb{R}^3$. Let X be a Banach space, $X_n \subseteq X$ be a finite set, $(T(t))_{t\geq 0}$ the C_0-semigroup, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, and $A \in \omega - OCP_n$ is a generator of C_0-semigroup. This paper consist of results of analytic semigroup on Stokes equation and homogeneous system with dynamic boundary conditions. Agmon *et al.* [1], estimated some boundary problems for solutions of elliptic partial differential equation. Akinyele *et al.* [2], further characterized ω-order reversing partial contraction mapping as a semigroup of linear operator. Amann [3], investigated linear and quasilinear parabolic problems. Balakrishnan [4], obtained an operator calculus for infinitesimal generators of semigroup. Banach [5], established and introduced the concept of Banach spaces. Bejanaru *et al.* [6], approximates controllability results and application to elliptic and parabolic systems with boundary conditions. Constantin *et al.* [7], deduced further Navier-Stokes equations. Engel and Nagel [8], obtained one-parameter semigroup for

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linear evolution equations. Frigon and O'Regan [9], introduced existence results for initial value problems in Banach spaces. Fujita and Kato [10], obtained some results on the Navier-Stokes initial-value problem. Rauf and Akinyele [11], introduced ω-order preserving partial contraction mapping and established its properties, also in [12], Rauf *et al.* deduced some results of stability and spectra properties on semigroup of linear operator. Vrabie [13], proved some results of C_0-semigroup and its applications. Walker [14], presented some dynamical systems and evolution. Yosida [15], established and proved some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [13]
A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.
Definition 2.2 (ω-OCP_n) [11]
A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in \mathrm{Dom}\alpha :$ $x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.
Definition 2.3 (Analytic Semigroup) [8]
We say that a C_0-semigroup $\{T(t); t \geq 0\}$ is analytic if there exists $0 < \theta \leq \pi$, and a mapping $S : \bar{\mathbb{C}}_\theta \to L(X)$ such that:
(i) $T(t) = S(t)$ for each $t \geq 0$;
(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \bar{\mathbb{C}}_\theta$;
(iii) $lim_{z_1 \in \bar{\mathbb{C}}_\theta, z_1 \to 0} S(z_1)x = x$ for $x \in X$; and
(iv) the mapping $z_1 \to S(z_1)$ is analytic from $\bar{\mathbb{C}}_\theta$ to $L(X)$. In addition, for each $0 < \delta < \theta$, the mapping $z_1 \to S(z_1)$ is bounded from $\mathbb{C}_\delta$ to $L(X)$, then the C_0-Semigroup $\{T(t); t \geq 0\}$ is called analytic and uniformly bounded.
Definition 2.4 (Compact Semigroup) [8]
A C_0-semigroup is compact if for each $t > 0$, $T(t)$ is a compact operator.
Definition 2.5 (Adjoint Semigroup)[13]
Let H be a real Hilbert space identified with its own topological dual. The operator $A : D(A) \subseteq H \to H$ is called:
(i) self-adjoint if $A = A^*$
(ii) skew-adjoint if $A = -A^*$
(iii) symmetric if $< Ax, y >=< x, Ay >$ for each $x, y \in D(A)$;
(iv) skew-symmetric if $< Ax, y >= - < x, Ay >$ for $x, y \in D(A)$.
Example 1
3×3 matrix $[M_m(\mathbb{C})]$, we have
for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.

Suppose we have

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \end{pmatrix}.$$

Example 2

The $H^{-1}(\Omega)$ setting. Let Ω be a nonempty and open subset in $\mathbb{R}^n$, let $X = H^{-1}(\Omega)$, and let us define $A : D(A) \subseteq X \to X$ by

$$\begin{cases} D(A) = H_0^1(\Omega) \\ \quad Au = \Delta u \end{cases}$$

for each $A \in \omega$-OCP_n and $u \in D(A)$. In that it follows that $H_0^1(\Omega)$ is endowed with the usual norm on $H^1(\Omega)$ defined by

$$\|u\|_{H^1(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\Delta\|^2_{L^2(\Omega)})^{1/2}.$$

Example 3

The $L^2(\Omega)$ setting. Let Ω be a nonempty and open subset in $\mathbb{R}^n$, let $X = L^2(\Omega)$ and let us consider the operator A on X defined by:

$$\begin{cases} D(A) = \{u \in H_0^1(\Omega); \Delta u \in L^2(\Omega)\} \\ \quad Au = \Delta u \end{cases}$$

for each $A \in \omega$-OCP_n and $u \in D(A)$.

