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ENTROPY SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS IN MUSIELAK FRAMEWORK WITHOUT SIGN CONDITION AND L^1 DATA

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Abstract. In the present paper, we study the existence of entropy solution for the strongly nonlinear unilateral parabolic inequalities in Musielak-Orlicz-spaces.

1. Introduction:

In this note we prove the existence of entropy solutions for a class of nonlinear parabolic unilateral problems of the type:

$$(\mathcal{P})\begin{cases} u \geq \zeta \quad \text{a.e. in } \Omega \times (0,T), \\ \frac{\partial b(u)}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u)\big) + g(x,t,u,\nabla u) = f \quad \text{in } Q, \\ u = 0 \quad \text{in } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \quad \text{in} \quad \Omega, \end{cases}$$

where $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions Operator defined on $D(A) \subset W_0^{1,x}L_\varphi(Q) \longrightarrow W^{-1,x}L_\psi(Q)$ where φ and ψ are two complementary Musielak-Orlicz functions. Let g be a Caratheodory function such that the growth condition

$$|g(x,t,s,\xi)| \leq P(x,t) + \rho(s)\varphi(x,|\xi|) \tag{1}$$

is satisfied, where $\rho : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a continuous non-decreasing function which belongs to $L^1(\mathbb{R})$ and $P(x,t)$ is a given non-negative function in $L^1(Q)$. The function $\zeta \in W_0^{1,x}E_\varphi(Q) \cap$

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$L^{\infty}(Q)$.

In the classical Sobolev spaces Dall'aglio-Orsina [23] and Porretta [32] proved the existence of solutions for the problem $(\mathcal{P})$, where $b(u)=u$ and g is a nonlinearity with the following "natural" growth condition (of order p):

$$|g(x, t, s, \xi)| \leq b(s)\left(|\xi|^{p}+c(x, t)\right), \tag{2}$$

and which satisfies the classical sign condition,

$$g(x, t, s, \xi) s \geq 0. \tag{3}$$

The right hand side f is assumed to belong to $L^{1}(Q)$. This result generalizes analogous one of Boccardo - Gallouët [20], see also [21, 22] for related topics.

$$|g(x, t, s, \xi)| \geq \beta|\xi|^{p} \text{ for } |s| \geq \gamma$$

In the framework of Orlicz-Sobolev spaces,in [2] the autors have studied the existence and uniqueness result to the nonlinear parabolic equations whose prototype is

$$\begin{cases}\frac{\partial b(u)}{\partial t}-\Delta_{M} u-\operatorname{div}\left(\bar{c}(x, t) \bar{M}^{-1} M\left(\frac{\alpha_{0}}{\lambda}|b(u)|\right)\right)=f \text{ in } Q_{T}, \\ u(x, t)=0 & \text{on } \partial \Omega \times(0, T), \\ b(u)(t=0)=b\left(u_{0}\right) & \text{in } \Omega.\end{cases} \tag{4}$$

where $-\Delta_{M} u=-\operatorname{div}\left((1+|u|)^{2} D u \frac{\log (e+D u)}{|D u|}\right), \bar{c} \in\left(L^{\infty}\left(Q_{T}\right)\right)^{N}$ and $M(t)=t \log (e+t)$ is an N -function. The data f and $b\left(u_{0}\right)$ in $L^{1}\left(Q_{T}\right)$ and $L^{1}(\Omega)$.
Another approach to define a suitable generalized solution is that of entropy solution which was introduced in [8] in the elliptic case and by Prignet [29] in the parabolic case.

Aharouch and Bennouna [5] have proved the existence and uniqueness of entropy solutions in the framework of Orlicz-Sobolev spaces $W_{0}^{1} L_{M}(\Omega)$ assuming the Δ_{2} condition on the N -function M.

In the generalized-Orlicz spaces, the work [6] is a continuation of [5] where AlHawmi, Benkirane, Hjiaj and Touzani proved the existence and uniqueness of entropy solution for

$$\begin{cases}-\operatorname{div}(a(x, u, \nabla u))=f & \text{in } \Omega \\ u(x)=0 & \text{on } \partial \Omega\end{cases}$$

where $\Phi=0$ and $\bar{M}$ satisfy the Δ_{2} -condition. Antontsev and Shmarev [7] proved theorems of existence and uniqueness of weak solutions of Dirichlet problem for a class of nonlinear parabolic equations with nonstandard anisotropic growth conditions in the variable exponent Lebesgue spaces. Equations of this class generalize the evolution $p(x,t)$ -Laplacian of the type

$$(5)\quad \begin{cases} \frac{\partial u}{\partial t} - \sum_i \frac{\partial}{\partial x_i}\left[a_i(x,t,u)\,|D_i u|^{p_i(x,t)-2}\,D_i u + b_i(x,t,u)\right] = 0 & \text{in } Q_T \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \cdot$$

In general Musielak-Sobolev spaces, the authors in [1] have proved the existence of solutions of the unilateral problem

$$Au - \operatorname{div}\Phi(x,u) + H(x,u,\nabla u) = \mu$$

where A is a Leray-Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega), \mu \in L(\Omega) + W^{-1}E_{\bar{M}}(\Omega)$, where M and $\bar{M}$ are two complementary Musielak-Orlicz functions and both the first and the second lower terms Φ and H satisfies only the growth condition and $u \geq \zeta$ where ζ is a measurable function, and further works can be found in[9, 10, 34, 12, 13, 14, 15, 16, 24, 25, 26, 27, 28], then our novelty in this paper, is to prove the existence of unilateral entropy solution for the problem $(\mathcal{P})$ in the general framework of Musielak-Orlicz spaces without the sign condition (3) and without the following coercivity condition

$$|g(x,t,s,\xi)| \geq \beta\varphi(x,|\xi|).$$

The paper is organized as follows: In Section 2 , we present some preliminaries and background,section 3 is devoted to essential assumptions, Section 4 concern some technical lemmas which will be needed later,and section 5 is to specify the definition of an entropy solution.In the final section 6, we give our main result and state the prove of an existence of solution.

2. **Preliminaries**

2.1. **Musielak-Orlicz functions.** Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

(a) $\varphi(x,\cdot)$ is an N-function for all $x \in \Omega$ (i.e. convex, strictly increasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$, for all $t > 0$, $\lim\limits_{t\to 0}\sup\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t} = 0$ and $\lim\limits_{t\to\infty}\inf\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t} = \infty$),

(b) $\varphi(\cdot,t)$ is a measurable function.

The function φ is called a Musielak–Orlicz function.

For a Musielak-orlicz function φ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t.$$

The Musielak-orlicz function φ is said to satisfy the Δ_2-condition if for some $k>0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t)\leq k\,\varphi(x,t)+h(x) \text{ for all } x\in\Omega \text{ and } t\geq 0. \tag{6}$$

When (6) holds only for $t\geq t_0>0$, then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ and we write $\gamma\prec\varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x\in\Omega$

$$\gamma(x,t)\leq\varphi(x,ct) \text{ for all } t\geq t_0, \quad (\text{ resp. for all } t\geq 0 \text{ i.e. } t_0=0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma\prec\prec\varphi$ if for every positive constant c we have

$$\lim_{t\longrightarrow 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0, \quad (\text{resp. } \lim_{t\longrightarrow\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0).$$

Remark 2.1. [18] *If $\gamma\prec\prec\varphi$ near infinity, then $\forall\varepsilon>0$ there exist $k(\varepsilon)>0$ such that for almost all $x\in\Omega$ we have*

$$\gamma(x,t)\leq k(\varepsilon)\varphi(x,\varepsilon t), \quad \textit{for all } t\geq 0. \tag{7}$$

2.2. **Musielak-Orlicz space:** For a Musielak-Orlicz function φ and a measurable function $u:\Omega\longrightarrow\mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u)=\int_\Omega\varphi(x,|u(x)|)\,dx.$$

The set $K_\varphi(\Omega)=\Big\{u:\Omega\longrightarrow\mathbb{R} \text{ measurable } / \ \rho_{\varphi,\Omega}(u)<\infty\Big\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega)=\big\{u:\Omega\longrightarrow\mathbb{R} \text{ measurable } \Big/\rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big)<\infty, \text{ for some } \lambda>0\big\}.$$

For a Musielak-Orlicz function φ we put: $\psi(x,s)=\sup_{t\geq 0}\{st-\varphi(x,t)\}$,
ψ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sens of Young with respect to the variable s.

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega}=\inf\left\{\lambda>0/\int_\Omega\varphi\Big(x,\frac{|u(x)|}{\lambda}\Big)\,dx\leq 1\right\},$$

which is called the Luxemburg norm and the so–called Orlicz norm by:

$$|||u|||_{\varphi,\Omega}=\sup_{\|v\|_\psi\leq 1}\int_\Omega|u(x)v(x)|\,dx,$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent [34].

We will also use the space $E_\varphi(\Omega)$ defined by

$$E_\varphi(\Omega) = \Big\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } \Big/ \rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) < \infty, \text{ for all } \lambda > 0\Big\}.$$

A Musielak function φ is called locally integrable on Ω if $\rho_\varphi(t\chi_E) < \infty$ for all $t > 0$ and all measurable $E \subset \Omega$ with $\text{meas}(E) < \infty$.

Let φ a Musielak function which is locally integrable. Then $E_\varphi(\Omega)$ is separable [34].

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \Big\{ u \in L_\varphi(\Omega) : \forall |\alpha| \leq m,\ D^\alpha u \in L_\varphi(\Omega) \Big\}.$$

and

$$W^m E_\varphi(\Omega) = \Big\{ u \in E_\varphi(\Omega) : \forall |\alpha| \leq m,\ D^\alpha u \in E_\varphi(\Omega) \Big\}.$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\Big(D^\alpha u\Big) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}$$

for $u \in W^m L_\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\Big(W^m L_\varphi(\Omega), \|\|^m_{\varphi,\Omega}\Big)$ is a Banach space if φ satisfies the following condition [34]:

(8) $$\text{there exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \geq c_0.$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

The space $W^m_0 L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. and the space $W^m_0 E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \Big\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \Big\}.$$

and

$$W^{-m}E_\psi(\Omega) = \Big\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \Big\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0.$$

For φ and her complementary function ψ, the following inequality is called the Young inequality [34]:

$$ts \le \varphi(x,t) + \psi(x,s), \quad \forall t, s \ge 0, x \in \Omega. \tag{9}$$

This inequality implies that

$$\||u\||_{\varphi,\Omega} \le \rho_{\varphi,\Omega}(u) + 1. \tag{10}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \le \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1. \tag{11}$$

$$\|u\|_{\varphi,\Omega} \ge \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \le 1. \tag{12}$$

For two complementary Musielak Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Hölder inequality [34]

$$\Big|\int_\Omega u(x)v(x)\,dx\Big| \le \|u\|_{\varphi,\Omega}\||v\||_{\psi,\Omega}. \tag{13}$$

3. **Essential Assumptions**

Throughout this paper, we assume that the following assumptions hold true: Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $T > 0$, we denote $Q = \Omega \times [0,T]$, and let φ and γ be two Musielak-Orlicz functions functions such that φ is locally integrable and $\gamma \prec\prec \varphi$.

