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# A CONDITION ON THE PROBABILITY THAT A GROUP ELEMENT FIXES A SET

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Abstract. Let $G$ be a finite non abelian group. The probability that a group element fixes a set is the probability that studies the number of orbits under group actions to the number of commuting elements of size two. The probability was computed and studied by several researchers. In this paper, we construct a new condition for the elements of the fixed set in which the set under the study is the set that consists of two commute elements $a$ and $b$, where lcm($|a|$, $|b|$)$\geq 2^i, 1 < i \leq 2n$. This condition is applied for the dihedral groups, where the conjugate action is used to determine the probability.

## 1. Introduction

The commutativity degree was established in 1968 by Erdos and Turan [1] who worked on symmetric groups. After years, the concept of the commutativity is used for finite groups and proved that the probability is less than or equal to 5/8 [2] and [3]. Recently, the commutativity is generalized by introducing the probability that a group element fixes a set [6]. The set that Omer et. all [6] established is the set that consists two commuting elements in which the least common multiple between the two elements is equal to two.

The researchers are then proceed the work on the probability that a group element fixes a set, where it is computed for the dihedral groups, quaternion groups and cyclic groups [6]. Omer *et all.* [6] found that the probability that a dihedral group element fixes the set; when $n$ is even and $\frac{n}{2}$ is odd, is equal to $\frac{12}{5n}$. The probability is computed under two different group actions, namely conjugation and regular [6]. In addition, Omer *et all.* [7] obtained the probability for symmetric groups and alternating groups, where the probability was $P_{S_n}(\Omega) = \frac{5}{|\Omega|}$, where $\Omega$ is the set of all subsets of commuting elements in the form

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of $(a, b)$, where $\text{lcm}(|a|, |b|)$ equals two. Omer *et all.* [7] computed the probability of an alternating group element fixes a set under conjugate action, where $P_{A_n}(\Omega) = \frac{2}{|\Omega|}$ and the probability is equal to half when the group acts on the set regularly.

In 2015, the probability that a group element fixes a set is found for metacyclic 2-groups [8]. Omer *et all.* [11] and [12] determined the probability for the metacyclic 2-groups of negative and positive types. The probability that a metacyclic 2-group of positive type fixes a set is equal to $\frac{5}{|\Omega|}$, when the group acts regularly on the set and equal to $\frac{2}{3}$ when the action is by conjugation [11].

The probability is then extended using the same set except for the commuting element $a$ and $b$, where neither $a$ or $b$ is a trivial element [13].

The probability was then associated to the graph theory, more precisely to the orbit graph [9], the isotropy graph [10].

As mentioned, the probability uses the orbits under group action on the set, thus the followings are some basic concepts related and used in this paper.

**Definition 1.** [4] *Let $G$ be a finite group, and let $g_1, g_2$ be elements in $G$. The elements $g_1, g_2$ are said to be conjugate if there is some $h$ in $G$ such that $g_2 = hg_1h^{-1}$. The set of all conjugates of $g_1$ is called the conjugacy classes of $g_1$.*

**Definition 2.** [5] *Let $G$ act on a set $S$, and $x \in S$. The orbit of $x$, denoted by $cl(x)$ is the subset*

$$cl(x) = \{gx : g \in G\} \subseteq S.$$

In case that a group $G$ acts on itself by conjugation, the orbit $\text{cl}(x)$ is

$$\{y \in G : y = axa^{-1} \text{ for some } a \in G\}.$$

## 2. Main Results

In this section, the condition that used for the probability that a group element fixes a set is constructed and the probability is then determined for the dihedral groups of order $2n$. We begin with the definition.