Example 4

Let $H = \{u \in [L^2(\mathbb{R}^3)]^3; \nabla.u = 0\}$, where the condition $\nabla.u = 0$ is understood in the sense of distributions over $\mathbb{R}^3$. One can easily see that, endowed with the standard inner product of the space $[L^2(\mathbb{R}^3)]$, defined by

$$< u, v >= \sum_{i=1}^{3} < u_i, v_i >_{L^2(\mathbb{R}^3)} .$$

Suppose H is a real Hilbert space and $A \in \omega$-OCP_n. We define $A : D(A) \subseteq H \to H$, called the Stokes operator on $\mathbb{R}^3$, by

$$\begin{cases} D(A) = \{u \in [H^2(\mathbb{R}^3)]^3 \cap H; \Delta u \in H\} \\ \quad Au = \Delta u \ for \ u \in D(A), \end{cases}$$

where $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$ in the sense of distributions over $\mathbb{R}^3$.

Theorem 2.1 [13]

The application $I - \Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the canonical isormorphism between $H_0^1(\Omega)$, endowed with the usual norm $H^1(\Omega)$, and its dual $H^{-1}(\Omega)$, endowed with usual dual norm. In addition, for each $u \in H_0^1(\Omega)$ and each $v \in L^2(\Omega)$ we have

$$< u, v >_{L^2(\Omega)} = < u, v >_{H_0^1(\Omega), H^1(\Omega)} .$$

Theorem 2.2 (Hille-Yoshida)[13]
A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed; and
ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}.$$

Lemma 2.3 [8]
Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary τ and let $\mu \geq 0$, $\lambda > 0$. Then for each $f \in L^2(\Omega)$ and $g \in L^2(\tau)$, elliptic problem

$$\begin{cases} \mu u - \Delta u = f \\ \lambda v + u_v = g \\ u_{|\tau} = v \end{cases}$$

has a unique solution $u \in H^{3/2}(\Omega)$ with $\Delta u \in L^2(\Omega)$, $u_{|\tau} \in H^1(\tau)$ and $u_v \in L^2(\tau)$.
Corollary 2.4 [8]
If $A : D(A) \subseteq H \to H$ is self-adjoint and generates a C_0-semigroup of contraction $\{T(t); t \geq 0\}$, then $\{T(t); t \geq 0\}$ is analytic.
Theorem 2.5 [13]
Assume that Ω is a nonempty, open and bounded subset in $\mathbb{R}^n$ whose boundary is of class C^1, $m \in \mathbb{N}$ and $p, q \in [1, +\infty)$. Then,

i. if $mp < n$ and $q < \frac{np}{n-mp}$, we have that $W^{m,p}(\Omega)$ is compactly imbedded in $L^q(\Omega)$;
ii. if $mp = n$ and $q \in [1, +\infty)$ is compactly imbedded in $L^q(\Omega)$; and
iii. if $mp > n$, then $W^{m,p}(\Omega)$ is compactly imbedded in $C(\overline{\Omega})$.

3. Main Results

This section presents results of analytic semigroup generated by ω-OCP_n with the consideration of Stokes equation and dynamic boundary condition problems:
Theorem 3.1
Suppose the operator $A \in \omega$-OCP_n, defined by the example 2 is the generator of a C_0-semigroup of contraction . In addition, A is self-adjoint and $\|.\|_{D(A)}$ is equivalent with the norm of the space $H^1(\Omega)$.
Proof:
By virtue of Theorem 2.1, we know that $I - \Delta$ is the canonical isomorphism between $H_0^1(\Omega)$,

endowed with usual norm of $H^1(\Omega)$, and its dual $H^{-1}(\Omega)$. Let us denote by $F = (I-\Delta)^{-1}$ which is an isometry between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Consequently

$$< u, v >_{H^{-1}(\Omega)} = < Fu, Fv >_{H_0^1(\Omega)} \tag{3.1}$$

for each $u, v \in H^{-1}(\Omega)$. Let $u, v \in H_0^1(\Omega)$. We have

$$\begin{aligned} < Fu, Fv >_{H_0^1(\Omega)} &= \int_\Omega \nabla u \nabla (Fv) dw + \int_\Omega uFv dw \\ &= \int_\Omega u(-\Delta(Fv)) dw + \int_\Omega uFv dw \\ &= \int_\Omega u(I-\Delta)(Fv) dw = < u, v >_{L^2(\Omega)} . \end{aligned} \tag{3.2}$$