$b : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly increasing C^1-function with $b(0) = 0$ and such that

$$0 < b_0 \le b_0(s) \le b_1 \quad \forall s \in \mathbb{R}, \tag{14}$$

where b_0 and b_1 are given real numbers.

Let $A : D(A) \subset W_0^{1,x}L_\varphi(Q) \longrightarrow W^{-1,x}L_\psi(Q)$ be a mapping given by

$$A(u) = -\text{div}(a(x,t,u,\nabla u)),$$

where $a : a(x,t,s,\xi) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e $(x,t) \in Q$ and for all $s \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$, $\xi \ne \xi'$:

$$|a(x,t,s,\xi)| \le \beta\Big(c(x,t) + \psi_x^{-1}\gamma(x,\nu|\xi|) + \psi_x^{-1}\varphi(x,\nu|\xi|)\Big), \tag{15}$$

$$\Big(a(x,t,s,\xi) - a(x,t,s,\xi')\Big)(\xi - \xi') > 0, \tag{16}$$

$$a(x,t,s,\xi).\xi \geq \alpha\varphi(x,|\xi|). \tag{17}$$

where $c(x,t)$ a positive function, $c(x,t) \in E_{\psi}(Q)$ and positive constants ν, β, α.
let $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a Caratheodory function satisfying for a.e. $(x,t) \in \Omega \times [0,T]$ and $\forall s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|g(x,t,s,\xi)| \leq P(x,t) + \rho(s)\varphi(x,|\xi|) \tag{18}$$

is satisfied, where $\rho : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a continuous non-decreasing function which belongs to $L^1(\mathbb{R})$ and $P(x,t)$ is a given non-negative function in $L^1(Q)$.

$$f \text{ is an element of } L^1(Q), \tag{19}$$

$$u_0 \text{ is an element of } L^1(\Omega). \tag{20}$$

Let us give the following lemma which will be needed later.

4. **Some technical Lemmas**

Lemma 4.1. [17]. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*
i) *There exist a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \geq c$,*
ii) *There exist a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{log(\frac{1}{|x-y|}}\right)}, \quad \forall t \geq 1. \tag{21}$$

iii)

$$\textit{If } D \subset \Omega \textit{ is a bounded measurable set, then } \int_D \varphi(x,1)dx < \infty. \tag{22}$$

iv) *There exist a constant $C > 0$ such that $\psi(x,1) \leq C$ a.e in Ω.*
Under this assumptions, $\mathcal{D}(\Omega)$ is dense in $L_{\varphi}(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_{\varphi}(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1L_{\varphi}(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_{\psi}(\Omega)$ on an element u of $W_0^1L_{\varphi}(\Omega)$ is well defined. It will be denoted by $< S, u >$.

Lemma 4.2. [?]. *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak- Orlicz function and let $u \in W_0^1L_{\varphi}(\Omega)$. Then $F(u) \in W_0^1L_{\varphi}(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e \text{ in } \{x \in \Omega : u(x) \in D\}. \\ 0 & a.e \text{ in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 4.3 (Poincaré inequality)**.** [36]*Let φ a Musielak Orlicz function which satisfies the assumptions of lemma 4.1, suppose that $\varphi(x,t)$ decreases with respect of one of coordinate of x.*
Then, that exists a constant $c>0$ depends only of Ω such that

$$\int_\Omega \varphi(x,|u(x)|)dx \leq \int_\Omega \varphi(x,c|\nabla u(x)|)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{23}$$

Lemma 4.4. [36] *Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

Lemma 4.5 (The Nemytskii Operator)**.** [9]. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|). \tag{24}$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$. Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x,u(x))$ is continuous from

$$\mathcal{P}\bigg(E_\varphi(\Omega), \frac{1}{k_2}\bigg) = \bigg\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\bigg\}.$$

into $L_\psi(\Omega)$.
Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\mathcal{P}\Big(E_\varphi(\Omega), \frac{1}{k_2}\Big)$ to $E_\gamma(\Omega)$.

Lemma 4.6. *Let φ be a Musielak function. Let $(u_n)_n$ be a sequence of $W^{1,x}L_\varphi(Q)$ such that*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi L_\psi)$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathcal{D}'(Q)$$

with $(h_n)_n$ bounded in $W^{-1,x}L_\psi(Q)$ and $(k_n)_n$ bounded in the space $\mathcal{M}(Q)$ set of measures on Q. then $u_n \longrightarrow u$ strongly in $L^1_{loc}(Q)$.
If further $u_n \in W_0^{1,x}L_\varphi(Q)$ then $u_n \longrightarrow u$ strongly in $L^1(Q)$.

Proof. It is easily adapted from that given in [22] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [35]. □

Theorem 4.1. [3]
Let φ be an Musielak-Orlicz function satisfies the assumption (21). If $u \in W^{1,x}L_\varphi(Q) \cap L^2(Q)$ (respectively $u \in W_0^{1,x}L_\varphi(Q) \cap L^2(Q)$) and
$\frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(Q) + L^2(Q)$, then there exists a sequence $(v_j) \in D(\bar{Q})$ (respec- tively $D(\bar{I}, D(\Omega))$) such that $v_j \longrightarrow u$ in $W^{1,x}L_\varphi(Q) \cap L^2(Q)$ and $\frac{\partial v_j}{\partial t} \longrightarrow \frac{\partial u}{\partial t}$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$ for the modular convergence.

5. Definition of an entropy solution

For $k > 0$ we define the truncation at height k: $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ by :

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{25}$$

We define

$$T_0^{1,\varphi}(Q) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable such that } T_k(u) \in W_0^{1,x}L_\varphi(Q)\ \forall k > 0 \Big\}.$$

Let ζ a measurable function with values in $\mathbb{R}$ such that

$$\zeta \in W_0^1 E_\varphi(Q) \cap L^\infty(Q), \quad \frac{\partial \zeta}{\partial t} \in L^1(Q) \quad \text{such that} \quad u_0 \geq \zeta$$

and let

$$K_\zeta = \left\{ u \in W_0^{1,x}L_\varphi(Q) : u \geq \zeta \text{ a.e. in } Q \right\}.$$

The definition of an entropy solution for problem $(\mathcal{P})$ can be stated as follows.

Definition 5.1. *A measurable function u defined on Q is an entropy solution of the problem $(\mathcal{P})$ if*

$$u \in T_0^{1,\varphi}(Q) \text{ and } u \geq \zeta \quad \text{a.e. in } \Omega \times (0,T), \tag{26}$$

and for all $v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$, $\frac{\partial v}{\partial t} \in W_0^{-1,x}L_\psi(Q)$ such that $v \geq \zeta$ a.e. in Q and $\forall k > 0$, $\tau \in (0,T)$

$$\begin{aligned} &\int_\Omega S_k(b(u(\tau)) - v(\tau)\,dx + \int_0^\tau \langle \frac{\partial v}{\partial t}, T_k(b(u) - v)\rangle\,dt \\ &\quad + \int_Q a(x,t,u,\nabla u)\nabla T_k(u - v)\,dx\,dt + \int_Q g(x,t,u,\nabla u)T_k(u-v)\,dx\,dt \\ &\quad \leq \int_Q f T_k(u-v)\,dx\,dt + \int_\Omega S_k(b(u_0) - v(0)\,dx, \end{aligned} \tag{27}$$

where $S_k(s) = \int_0^s T_k(r)\,dr$.

6. Statements of results

This section is devoted to the proof of the following existence theorem.

Theorem 6.1. *Assume that (14)...(20) hold true. Then, there exists at least one entropy solution u of the problem $(\mathcal{P})$ in the sense of Definition 5.1.*

Proof. The proof of Theorem 6.1 is divided into 7 steps.

Step 1 : Approximate problem.

For each $n > 0$, we define the approximation

$$b_n(s) = b\left(T_n(s)\right), \forall s \in \mathbb{R}, \tag{28}$$

$$a_n(x,t,s,\xi) = a\left(x,t,T_n(s),\xi\right) \quad \text{a.e. } (x,t) \in Q, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N \tag{29}$$

$$g_n(x,t,s,\xi) = \frac{g(x,t,s,\xi)}{1+\frac{1}{n}|g(x,t,s,\xi)|} \tag{30}$$

$$u_{0n} \in \mathcal{C}_0^\infty(\Omega) \text{ such that } b_n\left(u_{0n}\right) \rightarrow b\left(u_0\right) \text{ strongly in } L^1(\Omega) \tag{31}$$

$f_n \in L^1(Q)$ such that $f_n \rightarrow f$ strongly in $L^1(Q)$, and $\|f_n\|_{L^1(Q)} \leq \|f\|_{L^1(Q)}$.
Consider the nonlinear approximate problems

$$\begin{cases} \frac{\partial b_n(u_n)}{\partial t} - \operatorname{div}\left(a_n\left(x,t,u_n,\nabla u_n\right)\right) + g_n\left(x,t,u_n,\nabla u_n\right) & \\ +nT_n\left(u_n-\zeta\right)^-\left(u_n\right) = f_n & \text{in } Q, \\ u_n(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(x,0) = u_{0n} & \text{in } \Omega. \end{cases} \tag{32}$$

Since g_n is bounded for any fixed $n > 0$, there exists at last one solution $u_n \in W_0^{1,x}L_\varphi(Q)$ of (32) (see [33])

Step 2 : A priori estimates.