**Definition 3.** *Let $G$ be a group. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $lcm(|a|, |b|) \geq 2^i, 1 < i \leq 2n$. Let $\Omega$ be the set of all subsets of commuting elements of $G$ of size two, where $lcm(|a|, |b|) \geq 2^i, 1 < i \leq 2n$ and $G$ acts on $\Omega$. Then the probability that an element of a group fixes a set is:*

$$P_G(\Omega) = \frac{|\{(g, \omega) \in G \times \Omega | g\omega = \omega \text{ for } g \in G \text{ and } \omega \in \Omega\}|}{|\Omega||G|}.$$

The following theorem provides the probability that an element of a group fixes a set $\Omega$ in terms of the number orbits of $\Omega$ under group action of $G$ on $\Omega$.

**Theorem 1.** *Let $G$ be a finite group. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $lcm(|a|, |b|) \geq 2^i, 1 < i \leq 2n$. Let $\Omega$ be the set of all* *subsets of commuting elements of $G$ of size two and $G$ acts on $\Omega$. Then the probability that an element of a group fixes a set is given by:*

$$P_G(\Omega) = \frac{K(\Omega)}{|\Omega|},$$

*where $K(\Omega)$ is the number of orbits of $\Omega$ in $G$.*

The probability under the new condition is found for some special presentations of dihedral groups. The following proposition illustrates the elements of order $2^i, 1 < i \leq 2n$ in the dihedral groups.

**Proposition 1.** *Let $G$ be a dihedral group of order $2n$, $G \cong \langle a, b : a^n = 1, b^2 = 1, (ab)^2 = 1 \rangle$. Then, the elements of order $\geq 2^i, 1 \leq i \leq 2n$ are stated as follows:*

$$\begin{cases} a^{\frac{n}{2}}, a^i, 1 \leq i \leq 2n, a^i b, & \text{if } n \text{ is even and } \frac{n}{2} = 2^i, 1 < i \leq 2n, \\ a^{\frac{n}{4}}, a^i b, & \text{if } n \text{ is even and } \frac{n}{2} \neq 2^i, 1 < i \leq 2n, \\ a^i b, 0 \leq i \leq 2n, & \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* Determining the commuting elements that have orders greater than or equal to $2^i$ needs to study the cases of $n$. First, when $n$ is even and $\frac{n}{2} = 2^i, 1 < i \leq 2n$. If $\frac{n}{2} = 2^i$, the elements $|a^{\frac{n}{2}}| = 2$, where these elements are in the center of $G$. Also, the elements $a^i, 1 \leq i \leq 2n$ have the order $2^i$. Second, when $n$ is even and $\frac{n}{2} \neq 2^i, 1 < i \leq 2n$, hence their is one element of order $2^i$ is $|a^{\frac{n}{4}}|$. In the both cases $\frac{n}{2} = 2^i$ and $\frac{n}{2} \neq 2^i$, there are elements $|a^i b| = 2, 0 \leq i \leq 2n$. Third, in the case that $n$ is odd, the elements of order $2^i$ are $a^i b, 0 \leq i \leq 2n$. □

Th following proposition states the number of elements of $\Omega$ in $D_{2n}$, where $\Omega$ is the set of all subsets of commuting elements of size two in the form of $(a, b)$, where $\operatorname{lcm}(|a|, |b|) = 2^i, 1 < i \leq 2n$.

**Proposition 2.** *Let $G$ be a dihedral group of order $2n$, $G \cong \langle a, b : a^n = 1, b^2 = 1, (ab)^2 = 1 \rangle$. Then the elements of order $2^i, 1 \leq i \leq 2n$ as follows:*

$$|\Omega| = \begin{cases} \frac{7n-2}{2} + \sum_{\substack{i=2j \\ j=j+1}}^{\frac{n}{2}} (n-i), & \text{if } \frac{n}{2} \text{ is even and } \frac{n}{2} = 2^i, 1 < i \leq 2n, \\ \frac{5n+6}{2}, & \text{if } \frac{n}{2} \text{ is even and } \frac{n}{2} \neq 2^i, 1 < i \leq 2n, \\ \frac{5n+2}{2}, & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd,} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* Based on Proposition 1, when $n$ is even and $\frac{n}{2} = 2^i, 1 \leq i \leq 2n$ the elements of order greater than or equal to $2^i$ are in the form $(a^i, a^j), i \neq j, i < j \leq n-1$. Thus, the number of elements in this form are $\sum_{\substack{i=2 \\ j=j+1}}^{\frac{n}{2}} (n-i)$. Also, there are $n$ elements of the form of $(a^{\frac{n}{2}}, a^i b), 0 \leq i \leq 2n$. Moreover, $D_n$ consists of $(n-1)$ elements are in the form of