From (3.1), taking into that $F(I-\Delta) = I$, we deduced

$$\begin{aligned} < -\Delta u, v >_{H^{-1}(\Omega)} &= < u - \Delta u, v >_{H^{-1}(\Omega)} - < u, v >_{H^{-1}(\Omega)} \\ &= < F(u - \Delta u), Fv >_{H_0^1(\Omega)} - < u, v >_{H^{-1}(\Omega)} \\ &= < u, Fv >_{H^{-1}(\Omega)} - < u, v >_{H^{-1}(\Omega)} . \end{aligned}$$

From (3.2), we have

$$< \Delta u, v >_{H^{-1}(\Omega)} = < u, v >_{H^{-1}(\Omega)} - < u, v >_{L^2(\Omega)} . \tag{3.3}$$

Therefore A is symmetric. But $(I-A)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, and it follows that A is self-adjoint. Taking $u = v$ in (3.3), we obtained

$$< Au, v >_{H^{-1}(\Omega)} = \|u\|^2_{H^{-1}(\Omega)} - \|u\|^2_{L^2(\Omega)} \le 0. \tag{3.4}$$

Since $\lambda > 0$, we have $(\lambda I - A)^{-1} \in \mathcal{L}(H^{-1}(\Omega))$, while (3.4) implies that for $\lambda > 0$

$$< \lambda u - Au, v >_{H^{-1}(\Omega)} \ge \lambda \|u\|^2_{H^{-1}(\Omega)}.$$

Hence $\|R(\lambda; A)\|_{L(H^{-1}(\Omega))} \le \frac{1}{\lambda}$. Since $H_0^1(\Omega)$ is dense in $H^{-1}(\Omega)$, we are in the hypothesis of Theoerem 2.2, from where it follows that $A \in \omega\text{-}OCP_n$ generates a C_0-semigroup of contraction on $H^{-1}(\Omega)$. Finally by norm on $D(A)$ equivalent with $\|.\|_{D(A)}$ is equivalent with the space $H^{-1}(\Omega)$. Hence the proof is complete.

Theorem 3.2

The linear operator $A \in \omega\text{-}OCP_n$, defined by the example 3 is the infinitesimal generator of C_0-semigroup of contractions. Moreover, A is self-adjoint, and $(D(A), \|.\|_{D(A)})$ is continuously included in $H_0^1(\Omega)$. If Ω is bounded with C^1 boundary, then $(D(A), \|.\|_{D(A)})$ is compactly imbedded in $L^2(\Omega)$.

Proof:

Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, and $C_0^\infty(\Omega) \subseteq D(A)$, it follows that A is densely defined. Let $\lambda > 0$ and $f \in L^2(\Omega)$. Since $L^2(\Omega)$ is continuously imbedded in $H^{-1}(\Omega)$, and

$-\Delta : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is the duality mapping with respect to the gradient norm on $H_0^1(\Omega)$, we have

$$< Au, v >_{L^2(\Omega)} = < \nabla u, \nabla v >_{L^2(\Omega)} = < v, \Delta u >_{H_0^1(\Omega),\ H^1(\Omega)} . \tag{3.5}$$