Fixed $k > 0$ Let $\tau \in (0,T)$ and using $\exp\left(G\left(u_n\right)\right) T_k\left(u_n\right)^+ \chi_{(0,\tau)}$ as a test function in problem (32) where $G(s) = \displaystyle\int_0^s \frac{\rho(r)}{\alpha} dr$, We get

$$\begin{aligned} &\int_{Q_\tau} \frac{\partial b_n(u_n)}{\partial t} \exp\left(G\left(u_n\right)\right) T_k\left(u_n\right)^+ dxdt \\ &+ \int_{Q_\tau} a_n\left(x,t,u_n,\nabla u_n\right) \nabla\left(\exp\left(G\left(u_n\right)\right) T_k\left(u_n\right)^+\right) dxdt \\ &+ \int_{Q_\tau} g\left(x,t,u_n,\nabla u_n\right) \exp\left(G\left(u_n\right)\right) T_k\left(u_n\right)^+ dxdt \\ &+ \int_{Q_\tau} nT_n\left(u_n-\zeta\right)^-\left(u_n\right) \exp\left(G\left(u_n\right)\right) T_k\left(u_n\right)^+ dxdt \\ &\leq k \exp\left(\frac{\|\rho\|_{L^1}}{\alpha}\right) \|f_n\|_{L^1(Q_T)}. \end{aligned} \tag{33}$$

We take

$$\widetilde{T}_k(r) = \int_0^r \exp(G(s)) T_k(s)^+ ds,$$

then

$$\int_{Q_\tau} \frac{\partial b_n(u_n)}{\partial t} \exp\left(G\left(u_n\right)\right) T_k\left(u_n\right)^+ dxdt = \int_\Omega \widetilde{T}_k\left(b_n(u_n(\tau)\right) dx - \int_\Omega \widetilde{T}_k\left(b_n(u_n(0))\right) dx. \tag{34}$$

By definition we have

$$\int_\Omega \widetilde{T}_k\left(b_n(u_n(\tau)\right) dx \geq 0, \tag{35}$$

and

$$\int_\Omega \widetilde{T}_k\left(b_n(u_n(0))\right) dx \leq k \exp\left(\frac{\|\rho\|_{L^1}}{\alpha}\right) \|b(u_0)\|_{L^1(\Omega)}. \tag{36}$$

By using the hypothesis (18) one has

$$\begin{aligned} &\int_{Q_\tau} g_n\left(x,t,u_n,\nabla u_n\right) \exp\left(G\left(u_n\right)\right) T_k\left(u_n\right)^+ dxdt \\ &\leq k \exp\left(\frac{\|\rho\|_{L^1}}{\alpha}\right) \int_{Q_T} |P(x,t)| dxdt \\ &+ \int_{Q_\tau} \rho\left(u_n\right) \exp\left(G\left(u_n\right)\right) \varphi\left(x,\nabla u_n\right) T_k\left(u_n\right)^+ dxdt. \end{aligned} \tag{37}$$

Finally by combining (33),(34),(35),(36) and (37) we obtain:

$$\begin{aligned} &\int_{Q_\tau} a\left(x,t,u_n,\nabla u_n\right) \exp\left(G\left(u_n\right)\right) \nabla T_k\left(u_n\right)^+ dxdt \\ &+ \int_{Q_\tau} nT_n\left(u_n-\zeta\right)^- \exp\left(G\left(u_n\right)\right) T_k\left(u_n\right)^+ dxdt \\ &\leq k \exp\left(\frac{\|\rho\|_{L^1}}{\alpha}\right) (\|f_n\|_{L^1(Q_T)} + \int_Q |P(x,t)| dxdt + \|b(u_0)\|_{L^1(\Omega)}) \end{aligned} \tag{38}$$

In view of (17) one has $\forall n > 0$:

$$\begin{aligned} &\alpha \int_{Q_\tau} \varphi\left(x, \left|\nabla T_k\left(u_n\right)^+\right|\right) \exp\left(G\left(u_n\right)\right) dxdt \\ &+n \int_{Q_\tau} T_n\left(u_n-\zeta\right)^- T_k\left(u_n\right)^+ \exp\left(G\left(u_n\right)\right) dxdt \leq c_1 k. \end{aligned} \tag{39}$$

Thus,

$$\begin{aligned} &\alpha \int_{Q_\tau} \varphi\left(x, \left|\nabla T_k\left(u_n\right)^+\right|\right) \exp\left(G\left(u_n\right)\right) dxdt \\ &+c_2 n \int_{Q_\tau} T_n\left(u_n-\zeta\right)^- T_k\left(u_n\right)^+ \exp\left(G\left(u_n\right)\right) dxdt \leq c_1 c_2 k. \end{aligned}$$

with $c_2 = \frac{1}{\alpha}$

It follows that

$$0 \leq \int_{Q_\tau} nT_n\left(u_n - \zeta\right)^- \frac{T_k\left(u_n\right)^+}{k} \exp\left(G\left(u_n\right)\right) dxdt \leq c_1,$$

we deduce by Fatou's lemma as $k \to 0$ that

$$0 \leq \int_{\{u_n \geq 0\}} nT_n\left(u_n - \zeta\right)^- \exp\left(G\left(u_n\right)\right) dxdt \leq c_1.$$

Return to (39) we deduce easily

$$\int_{\{0 \leq u_n \leq k\}} a\left(x, t, u_n, \nabla u_n\right) \exp\left(G\left(u_n\right)\right) \nabla T_k\left(u_n\right) dxdt \leq kc_1c_2.$$

And as one has $\exp\left(G\left(u_n\right)\right) \geq 1$ for $0 \leq u_n \leq k$, then

$$\int_{\{0 \leq u_n \leq k\}} a\left(x, t, u_n, \nabla u_n\right) \nabla T_k\left(u_n\right) dxdt \leq kc_1c_2. \tag{40}$$

Thanks to (39) we have

$$\alpha \int_{Q_\tau} \varphi\left(x, \left|\nabla T_k\left(u_n\right)^+\right|\right) \exp\left(G\left(u_n\right)\right) dxdt \leq c_1ck, \tag{41}$$

we deduce that,

$$\alpha \int_{Q} \varphi\left(x, \left|\nabla T_k\left(u_n\right)^+\right|\right) dxdt \leq c_1c_2k, \tag{42}$$

and

$$0 \leq \int_{\{u_n \geq 0\}} nT_n\left(u_n - \zeta\right)^- \left(u_n\right) dxdt \leq c_1. \tag{43}$$

Now, by choosing $\exp\left(-G\left(u_n\right)\right) T_k\left(u_n\right)^- \chi_{(0,\tau)}$ in (32) with $k > 0$ and for every $\tau \in [0, T]$, we get

$$\begin{aligned}
&\int_0^T \left\langle \frac{\partial b_n\left(u_n\right)}{\partial t}; \exp\left(-G\left(u_n\right)\right) T_k\left(u_n\right)^- \chi_{(0,\tau)} dt \right\rangle \\
&- \int_{Q_\tau} a\left(x, t, u_n, \nabla u_n\right) \nabla u_n \exp\left(-G\left(u_n\right)\right) \chi_{\{-k \leq u_n \leq 0\}} dxdt \\
&+ \int_{Q_\tau} P(x,t) T_k\left(u_n\right)^- \exp\left(-G\left(u_n\right)\right) dxdt \\
&+ \int_{Q_\tau} nT_n\left(u_n - \zeta\right)^- T_k\left(u_n\right)^- \exp\left(-G\left(u_n\right)\right)\left(u_n\right) dxdt \\
&\geq \int_{Q_\tau} f_n T_k\left(u_n\right)^- \exp\left(-G\left(u_n\right)\right) dxdt,
\end{aligned} \tag{44}$$

we deduce that

$$\begin{aligned}
&\int_{Q_\tau} a\left(x,t,u_n,\nabla u_n\right)\nabla u_n \exp\left(-G\left(u_n\right)\right)\chi_{(-k\leq u_n\leq 0\}}dxdt\\
&\quad-\int_{Q_\tau} nT_n\left(u_n-\zeta\right)^- T_k\left(u_n\right)^- \exp\left(-G\left(u_n\right)\right)dxdt\\
&\leq k\exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right)\left(\|\gamma\|_{L^1(Q)}+\|f_n\|_{L^1(Q)}\right)\\
&\quad+\int_0^T\left\langle\frac{\partial b_n\left(u_n\right)}{\partial t};\exp\left(-G\left(u_n\right)\right)T_k\left(u_n\right)^-\chi_{(0,\tau)}\right\rangle dt.
\end{aligned}\tag{45}$$

Similarly to (36) we obtain

we take

$$\widetilde{T}_k(r)=\int_0^r \exp(-G(s))T_k(s)^- ds,$$

then

$$\int_{Q_\tau}\frac{\partial b_n(u_n)}{\partial t}\exp\left(G\left(u_n\right)\right)T_k\left(u_n\right)^+dxdt=\int_\Omega \widetilde{T}_k\left(b_n(u_n(\tau)\right)dx-\int_\Omega \widetilde{T}_k\left(b_n(u_n(0))\right)dx.\tag{46}$$

By definition we have

$$\int_\Omega \widetilde{T}_k\left(b_n(u_n(\tau)\right)dx\geq 0,\tag{47}$$

and

$$\int_\Omega \widetilde{T}_k\left(b_n(u_n(0))\right)dx\leq k\exp\left(\frac{\|\rho\|_{L^1}}{\alpha}\right)\|b(u_0)\|_{L^1(\Omega)},\tag{48}$$

and using same techniques, we obtain also

$$\begin{aligned}
&\int_{Q_\tau} a\left(x,t,u_n,\nabla u_n\right)\exp\left(-G\left(u_n\right)\right)\nabla T_k\left(u_n\right)dxdt\\
&\quad+c_2\int_{Q_\tau} nT_n(u-\zeta)^-\exp\left(-G\left(u_n\right)\right)T_k\left(u_n\right)^-dxdt\leq kc_1c_2
\end{aligned}.$$

It follow that

$$0\leq\int_{Q_\tau} nT_n\left(u_n-\zeta\right)^-\exp\left(-G\left(u_n\right)\right)\frac{T_k\left(u_n\right)^-}{k}dxdt\leq c_1,$$

we deduce by Fatou's lemma as $k\to 0$ that

$$0\leq\int_{\{u_n\leq 0\}} nT_n\left(u_n-\zeta\right)^-\exp\left(-G\left(u_n\right)\right)dxdt\leq c_1.$$