$(1, a^i), 0 \leq i \leq n-1$. In addition, there are $n$ elements of the form $(1, a^i b), 0 \leq i \leq n$ and $\frac{n}{2}$ elements are in the form of $(a^i b, a^{\frac{n}{2}+i} b), 0 \leq i \leq 2n$. Therefore, the number of commuting elements in this case are $|\Omega| = \frac{7n-2}{2} + \sum_{\substack{i=2j \\ j=j+1}}^{\frac{n}{2}} (n-i)$. Now, when $\frac{n}{2}$ is even and $\frac{n}{2} \neq 2^i$. Thus, the dihedral group based on the condition in Definition 3 consists of $n$ elements are in the form of $(1, a^i b), 0 \leq i \leq 2n$ and $n$ elements are in the form of $(a^{\frac{n}{2}}, a^i b), 0 \leq i \leq 2n$. Moreover, there are $\frac{n}{2}$ elements are in the form of $(a^i b, a^{\frac{n}{2}+i} b), 0 \leq i \leq 2n$, two elements are in the form of $(1, a^{\frac{n}{2}}), (1, a^{\frac{n}{4}})$ and only one element is in the form of $(a^{\frac{n}{2}}, a^{\frac{n}{4}})$. Hence, $|\Omega| = \frac{5n+6}{2}$. In the case that $n$ is even and $\frac{n}{2}$ is odd, there are $n$ elements are in the form of $(1, a^i b), 0 \leq i \leq n$, number of $n$ elements are in the form of $(a^{\frac{n}{2}}, a^i b), 0 \leq i \leq n$. In addition there are $\frac{n}{2}$ elements are in the form of $(a^i b, a^{\frac{n}{2}+i} b), 0 \leq i \leq n$ and only one elements is in the form of $(1, a^{\frac{n}{2}})$. Thus, $|\Omega| = \frac{5n+2}{2}$. Lastly, when $n$ is odd. Since $|Z(G)| = 1$, thus the elements of $\Omega$ are in the form $(1, a^i b), 0 \leq i \leq 2n$. The proof then follows. □

In the following context, the new condition stated in Theorem 1 is used to compute the probability that a dihedral group element fixes a set.

**Theorem 2.** *Let $G$ be a dihedral group $D_{2n}$ of order $2n$, where $n \in \mathbb{N}$. Let $S$ be a set of elements of $G$ of size two in the form of $(a, b)$ where $a$ and $b$ commute and $lcm(|a|, |b|) \geq 2^i, 1 \leq i \leq 2n$. Let $\Omega$ be the set of all subsets of commuting elements of size two. If $G$ acts on $\Omega$ by conjugation, then*

$$P_G(\Omega) = \begin{cases} \frac{K(\Omega)}{|\Omega|}, \text{ if } \frac{n}{2} \text{ is even and } \frac{n}{2} = 2^i, \text{ where } K(\Omega) = \frac{12+n}{2} + \sum_{\substack{i=2j \\ j=j+1}}^{\frac{n}{2}} (n-i), \\ \frac{8}{|\Omega|}, \text{ if } \frac{n}{2} \text{ is even and } \frac{n}{2} \neq 2^i, \\ \frac{14}{5n+2}, \text{ if } n \text{ is even and } \frac{n}{2} \text{ is odd}, \\ \frac{1}{n}, \text{ if } n \text{ is odd.} \end{cases}$$

*Proof.* Based on Proposition 2 and when $\frac{n}{2}$ is even and $\frac{n}{2} = 2^i$. Since $G$ acts on $\Omega$ by conjugation, then the orbits under the action are stated as follows:

There is one orbit in the form $\{(a^{\frac{n}{2}}, a^i b), 0 \leq i \leq 2n\}$, where $i$ is even, another one is in the form of $\{(a^{\frac{n}{2}}, a^i b), 0 \leq i \leq 2n\}$, where $i$ is odd. Also, there is one orbit of the form $\{(1, a^i b), 0 \leq i \leq 2n\}$ of size $\frac{n}{2}$, $0 \leq i \leq 2n, i$ is even and another one in the same form when $i$ is odd. In addition, there are two orbits in the form of $\{(a^i b, a^{\frac{n}{2}+i} b), 0 \leq i \leq 2n\}$ for both cases of $i$ even and odd. Moreover, there are $\frac{n}{2}$ orbits in the form of $\{(1, a^i)(1, a^{n-i}), 0 \leq i \leq n\}$ of size two and $\sum_{\substack{i=2j \\ j=j+1}}^{\frac{n}{2}} (n-i)$ orbits are in the form of $\{(a^i, b^j)(a^{n-i}, b^{n-j}), i \neq j, i < j \leq n\}$ of size two. Thus, there are $\frac{12+n}{2} + \sum_{\substack{i=2j \\ j=j+1}}^{\frac{n}{2}} (n-i)$ orbits in this case. Using Theorem 1, thus $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|}$, where $K(\Omega) = \frac{12+n}{2} + \sum_{\substack{i=2j \\ j=j+1}}^{\frac{n}{2}} (n-i)$. Second, $P_G(\Omega)$ is obtained in the case that $\frac{n}{2}$ is even and $\frac{n}{2} \neq 2^i$. According to Definition 3, the orbits under the conjugate action are described as follows: There is an orbit in the form of $\{(1, a^i b), 0 \leq i \leq 2n\}$, where $i$ is even, another one of the form $\{(1, a^i b), 0 \leq i \leq 2n\}$, where $i$ is odd. In addition, there are two orbits in the form of $\{(a^{\frac{n}{2}}, a^i b), 0 \leq i \leq 2n\}$, for both cases of $i$, namely even and odd. There are two orbits in the form of $\{(a^i b, a^{\frac{n}{2}+i} b), 0 \leq i \leq 2n\}$ for both cases of $i$. However, there is one orbit in the form of $\{(a^{\frac{n}{2}}, a^{\frac{n}{4}})\}$ and another orbit is in the form of $\{(1, a^{\frac{n}{2}}), (1, a^{\frac{n}{4}})\}$. Using Theorem 1 and Proposition 2, $P_G(\Omega) = \frac{16}{5n+6}$. Third, when $n$ is even and $\frac{n}{2}$ is odd. There is an orbit in the form of $\{(1, a^i b), 0 \leq i \leq 2n\}$, where $i$ is even and one orbit is in the form $\{(1, a^i b), 0 \leq i \leq 2n\}$, where $i$ is odd. Moreover, there are two orbits in the form of $\{(a^{\frac{n}{2}}, a^i b), 0 \leq i \leq 2n\}$, for both cases of $i$ even and odd. There are two orbits in the form of $\{(a^i b, a^{\frac{n}{2}+i} b), 0 \leq i \leq 2n\}$ for both cases of $i$. There is one orbit is in the form of $\{(1, a^{\frac{n}{2}})\}$. Thus, the number of orbits in this case are seven. According to Theorem 1 and Proposition 2, $P_G(\Omega) = \frac{7}{|\Omega|}$. Lastly, we find $P_G(\Omega)$ when $n$ is odd. There is only one orbit, namely $\{(1, a^i b), 0 \leq i \leq 2n\}$. Using Theorem 1 and Proposition 2, $P_G(\Omega) = \frac{1}{n}$. The proof then follows. □

## Conclusion

In this paper, the probability that a group element fixes a set is computed by establishing a new condition for the fixed set. The probability is determined for the dihedral groups, where the conjugate action is used to obtain the probability.

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