By Theorem 3.1, we know that for any $\lambda > 0$ and $f \in L^2(\Omega)$ (notice that $L^2(\Omega) \subset H^{-1}(\Omega)$), the equation

$$\lambda u - \Delta u = f \tag{3.6}$$

has a unique solution $u_\lambda \in H_0^1(\Omega) \subset L^2(\Omega)$. So, $\Delta u_\lambda = \lambda u_\lambda - f$ is in $L^2(\Omega)$, which shows that $u_\lambda \in D(A)$ and $\lambda u_\lambda - Au_\lambda = f$. Taking the L^2 inner product on both sides of (3.6) above by u_λ and taking into account that by (3.5), we have $< Au, u >_{L^2(\Omega)} \leq 0$ for each $u \in D(A)$, then we deduced that

$$\lambda \|u_\lambda\|^2_{L^2(\Omega)} \leq < f, u_\lambda >_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u_\lambda\|_{L^2(\Omega)},$$

which shows that $\|R(\lambda; A)\|_{L(X)} \leq \frac{1}{\lambda}$. Finally from (3.5) and Theorem 2.2, it follows that $A \in \omega\text{-}OCP_n$ is self-adjoint if and only if A is symmetric and $A \in \omega\text{-}OCP_n$ is skew-adjoint if and only if A is skew-symmetric, then it follows that A is self adjoint. Since both inclusions, then $D(A) \subset H_0^1 \subset L^2(\Omega)$ are continuous, and the latter is compact whenever Ω is bounded by Theorem 2.5 and this complete the proof.

Theorem 3.3

Let $X = H$ be a real Hilbert space and define $A : D(A) \subseteq H \rightarrow H$, called the Stokes operator on $\mathbb{R}^3$, by

$$\begin{cases} D(A) = \{u \in [(\mathbb{R}^3)]^3 \cap H; \Delta u \in H\} \\ \quad Au = \Delta u \ for \ u \in D(A), \end{cases} \tag{3.7}$$

where $A \in \omega\text{-}OCP_n$ and $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$ are in the sense of distributions over $\mathbb{R}^3$. Then the operator A defined in (3.7) is the infinitesimal generator of an analytic C_0-semigroup of contractions in H.

Proof:

Since $C_\sigma^\infty(\mathbb{R}^3; \mathbb{R}^3) = \{u \in [D(\mathbb{R}^3)]^3; \nabla . u = 0\}$ is both dense in H and included in $D(A)$, it follows that $A \in \omega\text{-}OCP_n$ is densely defined. In addition, from Theorem 3.2, we deduced that $A \in \omega\text{-}OCP_n$ is closed and symmetric. On the other hand, for each $\lambda > 0$ and each $f \in H$, the equation

$$\lambda u - \Delta u = f \tag{3.8}$$

has a unique solution $u \in [H^2(\mathbb{R}^3)]^3$. Let us denote by $v = \nabla . u$, which clearly belongs to $H^1(\mathbb{R}^3)$. Since $\nabla . f = 0$ in $H^{-1}(\mathbb{R}^3)$, it clearly follows that $\lambda v - \Delta v = 0$ in $H^{-1}(\mathbb{R}^3)$, which implies $v = 0$. So, $u \in D(A)$ and therefore $(0, +\infty) \subseteq \rho(A)$. For each $\lambda > 0$, we have

$$\|R(\lambda; A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{3.9}$$

By Theorem 2.2, we know that $A \in \omega\text{-}OCP_n$ is the infinitesimal generator of a C_0-semigroup of contractions on H. From Theorem 3.2, we have that A is self-adjoint, and by Definition 2.3, deduced that the semigroup generated by $A \in \omega\text{-}OCP_n$ is analytic and this complete the proof.

Theorem 3.4

Let $X = H$ be a real Hilbert space. Let us denote by $P : [L^2(\Omega)]^3 \to H$ the orthogonal projection on H and let us define the Stokes operator on Ω, $A : D(A) \subseteq H \to H$, by

$$\begin{cases} D(A) = [H^2(\Omega) \cap H_0^1(\Omega)]^3 \cap H \\ \quad Au = P(\Delta u), \ \ for\ u \in D(A). \end{cases}$$

The operator $A \in \omega\text{-}OCP_n$ defined as above is the generator of a compact and analytic C_0-semigroup of contractions on H.