And as one has $\exp\left(-G\left(u_n\right)\right)\geq 1$ since $-k\leq u_n\leq 0$, then

$$\int_{\{-k\leq u_n\leq 0\}} a\left(x,t,u_n,\nabla u_n\right)\nabla T_k\left(u_n\right)dxdt\leq kc_1c_2,\tag{49}$$

$$\int_{Q_\tau} \varphi\left(x, \left|\nabla T_k\left(u_n\right)^-\right|\right) dxdt \leq \frac{kc_1c_2}{\alpha}, \tag{50}$$

and

$$0 \leq \int_{\{u_n \leq 0\}} nT_n\left(u_n - \zeta\right)^- dxdt \leq c_1. \tag{51}$$

Combining now (40) and (49) we get,

$$\int_Q a\left(x,t,u_n,\nabla u_n\right) \nabla T_k\left(u_n\right) dxdt \leq kC_1. \tag{52}$$

Of the same with (42) and (50) we get,

$$\int_Q \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right) dxdt \leq kC_2. \tag{53}$$

we conclude that $T_k\left(\left(u_n\right)\right)$ is bounded in $W_0^{1,x}L_\varphi\left(Q\right)$ independently of n and for any $k>0$, so there exists a subsequence still denoted by u_n such that

$$T_k\left(u_n\right) \rightharpoonup \xi_k \quad \text{weakly in} \quad W_0^{1,x}L_\varphi\left(Q\right) \tag{54}$$

On the other hand, using (53), we have

$$\begin{aligned} \inf_{x\in\Omega} \varphi\left(x, \frac{k}{\delta}\right) \operatorname{meas}\left\{|u_n| > k\right\} &\leq \int_{|u_n|>k} \varphi\left(x, \frac{\left|T_k\left(u_n\right)\right|}{\delta}\right) dxdt \\ &\leq \int_Q \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right) dxdt \leq kC_2 \end{aligned}$$

Then

$$\operatorname{meas}\left\{|u_n| > k\right\} \leq \frac{kC_2}{\inf_{x\in\Omega} \varphi\left(x, \frac{k}{\delta}\right)}$$

for all n and for all k.

Assuming that there exists a positive function M such that $\lim_{t\to\infty} \frac{M(t)}{t} = +\infty$ and $M(t) \leq ess \inf_{x\in\Omega}\varphi(x,t), \forall t \geq 0$. Thus, we get

$$\lim_{k\to\infty} \operatorname{meas}\left\{|u_n| > k\right\} = 0 \tag{55}$$

Let $\lambda > 0$ then

$$\begin{gathered} \operatorname{meas}\{\ \{u_m - u_n| > \lambda\} \leq \operatorname{meas}\left\{|u_m| > k\right\} \\ +\operatorname{meas}\{\ \{u_n| > k\} + \operatorname{meas}\left\{\left|T_k\left(u_m\right) - T_k\left(u_n\right)\right| > \lambda\right\} \end{gathered}$$

By (54) we can assume that $T_k\left(u_n\right)$ is a Cauchy sequence in measure in Q_T and using (55) we deduce that for any $\epsilon > 0$ there exists some $k(\epsilon) > 0$ such that

$$\operatorname{meas}\left\{|u_m - u_n| > \lambda\right\} \leq \epsilon \quad \text{for all} \quad n,m > N_{k(\epsilon),\lambda}$$

Which means that u_n is a Cauchy sequence in measure in Q, thus converge almost every where to some measurable function u

For $k < n$, let $g_k \in W^{2,\infty}(\mathbb{R})$, such that g_k', has a compact support $\operatorname{supp}(g_k') \subset [-k,k]$. We multiply (32) by $g_k'(u_n)$, to obtain in $\mathcal{D}'(Q)$

$$\frac{\partial B^n_{gk}(u_n)}{\partial t} = div\left(g_k'(u_n)\left(a_n(u_n,\nabla u_n)\right)\right) - g_k''(u_n)\left(a_n(u_n,\nabla u_n)\right) + f_n g_k'(u_n), \tag{56}$$

where $B^n_{gk}(r) = \displaystyle\int_0^r g_k'(s)\frac{\partial b_n(s)}{\partial s}ds$.

Then, we show that

$$\left(B^n_{gk}(u_n)\right) \text{ is bounded in } W_0^{1,x}L_\varphi(Q), \tag{57}$$

and

$$\left(\frac{\partial B^n_{g_k}(u_n)}{\partial t}\right) \text{ is bounded in } L^1(Q) + W^{-1,x}L_\psi(Q). \tag{58}$$

independently of n.

Indeed, first we have

$$\left|\nabla B^n_{g_t}(u_n)\right| \le b_1\left|\nabla T_k(u_n)\right| \|g_k'\|_{L^\infty(R)} \text{ a.e. in } Q,$$

and using (52) we obtain (57). To show that (58) holds true, since supp(g_k') and supp (g_k'') are both included in $[-k,k], u_n$ may be replaced by $T_k(u_n)$ in each of these terms. As a consequence, each term in the right hand side of (56) is bounded either in $W^{-1,x}L_\psi(Q)$ or in $L^1(Q)$ which shows that (58) holds true.

Arguing again as in (16) estimates (57),(58) and the following remark, imply that, for a subsequence, still indexed by n

$$b_n(u_n) \to b(u) \text{ a.e in } Q, \quad b(u) \in L^\infty\left(0,T,L^1(\Omega)\right), \tag{59}$$

where u is a measurable function defined on Q.

Remark 6.1.

For every $g \in W^{2,\infty}(\mathbb{R})$, nondecreasing function such that supp(g') $\subset [-k,k]$ and (14), we have $b_0\left|g(r) - g(r')\right| \le \left|B_g(r) - B_g(r')\right| \le b_1\left|g(r) - g(r')\right|$ for every in $\mathbb{R}$.

Step 3 : Boundedness of $a(x,t,T_k(u_n),\nabla T_k(u_n))$ ***in*** $(L_\psi(Q))^N$.

Let $\lambda > 0$ then,

$$\begin{gathered}\text{meas }\{|u_m - u_n| > \lambda\} \leq \text{meas}\{|u_m| > k\} \\ +\text{meas}\{|u_n| > k\} + \text{meas}\{|T_k(u_m) - T_k(u_n)| > \lambda\}\end{gathered}$$

By (54) we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Q and using (55) we deduce that for any $\epsilon > 0$ there exists some $k(\epsilon) > 0$ such that

$$\text{meas}\{|u_m - u_n| > \lambda\} \leq \epsilon \quad \text{for all} \quad n, m > N_{k(\epsilon),\lambda}$$

Which means that u_n is a Cauchy sequence in measure in Q, thus converge almost every where to some measurable function u.

Let $w \in (E_\varphi(\Omega))^N$ be arbitrary. By condition (16) we have,

$$(a(x,t,u_n,\nabla u_n) - a(x,t,u_n,w))(\nabla u_n - w) > 0$$

then

$$\begin{aligned}\int_{\{|u_n|\leq k\}} a(x,t,u_n,\nabla u_n)\, w dxdt \leq & \int_{\{|u_n|\leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \\ & + \int_{\{|u_n|\leq k\}} a(x,t,u_n,w)(w - \nabla u_n)\, dxdt.\end{aligned}$$

by (15) we have for $\nu > \beta$

$$\begin{aligned}\int_{\{|u_n|\leq k\}} \psi_x\left(x, \frac{a\left(x,t,u_n,\frac{w}{k_2}\right)}{3\nu}\right) dxdt \leq & \frac{\beta}{3\nu}\int_Q [\psi(x,a_0(x,t)) + \gamma(x,k_1|T_k(u_n))]\, dxdt \\ & + \frac{\beta}{3\nu}\int_Q [\varphi(x,|w|)] dxdt \\ \leq & \frac{\beta}{3\nu}\left[\int_Q \psi(x,a_0(x,t)) + \gamma(x,k_1 k)\, dxdt\right] \\ & + \frac{\beta}{3\nu}\left[\int_Q \varphi(x,|w|) dxdt\right].\end{aligned} \tag{60}$$

Thus $\left\{a\left(x,t,T_k(u_n),\frac{w}{k_2}\right)\right\}$ is bounded in $(L_\psi(\Omega))^N$. By (52), (60) and by the theorem of Banach-Steinhaus, the sequence $\{a(x,t,T_k(u_n),\nabla T_k(u_n))\}$ remains bounded in $(L_\psi(\Omega))^N$ and we conclude,

(61)
$$a_n(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varpi_k \text{ in } (L_\psi(Q))^N, \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi) \text{ for some } \varpi_k \in (L_\psi(Q))^N.$$

Consequently,

$$T_k(u_n) \rightharpoonup \text{ weakly } T_k(u) \text{ in } W_0^{1,x}L_\varphi(Q) \text{for the topology } \sigma\left(\prod L_\varphi, \prod E_\psi\right). \tag{62}$$

Step 4 : Almost everywhere convergence of the gradients.

Taking $Z_m(u_n) = T_1(u_n - T_m(u_n))^-$ as a test function in the approximate problem (32) we get

$$\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n dxdt \le C\Big(\int_Q f_n Z_m(u_n)\,dxdt + \int_{\{|u_{0n}|>m\}} |b_n(u_{0n})|\,dxdt\Big)$$

where $C > 0$.

Passing to the limit as $n \to +\infty$, using the pointwise convergence of u_n and strongly convergence in $L^1(Q)$ of f_n we get

$$\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a_n(x,t,u_n,\nabla u_n)\nabla u_n dxdt \le C\Big(\int_Q fZ_m(u)dxdt + \int_{\{|u_0|>m\}} |b(u_0)|\,dxdt\Big).$$

Owning to Lebesgue's theorem and passing to the limit as $m \to +\infty$, in the all terms of the right-hand side, we get

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{63}$$

Finally, for the almost everywhere convergence of the gradients we use the following lemma

Lemma 6.1. *Under the Assumptions* (14)-(20)*, let* (z_n)*be a sequence in* $W_0^{1,x}L_\varphi(Q)$ *such that:*

$$z_n \to z \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \tag{64}$$

$$(a(x,t,z_n,\nabla z_n)) \text{ is bounded in } (L_\psi(Q))^N \tag{65}$$

$$\int_Q [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dxdt \to 0 \tag{66}$$

as n *and* s *tend to* $+\infty$*, and where* χ_s *is the characteristic function of* $Q^s = \{x \in Q; |\nabla z| \le s\}$.