Proof:

Since $C_\sigma^\infty(\Omega; \mathbb{R}^3) = \{u \in [D(\Omega)]^3; \nabla.u = 0\}$, which is dense in H, is included in $D(A)$, and its follows that $A \in \omega\text{-}OCP_n$ is densely defined. From Theorem 3.2, we have that A is closed and symmetric and thus self-adjoint. On the other hand, for each $\lambda > 0$ and $f \in H$, the equation

$$\lambda u - P(\Delta u) = f \tag{3.10}$$

has a unique solution $u \in H$ which in addition satisfies

$$\|u\| \leq \frac{1}{\lambda}\|f\|. \tag{3.11}$$

Therefore $(0, +\infty) \subseteq \rho(A)$ and $\|R(\lambda; A)\|_{L(H)} \leq \frac{1}{\lambda}$. By the virtue of Theorem 2.2, it follows that $A \in \omega\text{-}OCP_n$ generates a C_0-semigroup of contractions on H. Since A is self-adjoint, then by Theorem 3.3, the semigroup is analytic. Finally by Rellich-Kondrachov Theorem [16], for each $\lambda > 0$, $R(\lambda; A)$ is compact and this achieves the proof.

Theorem 3.5

Suppose $A \in \omega\text{-}OCP_n$. The operator

$$A : D(A) \subseteq L^2(\Omega) \times L^2(\tau) \to L^2(\Omega) \times L^2(\tau)$$

defined by

$$\begin{cases} \quad D(A) = \{(u, v) \in L^2(\Omega) \times L^2(\tau); \Delta u \in L^2(\Omega), u_v \in L^2(\tau), u_{|\tau} = v\} \\ A(u, v) = (\Delta u - u_v), \ \ for\ each\ (u, v) \in D(A), \end{cases}$$

is the infinitesimal generator of a compact and analytic C_0-semigroup of contractions.

Proof:

Clearly $D(A)$ is dense in $L^2(\Omega) \times L^2(\tau)$, because each function $L^2(\tau)$ can be approximated with functions of class C^2 on τ, and each function in $L^2(\Omega)$ can be approximated with function of class $C^2(\bar{\Omega})$ whose restrictions to the boundary are preassigned C^2 functions on τ. Moreover, let us observe that for each (u, v),$(\eta, \vartheta) \in D(A)$, we have

$$< A(u,v),(\eta,\vartheta) >=< \Delta u, \eta > - < u_v, \vartheta >_{L^2(\tau)} = - < \nabla u, \nabla, \eta >$$
$$=< u, \Delta\eta > - < u, \eta_v >_{L^2(\tau)} =< (u,v), A(\eta,\vartheta) > .$$

Consequently A is symmetric. In addition, for each $\lambda > 0$, the equation $(\lambda I - A)(u, v) = (f, g)$ rewrites equivalently under the form

$$\begin{cases} \lambda u - \Delta u = f \\ \lambda v + u_v = g \\ u_{|\tau} = v. \end{cases} \tag{3.12}$$

From Lemma 2.3, we know that for each $(f, g) \in L^2(\Omega) \times L^2(\tau)$, the above problem has a unique solution u satisfying $u \in H^{3/2}$, $\Delta \in L^2(\Omega)$, $u_{|\tau} \in H^1(\tau)$ and $u_v \in L^2(\tau)$. Taking the $L^2(\Omega)$-inner product on both sides of the first two equations in (3.12) by u and respectively by v and taking into consideration the third equation, we deduced

$$\begin{cases} \lambda\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} - < u_v, u >_{L^2(\tau)} =< f, u >_{L^2(\Omega)} \\ \lambda\|v\|^2_{L^2(\tau)} + < u_v, u >_{L^2(\tau)} =< g, v >_{L^2(\tau)} . \end{cases} \tag{3.13}$$

Adding up the two equations in (3.13) and using Cauchy-Schwarz inequality, we obtained

$$\lambda\|(u, v)\|^2_{L^2(\Omega)\times L^2(\tau)} \leq \|(f, g)\|_{L^2(\Omega)\times L^2(\tau)} \|(u, v)\|_{L^2(\Omega)\times L^2(\tau)}.$$

a From this, we deduced on one hand that $(0, +\infty) \subseteq \rho(A)$ and, on the other hand, that for each $\lambda > 0$, we have $\|(\lambda I - A)^{-1}\|_{L(X)} \leq \frac{1}{\lambda}$. By virtue of Theorem 2.2, it follows that $A \in \omega$-OCP_n generates C_0-semigroup of contractions and from Theorem 3.2, we conclude that A is self-adjoint. Since $D(A)$, endowed with the graph-norm is compactly imbedded in $L^2(\Omega) \times L^2(\tau)$ and it follows that the semigroup generated by A is compact, then the proof is complete.