Then,

$$\nabla z_n \to \nabla z \text{ a.e. in } Q \tag{67}$$

$$\lim_{n\to+\infty}\int_Q a(x,t,z_n,\nabla z_n)\nabla z_n dxdt = \int_Q a(x,t,z,\nabla z)\nabla z dxdt \tag{68}$$

$$\varphi(x,|\nabla z_n|) \to \varphi(x,|\nabla z|) \text{ in } L^1(Q). \tag{69}$$

Proof: It is easily adapted from that given in [11].

Let $v_j \in D(Q)$ be a sequence such that $v_j \to u$ in $W_0^{1,x} L_\varphi(Q)$ for the modular convergence. This specific time regularization of $T_k(v_j)$ (for fixed $k \geq 0$) is defined as follows. Let $(\alpha_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$\alpha_0^\mu \in L^\infty(\Omega) \cap W_0^1 L_M(\Omega) \text{ for all } \mu > 0 \tag{70}$$

$$\|\alpha_0^\mu\|_{L^\infty(\Omega)} \leq k \quad \forall \mu > 0$$

$$\alpha_0^\mu \to T_k(u_0) \text{ a.c. in } \Omega \text{ and } \frac{1}{\mu}\|\alpha_0^\mu\|_{M,\Omega} \to 0, \text{ as } \mu \to +\infty$$

For fixed $k, \mu > 0$, let us consider the unique solution $T_k(v_j)_\mu \in L^\infty(Q) \cap W_0^{1,x} L_M(Q)$ of the monotone problem:

$$\begin{gathered}\frac{\partial T_k(v_j)_\mu}{\partial t} + \mu\left(T_k(v_j)_\mu - T_k(v_j)\right) = 0 \text{ in } D'(Q), \\ T_K(v_j)_\mu(t=0) = \alpha_0^\mu \text{ in } \Omega.\end{gathered} \tag{71}$$

Remark that due to,

$$\frac{\partial T_k(v_j)_\mu}{\partial t} \in W_0^{1,x} L_\varphi(Q)$$

We just recall that,

$$(T_k(v_j))_\mu \to T_k(u) \text{ a.e. in } Q, \text{ weakly-* in } L^\infty(Q)$$

$(T_k(v_j))_\mu \to (T_k(u))_\mu$ in $W_0^{1,x} L_\varphi(Q_T)$ for the modular convergence as $j \to +\infty$.

$(T_k(u))_\mu \to T_k(u)$ in $W_0^{1,x} L_\varphi(Q_T)$ for the modular convergence as $\mu \to +\infty$.

$\left\|(T_k(v_j))_\mu\right\|_{L^\infty(QT)} \leq \max\left(\|(T_k(u))\|_{L^\infty(Q_T)}, \quad \|\alpha_0^\mu\|_{L^\infty(\Omega)}\right) \leq k$ for all $\mu > 0$, and for all $k > 0$. We introduce a sequence of increasing $\mathbf{C}^1(\mathbb{R})-$ functions S_m such that

$$S_m(r) = 1 \text{ for } |r| \leq m, S_m(r) = m + 1 - |r|, \text{ for } m \leq |r| \leq m+1, S_m(r) = 0$$

for $|r| \geq m+1$ for any $m \geq 1$. And we denote by $\epsilon(n, \mu, \eta, j, m)$ the quantities such that

$$\lim_{m \to +\infty} \lim_{j \to +\infty} \lim_{\eta \to +\infty} \lim_{\mu \to +\infty} \lim_{n \to +\infty} \epsilon(n, \mu, \eta, j, m) = 0.$$

The main estimate is

Lemma 6.2. *We have*

$$\int_0^T \left\langle \frac{\partial b_n(u_n)}{\partial t}, T_\eta\left(u_n - (T_k(v_j))_\mu\right)^+ \exp(G(u_n)) S_m'(u_n)\right\rangle \geq w(n, \mu, \eta, j), \quad \forall m \geq 1$$

Proof : See [4].

For fixed $k \geq 0$, let $W_{\mu,\eta}^{n,j} = T_\eta \left(T_k(u_n) - T_k(v_j)_\mu\right)^+$ and $W_{\mu,\eta}^{j} = T_\eta(T_k(u) - T_k(v_j)_\mu)^+$

Multiplying the approximating equation by $\exp(G(u_n)))\, W_{\mu,\eta}^{n,j} S_m(u_n)$ and using the same technique in step 2 we obtain:

$$\left\langle \frac{\partial b_n(u_n)}{\partial t}, \exp(G(u_n))\, W_{\mu,\eta}^{n,j} S_m(u_n) \right\rangle \tag{72}$$

$$+ \int_Q a_n(x,t,u_n,\nabla u_n) \exp(G(u_n))\, \nabla\left(W_{\mu,\eta}^{n,j}\right) S_m(u_n)\, dxdt \tag{73}$$

$$+ \int_Q a_n(x,t,u_n,\nabla u_n)\, \nabla u_n \exp(G(u_n))\, W_{\mu,\eta}^{n,j} S'_m(u_n)\, dxdt \tag{74}$$

$$\leq \int_Q f_n \exp(G(u_n))\, W_{\mu,\eta}^{n,j} S_m(u_n)\, dxdt + \int_Q P(x,t) \exp(G(u_n))\, W_{\mu,\eta}^{n,j} S_m(u_n)\, dxdt \tag{75}$$

Now we pass to the limit in (72),(73),(74),and in (75) for k real number fixed.

By lemma 6.2 we have for any fixed $k \geq 0$

$$\int_Q \frac{\partial b_n(u_n)}{\partial t} \exp(G(u_n))\, W_{\mu,\eta}^{n,j} S_m(u_n)\, dxdt \geq \epsilon(n,\mu,\eta,j) \quad \text{for any } m \geq 1 \tag{76}$$

For the term (74):
we have

$$\begin{aligned}
&\int_Q a_n(x,t,u_n,\nabla u_n)\, S'_m(u_n)\, \nabla u_n \exp(G(u_n)) \exp(G(u_n))\, W_{\mu,\eta}^{n,j} dxdt \\
&= \int_{m \leq |u_n| \leq m+1} a_n(x,t,u_n,\nabla u_n)\, S'_m(u_n)\, \nabla u_n \exp(G(u_n))\, W_{\mu,\eta}^{n,j} dxdt \\
&\leq \eta C \int_{m \leq |u_n| \leq m+1} a_n(x,t,u_n,\nabla u_n)\, \nabla u_n dxdt
\end{aligned}$$

Using (63), we get

$$\int_Q a_n(x,t,u_n,\nabla u_n)\, S'_m(u_n)\, \nabla u_n \exp(G(u_n))\, W_{\mu,\eta}^{n,j} dxds \leq \epsilon(n,\mu,m).$$

Concerning the term (75):
Since $S_m(r) \leq 1$ and $W_{\mu,\eta}^{n,j} \leq \eta$ we get

$$\int_Q f_n S_m(u_n) \exp(G(u_n))\, W_{\mu,\eta}^{n,j} \quad dxdt \leq \epsilon(n,\eta), \tag{77}$$

$$\int_Q P(x,t)\exp\left(G\left(u_n\right)\right)W_{\mu,\eta}^{n,j}S_m\left(u_n\right)dxdt \leq C\eta. \tag{78}$$

For the term (73):

$$\begin{aligned}
&\int_Q a_n\left(x,t,u_n,\nabla u_n\right)S_m\left(u_n\right)\exp\left(G\left(u_n\right)\right)\nabla W_{\mu,\eta}^{n,j}dxdt\\
=&\int_{\left\{[u_n|\leq k\right\}\cap\left\{0\leq T_k(u_n)-T_k(v_j)_\mu\right)\leq\eta\right\}} a_n\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)S_m\left(u_n\right)\exp\left(G\left(u_n\right)\right)\\
&\times\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)_\mu\right)dxdt\\
&-\int_{\{|u_n|>k\}\cap\left\{0\leq T_k(u_n)-T_k(v_j)_\mu\right)\leq\eta\right\}} a_n\left(x,t,u_n,\nabla u_n\right)S_m\left(u_n\right)\\
&\times\exp\left(G\left(u_n\right)\right)\nabla T_k\left(v_j\right)_\mu dxdt
\end{aligned} \tag{79}$$

since $a_n\left(x,t,T_{k+\eta}\left(u_n\right),\nabla T_{k+\eta}\left(u_n\right)\right)$ is bounded in $\left(L_\psi\left(Q\right)\right)^N$, there exist some $\varpi_{k+\eta}\in\left(L_\psi\left(Q\right)\right)^N$ such that

$$a_n\left(x,t,T_{k+\eta}\left(u_n\right),\nabla T_{k+\eta}\left(u_n\right)\right)\rightharpoonup\varpi_{k+\eta}\text{weakly in }\left(L_\psi\left(Q\right)\right)^N.$$

Consequently,

$$\begin{aligned}
&\int_{\{|u_n|>k\}\cap\left\{0\leq T_k(u_n)-T_k(v_j)_\mu\right)\leq\eta\right\}} a_n\left(x,t,u_n,\nabla u_n\right)S_m\left(u_n\right)\exp\left(G\left(u_n\right)\right)\nabla T_k\left(v_j\right)_\mu dxdt\\
=&\int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(v_j)\mu)\leq\eta\}} S_m(u)\exp(G(u))\nabla T_k\left(v_j\right)_\mu\varpi_{k+\eta}dxdt+\epsilon(n)
\end{aligned} \tag{80}$$

where we have used the fact that

$$\begin{aligned}
&S_m\left(u_n\right)\exp\left(G\left(u_n\right)\right)\nabla T_k\left(v_j\right)_\mu\Big)\chi_{\{|u_n|>k\}\cap\left\{0\leq T_k(u_n)-T_k(v_j)_\mu\right)\leq\eta\right\}}\\
&\rightarrow S_m(u)\exp(G(u))\nabla T_k\left(v_j\right)_\mu\Big)\chi_{\{|u|>k\}\cap\left\{0\leq T_k(u)-T_k(v_j)_\mu\right)\leq\eta\right\}}
\end{aligned}$$

strongly in $\left(E_\varphi\left(Q\right)\right)^N$.