Theorem 3.6

The Operator $A : D(A) \subseteq L^2(\tau) \to L^2(\tau)$, defined by

$$\begin{cases} D(A) = \{y \in H^{1/2}(\tau);\ for\ which\ u = \zeta(y)\ satisfies\ u_v \in L^2(\tau)\} \\ Ay = u_v\ for\ each\ y \in D(A)\ and\ A \in \omega - OCP_n, \end{cases}$$

is the infinitesimal generator of an analytic and compact C_0-semigroup of contractions in $L^2(\tau)$.

Proof:

Let $y \in H^{1/2}(\tau)$ and let us consider the non-homogeneous elliptic problem

$$\begin{cases} -\Delta u = 0\ in\ \Omega \\ u = y\ on\ \tau, \end{cases} \tag{3.14}$$

with a unique solution $u \in H^1(\Omega)$ and let the solution be denoted by $u = \zeta(y)$. Let us observe that, for each $y, z \in D(A)$, we have

$$< Ay, z >_{L^2(\tau)} = - < \nabla u, \nabla v >_{L^2(\tau)} - < \Delta u, v >_{[H^1(\Omega)]^*, H^1(\Omega)}$$

and

$$< y, Az >_{L^2(\tau)} = - < \nabla u, \nabla v >_{L^2(\tau)} - < \Delta v, u >_{[H^1(\Omega)]^*, H^1(\Omega)},$$

where u satisfies (3.14), while v is the solution of a similar equation with y substituted by z. Here $< .,. >_{[H^1(\Omega)], H^1(\Omega)}$ is the usual duality between $H^1(\Omega)$ and its topological dual $[H^1(\Omega)]^*$. Since $\Delta u = \Delta v = 0$, it follows that

$$< Ay, z >_{L^2(\tau)} = - < \nabla u, \nabla v >_{L^2(\tau)} = < y, Az >_{L^2(\tau)}$$

for each $y, z \in D(A)$ and $A \in \omega\text{-}OCP_n$. So, A is symmetric. We shall prove next that A is self-adjoint in $H = L^2(\tau)$ and satisfies the condition in Theorem 2.2. To this aim, we shall show that, for each $\lambda > 0$, $(\lambda I - A)^{-1} \in \mathcal{L}(H)$ and $\|(\lambda I - A)\|_{L(H)} \leq \frac{1}{\lambda}$. One may easily see that $\lambda y - Ay = f$ if and only if $y = u_{|\tau}$, where u is the unique $H^1(\Omega)$ solution of the elliptic problem

$$\begin{cases} -\Delta u = 0 \ in \ \Omega \\ \lambda u + u_v = f \ on \ \tau. \end{cases} \tag{3.15}$$

Moreover, $(\lambda I - A)$ is surjective if and only if for each $f \in L^2(\tau)$, the unique solution in $H^1(\Omega)$ of (3.15) satisfies

$$u \in L^2(\tau) \ and \ u_v \in L^2(\tau), \tag{3.16}$$

and this follows from Lemma 2.3. Finally, we have to show that, for each $\lambda > 0$ and $A \in \omega\text{-}OCP_n$, we have

$$\|(\lambda I - A)^{-1}\|_{L(H)} \leq \frac{1}{\lambda}. \tag{3.17}$$

To this aim, let us multiply both sides of the first equation in (3.15) by u and integrating over Ω yields

$$\|\nabla u\|^2_{L^2(\Omega)} - < u_v, u >_{L^2(\tau)} = 0.$$

Taking into consideration the boundary condition in (3.15) and substituting u_v by $f - \lambda u$, we deduced

$$\|\nabla u\|^2_{L^2(\Omega)} + \lambda \|u\|^2_{L^2(\tau)} \leq \|f\|_{L^2(\tau)} \|u\|_{L^2(\tau)}. \tag{3.18}$$

From (3.18), we observed that $(\lambda I - A) = f$ if and only if $y = u_{|\tau}$, then we obtained (3.17). Therefore A satisfies all the conditions in Theorem 2.2, and combined with corollary 2.4, its follows that A generates and analytic semigroup. Hence the proof is complete.

Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping ($\omega\text{-}OCP_n$) as a semigroup of linear operator generates a compact and analytic results..

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