Letting $j\to+\infty$, we obtain

$$\begin{aligned}
&\int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(v_j)\mu)\leq\eta\}} S_m(u)\exp(G(u))\nabla T_k\left(v_j\right)_\mu\varpi_{k+\eta}dxdt\\
=&\int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(u)_\mu)\leq\eta\}} S_m(u)\exp(G(u))\nabla T_k(u)_\mu\varpi_{k+\eta}dxdt+\epsilon(n,j)
\end{aligned}$$

One easily has,

$$\int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(u)_\mu)\leq\eta\}} S_m(u)\exp(G(u))\nabla T_k(u)_\mu\varpi_{k+\eta}dxdt=\epsilon(n,j,\mu)$$

By (72)-(80) we obtain

$$\int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j)\mu)|\leq\eta\}} a_n\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)S_m\left(u_n\right)\exp\left(G\left(u_n\right)\right)$$

$$\times\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)_\mu\right)dxdt\leq C\eta+\epsilon(n,j,\mu,m),$$

we know that $\exp\left(G\left(u_n\right)\right)\geq 1$ and $S_m\left(u_n\right)=1$ for $|u_n|\leq k$ then,

$$\int_{\{|u_n|\leq k\}\cap\left\{0\leq T_k(u_n)-T_k(v_j)_\mu\right)|\leq\eta\}} a_n\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)_\mu\right)dxdt$$

$$\leq C\eta+\epsilon(n,j,\mu,m). \tag{81}$$

Now, let us prove that:

$$\int_Q\left[a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,t,T_k\left(u_n\right),\nabla T_k(u)\right)\right]\left[\nabla T_k\left(u_n\right)-\nabla T_k(u)\right]dxdt\to 0 \tag{82}$$

Setting for $s>0, Q^s=\{(x,t)\in Q:|\nabla T_k(u)|\leq s\}$ and $Q_j^s=\{(x,t)\in Q:|\nabla T_k\left(v_j\right)|\leq s\}$ and denoting by χ^s and χ_j^s the characteristic functions of Q^s and $\bar{Q}_j^s$ respectively, we deduce that letting $0<\delta<1$, define

$$\Theta_{n,k}=\left(a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,t,T_k\left(u_n\right),\nabla T_k(u)\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k(u)\right)$$

For $s>0$, we have

$$\begin{aligned}0&\leq\int_{Q^s}\Theta_{n,k}^\delta dxdt\\&=\int_{Q^s}\Theta_{n,k}^\delta\chi_{\left|T_k(u_n)-T_k(v_j)_\mu\right|\leq\eta)}dxdt\\&\quad+\int_{Q^s}\Theta_{n,k}^\delta\chi_{\left|T_k(u_n)-T_k(v_j)_\mu\right|>\eta)}dxdt.\end{aligned}$$

The first term of the right-side hand, with the Holder inequality,

$$\begin{aligned}\int_{Q^s}\Theta_{n,k}^\delta\chi_{\left|T_k(u_n)-T_k(v_j)_\mu\right|\leq\eta)}dxdt&\leq\left(\int_{Q^s}\Theta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)\mu|\leq\eta)}dxdt\right)^\delta\left(\int_{Q^s}dxdt\right)^{1-\delta}\\&\leq C_1\left(\int_{Q^s}\Theta_{n,k}\chi_{|T_k(u_n)-T_k(v_j)\mu|\leq\eta)}dxdt\right)^\delta.\end{aligned}$$

Also using the Holder inequality, the second term of the right-side hand is

$$\int_{Q^s}\Theta_{n,k}^\delta\chi_{|T_k(u_n)-T_k(v_j)\mu|>\eta)}dxdt\leq\left(\int_{Q^s}\Theta_{n,k}dxdt\right)^\delta\left(\int_{|T_k(u_n)-T_k(v_j)\mu|>\eta)}dxdt\right)^{1-\delta},$$

since $a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)$ is bounded in $\left(L_\psi\left(Q\right)\right)^N$, while $\nabla T_k\left(u_n\right)$ is bounded in $\left(L_\varphi\left(Q\right)\right)^N$, then,

$$\int_{Q^s} \Theta_{n,k}^{\delta} \chi_{|T_k(u_n)-T_k(v_j)\mu|>\eta)} dxdt \leq C_2 \operatorname{meas}\left\{(x,t) \in Q : \left|T_k(u_n) - T_k(v_j)_\mu\right| > \eta\right\}^{1-\delta}.$$

We obtain,

$$\begin{aligned}\int_{Q^s} \Theta_{n,k}^{\delta} dxdt \leq C_1 &\left(\int_{Q^*} \Theta_{n,k} \chi_{\left|T_k(u_n)-T_k(v_j)_\mu\right| \leq \eta)} dxdt\right)^{\delta} \\ &+ C_2 \operatorname{meas}\left\{(x,t) \in Q : \left|T_k(u_n) - T_k(v_j)_\mu\right| > \eta\right\}^{1-\delta}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{Q^s} \Theta_{n,k} \chi_{\left|T_k(u_n)-T_k(v_j)_\mu\right| \leq \eta)} dxdt \\ &\leq \int_{\left|T_k(u_n)-T_k(v_j)_\mu\right| \leq \eta)} \left(a\left(x,t,T_k(u_n), \nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n), \nabla T_k(u)\chi_s\right)\right) \\ &\qquad \times \left(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\right) dxdt.\end{aligned}$$

For each $s > r, r > 0$, one has

$$\begin{aligned}0 &\leq \int_{Q^r \cap \left\{\left|T_k(u_n)-T_k(v_j)_\mu\right| \leq \eta)\right\}} \left(a\left(x,t,T_k(u_n), \nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n), \nabla T_k(u)\right)\right) \\ &\qquad \times \left(\nabla T_k(u_n) - \nabla T_k(u)\right) dxdt \\ &\leq \int_{Q^s \cap \{|T_k(u_n)-T_k(v_j)\mu| \leq \eta)\}} \left(a\left(x,t,T_k(u_n), \nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n), \nabla T_k(u)\right)\right) \\ &\qquad \times \left(\nabla T_k(u_n) - \nabla T_k(u)\right) dxdt \\ &= \int_{Q^e \cap \{|T_k(u_n)-T_k(v_j)\mu| \leq \eta)\}} \left(a\left(x,t,T_k(u_n), \nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n), \nabla T_k(u)\chi_s\right)\right) \\ &\qquad \times \left(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\right) dxdt \\ &\leq \int_{Q \cap \{|T_k(u_n)-T_k(v_j)\mu| \leq \eta)\}} \left(a\left(x,t,T_k(u_n), \nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n), \nabla T_k(u)\chi^s\right)\right) \\ &\qquad \times \left(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\right) dxdt \\ &= \int_{\left|T_k(u_n)-T_k(v_j)_\mu\right| \leq \eta} \left(a\left(x,t,T_k(u_n), \nabla T_k(u_n)\right) - a\left(x,t,T_k(u_n), \nabla T_k(v_j)\chi_j^s\right)\right) \\ &\qquad \times \left(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right) dxdt \\ &\quad + \int_{|T_k(u_n)-T_k(v_j)\mu| \leq \eta} a\left(x,t,T_k(u_n), \nabla T_k(u_n)\right)\left(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\right) dxdt \\ &\quad + \int_{\left|T_k(u_n)-T_k(v_j)_\mu\right| \leq \eta} \left(a\left(x,t,T_k(u_n), \nabla T_k(v_j)\chi_j^s\right) - a\left(x,t,T_k(u_n), \nabla T_k(u)\chi^s\right)\right) \\ &\qquad \nabla T_k(u_n) dxdt \\ &\quad - \int_{\left|T_k(u_n)-T_k(v_j)_\mu\right| \leq \eta} a\left(x,t,T_k(u_n), \nabla T_k(v_j)\chi_j^s\right) \nabla T_k(v_j)\chi_j^s dxdt\end{aligned}$$

$$+\int_{|T_k(u_n)-T_k(v_j)\mu|\leq\eta} a\left(x,t,T_k\left(u_n\right),\nabla T_k(u)\chi^s\right)\nabla T_k(u)\chi^s dxdt$$
$$=I_1(n,j,s)+I_2(n,j)+I_3(n,j)+I_4(n,j,\mu)+I_5(n,\mu).$$

We go to the limit as n, j, μ, and $s\to+\infty$

$$I_1=\int_{|T_k(u_n)-T_k(v_j)\mu|\leq\eta} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)_\mu\right)dxdt$$
$$-\int_{|T_k(u_n)-T_k(v_j)\mu|\leq\eta} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k\left(v_j\right)_\mu\right)dxdt$$
$$-\int_{\left|T_k(u_n)-T_k(v_j)_\mu\right|\leq\eta} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right))dxdt.$$

Using (81), the first term of the right-hand side, we get

$$\int_{|T_k(u_n)-T_k(v_j)|\leq\eta} a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)_\mu\right)dxdt$$
$$\leq C\eta+\epsilon(n,m,j,s)-\int_{|u|>k\cap|T_k(u)-T_k(v_j)\mu|\leq\eta} a\left(x,t,T_k(u),0\right)\nabla T_k\left(v_j\right)_\mu dxdt$$
$$\leq C\eta+\epsilon(n,m,j,\mu).$$

The second term of the right-hand side tends to

$$\int_{|T_k(u)-T_k(v_j)|\leq\eta}\varpi_k\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k\left(v_j\right)_\mu\right)dxdt,$$

since $a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)$ is bounded in $\left(L_\psi\left(Q\right)\right)^N$, there exist some $\varpi_k\in\left(L_\psi\left(Q\right)\right)^N$ such that (for a subsequence still denoted by u_n

$$a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\to\varpi_k\quad\text{in }\left(L_\varphi\left(Q\right)\right)^N\text{ for }\quad\sigma\left(\Pi L_\psi,\Pi E_\varphi\right).$$

In view of the fact that

$$\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k\left(v_j\right)_\mu\right)\chi_{|T_k(u_n)-T_k(v_j)\mu|\leq\eta}$$
$$\to\left(\nabla T_k\left(v_j\right)\chi_j^s-\nabla T_k\left(v_j\right)_\mu\right)\chi_{|T_k(u)-T_k(v_j)|\leq\eta},$$

strongly in $\left(E_\varphi\left(Q\right)\right)^N$ as $n\to+\infty$.

The third term of the right-hand side tends to

$$\int_{\left|T_k(u)-T_k(v_j)_\mu\right|\leq\eta} a\left(x,t,T_k(u),\nabla T_k\left(v_j\right)\chi_j^s\right)\left(\nabla T_k(u)-\nabla T_k\left(v_j\right)\chi_j^s\right)dxdt,$$

since

$$a\left(x,t,T_k\left(u_n\right),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\chi_{\left|T_k(u_n)-T_k(v_j)_\mu\right|\leq\eta}$$
$$\to a\left(x,t,T_k(u),\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\chi_{\left|T_k(u)-T_k(v_j)_\mu\right|\leq\eta}$$

in $\left(E_\psi\left(Q\right)\right)^N$. while

$$\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\chi_j^s\right)\right)\to\left(\nabla T_k(u)-\nabla T_k\left(v_j\right)\chi_j^s\right)\right)$$

in $(L_\varphi(Q))^N$ for $\sigma(\Pi L_\psi, \Pi E_\varphi)$ Passing to limit as $j \to +\infty$ and $\mu \to +\infty$ and using Lebesgue's theorem, we have

$$I_1 \leq C\eta + \epsilon(n, j, s, \mu)$$

For what concerns I_2, by letting $n \to +\infty$, we have

$$I_2 \to \int_{\left|T_k(u)-T_k(v_j)_\mu\right| \leq \eta)} \varpi_k \left(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\right) dxdt,$$

since $a(x,t,T_k(u_n), \nabla T_k(u_n)) \to \varpi_k$ in $(L_\psi(Q))^N$, for $\sigma(\Pi L_\psi, \Pi E_\varphi)$, while

$$\begin{aligned} &\left(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\right)\chi_{\left|T_k(u_n)-T_k(v_j)_\mu\right|\leq\eta} \\ &\to \left(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\right)\chi_{\left|T_k(u)-T_k(v_j)_\mu\right|\leq\eta}, \end{aligned}$$

strongly in $(E_\varphi(Q))^N$.

Passing to limit $j \to +\infty$, and using Lebesgue's theorem, we have

$$I_2 = \epsilon(n, j).$$

Similar ways as above give

$$\begin{aligned} I_3 &= \epsilon(n,j). \\ I_4 &= \int_{|T_k(u)-T_k(u)_\mu| \leq \eta)} a(x,t,T_k(u), \nabla T_k(u))\nabla T_k(u)dxdt + \epsilon(n,j,\mu,s,m). \\ I_5 &= \int_{|T_k(u)-T_k(u)_\mu| \leq \eta)} a(x,t,T_k(u), \nabla T_k(u))\nabla T_k(u)dxdt + \epsilon(n,j,\mu,s,m). \end{aligned}$$

Finally, we obtain,

$$\int_{Q^*} \Theta_{n,k} dxdt \leq C_1(C\eta + \epsilon(n,\mu,\eta,m))^\delta + C_2(\epsilon(n,\mu,))^{1-\delta}.$$

Which yields, by passing to the limit sup over n, j, μ, s and η

$$\int_{Q^r} \left[(a(x,t,T_k(u_n), \nabla T_k(u_n)) - a(x,t,T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\right]^\delta dxdt = \epsilon(n).$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Q^r, and since r is arbitrary,

$$\nabla u_n \to \nabla u, \quad \text{a.e. in} \quad Q.$$

Step 5 : Equi-integrability of the nonlinearities.

We shall prove that $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ strongly in $L^1(\Omega)$.
Consider $\vartheta_0(u_n) = \int_0^{u_n} \rho(s)\chi_{\{s>h\}} ds$ and multiply (32) by $\exp(G(u_n))\vartheta_0(u_n)$, we get

$$\begin{aligned}\int_\Omega \widetilde{T}_h(b_n(u_n)(T))dx &+ \int_Q a(x,u_n,\nabla u_n)\nabla(\exp(G(u_n))\vartheta_0(u_n))\,dxdt \\ &+ \int_Q g_n(x,t,u_n,\nabla u_n)\exp(G(u_n))\vartheta_0(u_n)\Big)\,dxdt \\ \leq \left(\int_h^{+\infty}\rho(s)dx\right)\exp\left(\frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha}\right)&\Big[\|f\|_{L^1(Q)} + \|b(u_0)\|_{L^1(\Omega)} + \|P(.,.)\|_{L^1(Q)}\Big].\end{aligned}$$

$$\text{where } \widetilde{T}_h(r) = \int_0^r \vartheta_0(s)\exp(G(s))ds \geq 0,$$

then using same technique in previous step we can have

$$\int_{\{u_n>h\}} \rho(u_n)\varphi(x,\nabla u_n)\,dxdt \leq C\left(\int_h^{+\infty}\rho(s)dx\right).$$

since $\rho \in L^1(\mathbb{R})$, we get

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} \rho(u_n)\varphi(x,\nabla u_n)\,dxdt = 0$$

Similarly, let $\vartheta_0(u_n) = \int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}dx$ in (32) we have also

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} \rho(u_n)\varphi(x,\nabla u_n)\,dxdt = 0$$

We conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} \rho(u_n)\varphi(x,\nabla u_n)\,dxdt = 0. \tag{83}$$

Let $D \subset \Omega$ then

$$\begin{aligned}\int_D \rho(u_n)\varphi(x,\nabla u_n)\,dxdt \leq \max_{\{|u_n|\leq h\}}(\rho(x))&\int_{D\cap\{|u_n|\leq h\}}\varphi(x,\nabla u_n)\,dxdt \\ &+ \int_{D\cap\{|u_n|>h\}}\rho(u_n)\varphi(x,\nabla u_n)\,dxdt.\end{aligned}$$

Consequently $\rho(u_n)\varphi(x,\nabla u_n)$ is equi-integrable. Then $\rho(u_n)\varphi(x,\nabla u_n)$ converge to $\rho(u)\varphi(x,\nabla u)$ strongly in $L^1(\mathbb{R})$. By (18) we get

$$g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u) \text{ strongly in } L^1(Q). \tag{84}$$

Step 6 : Passage to the limit.

Now, we show that u satisfies (27)

Firstly show that $u \geq \zeta$ a.e. in Q from (43) and (51) we get

$$0 \leq \int_Q T_n (u_n - \zeta)^- \, dxdt \leq \frac{c_1}{n}$$

Let n tends to $+\infty$ we obtain

$$\int_Q (u - \zeta)^- dxdt = 0$$

then

$$(u - \zeta)^- = 0 \text{ a.e. in } Q.$$

Secondly passing Now to the limit in (85) to show that u satisfies the equation satisfies (27)

Let $v \in W_0^1 L_\varphi (Q) \cap L^\infty (Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x} L_\psi (Q) + L^1 (Q)$, then by theorem 4.1 we can take

$$\begin{gathered} \bar{v} = v \text{ on } Q \\ \bar{v} \in W^{1,x} L_\varphi(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R}) \\ \tfrac{\partial \bar{v}}{\partial t} \in W^{-1,x} L_\psi (Q) + L^1 (Q) \end{gathered}$$

and there exists $v_j \in \mathcal{D}(\Omega \times \mathbb{R})$ such that

$$v_j \to \bar{v} \quad \text{in} \quad W_0^{1,x} L_\varphi(\Omega \times \mathbb{R}) \quad \text{and} \quad \frac{\partial v_j}{\partial t} \to \frac{\partial \bar{v}}{\partial t} \in W^{-1,x} L_\psi (Q) + L^1 (Q).$$

for the modular convergence in $W_0^1 L_\varphi (Q)$, with

$$\|v_j\|_{L^\infty(Q)} \leq (N + 2)\|v\|_{L^\infty(Q)}$$

Pointwise multiplication of the approximate equation (32) by $T_k (u_n - v_j)$, we get

$$\text{(85)} \qquad \begin{cases} \displaystyle\int_0^\tau < \frac{\partial b_n(u_n)}{\partial s}, T_k (u_n - v_j) > ds + \int_Q a_n (x, s, u_n, \nabla u_n) \nabla T_k (u_n - v_j) \, dxds \\ \displaystyle + \int_Q T_n (u_n - \zeta)^- \, sg_{\frac{1}{n}} (u_n) T_k (u_n - v_j) \, dxds \\ \displaystyle + \int_Q^T g_n (x, s, u_n, \nabla u_n) \nabla T_k (u_n - v_j) \, dxds = \int_Q f_n T_k (u_n - v_j) \, dxds \end{cases}$$

We pass to the limit as in (85), n tend to $+\infty$ and j tend to $+\infty$. limit of the first term of (85):

The first term can be written

$$\begin{aligned}\int_0^\tau < \frac{\partial b_n(u_n)}{\partial s}, T_k\left(u_n - v_j\right) > ds = & \int_0^\tau < \frac{\partial\left(b_n(u_n) - v_j\right)}{\partial s}, T_k\left(u_n - v_j\right) > ds \\ & + \int_0^\tau < \frac{\partial v_j}{\partial s}, T_k\left(b_n(u_n) - v_j\right) > ds \\ = & S_k\left(b_n(u_n)(\tau) - v_j(\tau)\right) - S_k\left(b_n(u_n)(0) - v_j(0)\right) \\ & + \int_0^\tau < \frac{\partial v_j}{\partial s}, T_k\left(u_n - v_j\right) > ds\end{aligned}$$

We pass to the limit as $n \to +\infty$ and $j \to +\infty$ we can easily deduce

$$\begin{aligned}\int_0^\tau < \frac{\partial b_n(u_n)}{\partial s}, T_k\left(u_n - v_j\right) > ds \to & \int_\Omega S_k\left(b_n(u_n)(\tau) - v(\tau)\right) dx - \int_\Omega S_k\left(b_n(u_n)(0) - v(0)\right) dx \\ & + \int_0^\tau < \frac{\partial v}{\partial s}, T_k\left(b(u) - v\right) > ds\end{aligned}$$

– We can follow same way in [19] to prove that

$$\begin{aligned}& \liminf_{j\to\infty} \liminf_{n\to\infty} \int_Q a\left(x, s, u_n, \nabla u_n\right) \nabla T_k\left(u_n - v_j\right) dxds \\ \geq & \int_Q a(x, s, u, \nabla u) \nabla T_k(u - v) dxds\end{aligned}$$

– Limit of $g_n\left(x, s, u_n, \nabla u_n\right) T_k\left(u_n - v_j\right)$:
Since $g_n\left(x, s, u_n, \nabla u_n\right)$ converge strongly to $g(x, t, u, \nabla u)$ in $L^1(Q)$. and the pointwise convergence of u_n to u as $n \to +\infty$, it is possible to prove that $g_n\left(x, s, u_n, \nabla u_n\right) T_k\left(u_n - v_j\right)$ converge to $g(x, s, u, \nabla u) T_k\left(u - v_j\right)$ in $L^1(Q)$ and

$$\lim_{j\to\infty} \int_Q g(x, s, u, \nabla u) T_k\left(u - v_j\right) dxds = \int_Q g(x, s, u, \nabla u) T_k(u - v) dxds$$

– Since f_n converge strongly to f in $L^1(Q)$, and

$$T_k\left(u_n - v_j\right) \to T_k\left(u - v_j\right) \text{weakly* in } L^\infty(Q),$$

we have

$$\int_Q f_n T_k\left(u_n - v_j\right) dxds \to \int_Q f T_k\left(u - v_j\right) dxds,$$

as $n \to \infty$ and also we have

$$\int_Q f T_k\left(u - v_j\right) dxds \to \int_Q f T_k(u - v) dxds,$$

as $j \to \infty$.

Finally we know that

$$\int_Q T_n\left(u_n-\zeta\right)^- sg_{\frac{1}{n}}\left(u_n\right)T_k\left(u_n-v_j\right)dxds \geq 0,$$

thus

$$\begin{cases} \displaystyle\int_\Omega S_k(b(u(\tau))-v(\tau))dx+\int_0^\tau <\frac{\partial v}{\partial s}, T_k(b(u)-v)> ds \\ \displaystyle+\int_Q a(x,s,u,\nabla u)\nabla T_k(u-v)dxds+\int_Q g(x,s,u,\nabla u)T_k(u-v)dxds \\ \displaystyle\leq \int_Q fT_k(u-v)dxds-\int_\Omega S_k\left(b(u_0)-v(x,0)\right)dx \end{cases}$$

As a conclusion, the proof of Theorem (6.1) is complete.

□

References

[1] A.Aberqi, j.Bennouna and M.Elmassoudi: *Nonlinear alliptic equations with measure data in orlicz spaces* Gulf Journal of Mathematics Vol 6, Issue 4 (2018) 79-100.

[2] A. Aberqi,J. Bennouna, M. Elmassoudi, M. Hammoumi, *Existence and uniqueness of a renormalized solution of parabolic problems in Orlicz spaces* Monatshefte fur Mathematik https://doi.org/10.1007/s00605-018-01260-8.

[3] Ahmed Oubeid, M. L., Benkirane, A., Sidi El Vally, M.: *Strongly nonlinear parabolic prob- lems in Musielak-Orlicz-Sobolev spaces*, Bol. Soc. Paran. Mat.v. 33 1 191-23, (2015).

[4] Y.Akdim , J.Bennouna , M. Mekkour: *Solvability of degenerate parabolic equations without sign con- dition and three unbounded nonlinearities, Electronic Journal of Differential Equa- tions, No.03, 1-25, (2011).*

[5] L. Aharouch, , J.Bennouna : *Existence and uniqueness of solutions of unilateral problems in Orlicz spaces.* Nonlinear Anal. $72, 3553-3565(2010)$

[6] M. Al-Hawmi, A.Benkirane, H. Hjiaj, A. Touzani, : *Existence and uniqueness of entropy solution for some nonlinear elliptic unilateral problems in Musielak-Orlicz-Sobolev spaces,*Ann. Univ. Craiova Math. Comput. Sci. Ser. $44(1), 1-20(2017)$

[7] S.Antontsev, S. Shmarev, : *Anisotropic parabolic equations with variable nonlinearity.* Publ. Math. $53, 355-399(2009)$

[8] P.Benilan, L. Boccardo, T. Gallouet, R.Gariepy, M.Pierre, J. Vazquez : *An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations,* Ann. Scuola Norm. Super. Pisa $22, 241-273$ (1995)

[9] M.Ait Khellou, A. Benkirane : *Elliptic inequalities with L^1 data in Musielak-Orlicz spaces,* Monatsh Math, 183, 1–33, (2017).

[10] M.A. Ait Kellou, A.Benkirane ,S.M. Douiri : *An inequality of type Poincarke in Musielak spaces and applications to some nonlinear elliptic problems with L^1 -data, Complex Variablos and Elliptic Equations* $60, 1217-1242,$ (2015)

[11] E. Azroul, H. Redwane, M. Rhoudaf : *Existence of a renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces,* Port. Math. 66 (1) , 29-63, (2009).

[12] A. Benkirane, Y. El Hadfi and M. El Moumni.: Existence results for nonlinear parabolic equations with two lower order terms and L^1-data. *Ukrain. Mat. Zh.* 71 , no. 5, (2019) 610–630.

[13] A. Benkirane, Y. El Hadfi and M. El Moumni.: Renormalized solutions for nonlinear parabolic problems with L^1- data in orlicz-sobolev spaces. *Bulletin of Parana's Mathematical Society* (3s.) Volume 35, Number 1, (2017), pp: 57–84 .

[14] A. Benkirane, B. El Haji and M. El Moumni: *On the existence of solution for degenerate parabolic equations with singular terms,* Pure and Applied Mathematics Quarterly Volume 14, Number 3-4, 591-606(2018).

[15] A. Benkirane, M. El Moumni and A. Fri. An approximation of Hedberg's type in Sobolev spaces with variable exponent and application. *Chinese Journal of Mathematics*, Volume 2014 , Article ID 549051, (2014)7 pages .

[16] A. Benkirane, M. El Moumni and A. Fri. Renormalized solution for strongly nonlinear elliptic problems with lower order terms and L^1-data. *Izvestiya RAN: Ser. Mat.* (2017)**81**:3 3–20 .

[17] A. Benkirane, M. Sidi El Vally , (Ould Mohamedhen Val): *Some approximation properties in Musielak-Orlicz- Sobolev spaces,* Thai.J. Math. 10, 371-381, (2012).

[18] A. Benkirane, M.Sidi El Vally , (Ould Mohamedhen Val): *Variational inequalities in Musielak-Orlicz-Sobolev spaces,* Bull. Belg. Math. Soc. Simon Stevin 21, 787-811, (2014).

[19] A. Benkirane , J. Bennouna : *Existence of entropy solutions for some nonlinear problems in orlicz spaces,* Nonlinear Analysis 54, 85-102, (2001).

[20] L. Boccardo, T. Gallouët : *Nonlinear elliptic and parabolic equations involving measure data.* J. Funct. Anal. 87, 149-169 (1989)

[21] L. Boccardo, F.Murat, : *Strongly nonlinear Cauchy problems with gradient dependent lower order nonlinearity.* Pitman Res. Notes Math. 208, 247-254 (1989)

[22] L. Boccardo, , F. Murat : *Almost everywhere convergence of the gradients,* Nonlinear Anal. 19 (6) 581-597, (1992).

[23] A. DallAglio, L. Orsina : Nonlinear parabolic equations with natural growth conditions and L^1 data. Nonlinear Anal. 27, 5973 (1996).

[24] B. El Haji and M. El Moumni and K. Kouhaila: *On a nonlinear elliptic problems having large monotonocity with L^1-data in weighted Orlicz-Sobolev spaces*, Moroccan J. of Pure and Appl. Anal. (MJPAA) Vol 5(1), 104-116(2019).

[25] M. El Moumni: *Entropy solution for strongly nonlinear elliptic problems with lower order terms and L^1-data*, Annals of the University of Craiova - Mathematics and Computer Science Series. Volume 40, Number 2 , pp: 211–225 (2013).

[26] M. El Moumni: *Nonlinear elliptic equations without sign condition and L^1-data in Musielak-Orlicz-Sobolev spaces*, Acta. Appl. Math. 159:95–117 (2019).

[27] M. El Moumni: *Renormalized solutions for strongly nonlinear elliptic problems with lower order terms and measure data in Orlicz-Sobolev spaces*, Iran. J. Math. Sci. Inform. Vol 14, No 1, pp 95–119 (2019).

[28] N. El Amarti, B. El Haji and M. El Moumni: *Existence of renomalized solution for nonlinear Elliptic boundary value problem without Δ_2 -condition* SeMA 77, 389?414 (2020). https://doi.org/10.1007/s40324-020-00224-z.

[29] A. Prignet : *Existence and uniqueness of entropy solutions of parabolic problems with L^1 data.* Nonlinear Anal. $28, 1943-1954(1997)$

[30] H. Redwane : *Existence of a solution for a class of parabolic equations with three unbounded nonlinearities,* Adv. Dyn. Syst. Appl., 2, (2007), 241-264.

[31] H. Redwane. *Existence Results for a class of parabolic equations in Orlicz spaces.* Electronic Journal of Qualitative Theory of Diferential Equations (2010), No. 2, 1-19.

[32] A. Porretta: Existence results for strongly nonlinear parabolic equations via strong convergence of truncations. Ann. Math. Pura Appl. (IV) 177, 143-172 (1999).

[33] R. Landes : *On the existence of weak solutions for quasilinear parabolic initial-boundary value problems*, Proc. Roy. Soc. Edinburgh sect. A. **89** (1981) 137–217.

[34] J. Musielak : *Modular spaces and Orlicz spaces,* Lecture Notes in Math. 10-34 (1983).

[35] J. Simon : *Compact sets in the space* $L^p(0,T,B)$*,* Ann. Mat. Pura. Appl. 146, 65-96, (1987).

[36] A. Talh, A. Benkirane : *Strongly nonlinear elliptic boundary value problems in Musielak-Orlicz spaces,* Monatsh Math, 184, 1–32, (2017).