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A NOTE ON $(\omega + n)$-PROJECTIVE $QTAG$-MODULES

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ABSTRACT. In this paper $(\omega + n)$-projectives are studied with the help of h-pure and nice submodules. Here we show that if M is a $QTAG$-module of length not exceeding ω and K is its $(\omega + n)$-projective submodule for $n \in \mathbb{N} \cup \{0\}$ such that M/K is countably generated, then M is also $(\omega + n)$-projective.

1. INTRODUCTION

Let R be any ring. Consider the following two conditions on a module M_R:

(I) Every finitely generated submodule of every homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

A module M_R satisfies (I) and (II) is called a TAG-module, and a module satisfying condition (I) only is called a $QTAG$-module. The study of $QTAG$-modules and their structure began with work of Singh in [18]. Through a number of papers it has been seen that the structure theory of these modules is similar to that of torsion abelian groups and that these modules occur over any ring. Here the rings are almost restriction free and the $QTAG$-modules satisfy a simple condition. Several authors worked extensively on these modules and studied different notions and structures on $QTAG$-modules. It is interesting to note that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules [15].

Key words and phrases. QTAG-modules, $(\omega + n)$-projectives, h-pure submodules, nice submodules.

Everywhere in the text of the present article; let it be agreed that all the rings are associative with unity ($1 \neq 0$) and modules are unital $QTAG$-modules. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) : y \in M,\ x \in yR \text{ and } y \text{ uniform} \right\}$ are the exponent and height of x in M, respectively. $H_n(M)$ denotes the submodule of M generated by the elements of height at least n and $H^n(M)$ is the submodule of M generated by the elements of exponents at most n. Let us denote by M^1, the submodule of M, containing elements of infinite height. The module M is h-divisible [9] if $M = M^1 = \bigcap\limits_{n=0}^{\infty} H_n(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height. The module M is called separable [2] if $M^1 = 0$. Moreover, M is said to be bounded [17], if there exists an integer n such that $H_M(x) = n$ for every uniform element $x \in M$.

A submodule N of M is h-pure in M if $N \cap H_n(M) = H_n(N)$, for every integer $n \geq 0$. A submodule $B \subseteq M$ is a basic submodule [9] of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A fully invariant submodule $L \subseteq M$ is large [1], if $L + B = M$, for every basic submodule B in M. A submodule $N \subset M$ is nice [10] in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height.

Imitating [14], the submodules $H_n(M), n \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules [13] are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap\limits_{n=0}^{\infty} (N + H_n(M))$. Therefore, the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$ and h-dense in M if $\overline{N} = M$.

The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$ and a submodule S of $Soc(M)$ is called a subsocle of M. A module M is said to be quasi-complete [2], if the closure $\overline{N}$ of every h-pure submodule N of M, is h-pure in M and a module M is called h-pure-complete [8], if for every subsocle S of M there exists an h-pure submodule N of M such that $S = Soc(N)$.

An h-reduced module M is totally projective if it has a collection $\mathcal{N}$ of nice submodules such that (i) $0 \in \mathcal{N}$ (ii) if $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\sum\limits_{i\in I} N_i \in \mathcal{N}$ (iii) given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated. Call a collection $\mathcal{N}$ of nice submodules of M which satisfies conditions (i), (ii) and (iii) a nice system (see [11]) for M.

In what follows, all notations and notions are standard and will be in agreement with those used in [5, 6]. Many results of this paper are the generalization of [3, 4]. As usual, for any module M, M_r denotes the h-reduced part of M.

2. Results

Recall that a $QTAG$-module M is $(\omega+1)$-projective if there exists submodule $N \subset H^1(M)$ such that M/N is a direct sum of uniserial modules and a $QTAG$-module M is $(\omega+n)$-projective if there exists a submodule $N \subset H^n(M)$ such that M/N is a direct sum of uniserial modules [11].

Before stating and proving our main attainments, we shall list some important results that will be used in the sequel without a concrete referring.

Theorem 2.1. [7, Theorem 1.1]. *If M is an h-reduced $QTAG$-module so that M/K is countably generated for some totally projective submodule K of M, then M is totally projective.*

Following is the immediate consequence of the above result.

Corollary 2.1. [7, Corollary 1.2]. *Suppose M is a separable $QTAG$-module with a submodule K such that M/K is countably generated. Then M is a direct sum of uniserial modules if and only so does K.*

In [7], it was obtained an affirmation by dropping off the limitation on M to be separable but incorporating the additional restriction on K to be nice in M; thereby K is not h-dense in M if K is a proper submodule different from M.

Theorem 2.2. [7, Theorem 2.12]. *Suppose M is an h-reduced $QTAG$-module of length not exceeding $(\omega+n)$ for some nonnegative integer n with a h-pure and nice submodule K such that M/K is countably generated. Then M is $(\omega+n)$-projective if and only if K is $(\omega+n)$-projective.*

The following remark is of usefulness.

Remark 2.1. [7, Remark 2.10]. *The $QTAG$-module M is $(\omega+n)$-projective for $n \in \mathbb{N}\cup\{0\}$ if and only if there is $K \subseteq H^n(M)$ with the property M/K is a direct sum of uniserial modules.*

Although the properties of $(\omega+1)$-projective modules are not always preserved by the $(\omega+n)$-projective ones over $n \geq 2$ (see [16]), the purpose of this investigation is to enhance and improve the $(\omega+n)$-projective modules $\forall\, n \geq 0$ by deleting the restriction on h-density.

Therefore, we omit the limitation from Theorem 2.2 on K to be nice in M but, however, only when the whole module M is separable.

Theorem 2.3. *Suppose that M is a separable $QTAG$-module and K is a submodule of M such that it satisfies the following conditions:*

(a) *K is h-pure in M;*

(b) *M/K is countably generated.*

Then, for each non-negative integer n, M is $(\omega+n)$-projective if and only if K is $(\omega+n)$-projective.

Proof. The necessity is straightforward since each submodule of a $(\omega+n)$-projective module is again $(\omega+n)$-projective. We now concentrate on the more difficult converse implication. Consulting with Remark 2.1 for $(\omega+n)$-projectivity, given $L \subseteq H^n(K)$ such that K/L is a direct sum of uniserial modules for an arbitrary fixed natural number n. Thus L is nice in K and $K^1 \subseteq L$; actually within the current case $K^1 = 0$ since M is separable.

Letting $\overline{L} = \cap_{t<\omega}(L + H_t(M))$ be the closure of L in M with respect to the h-topology of M, we clearly observe that $H_n(\overline{L}) \subseteq M^1 = 0$ hence $\overline{L} \subseteq H^n(M)$. Moreover, we have

$$\begin{aligned} K \cap \overline{L} &= \cap_{t<\omega}[L + (K \cap H_t(M))], \\ &= \cap_{t<\omega}(L + H_t(K)), \\ &= L + K^1, \\ &= L. \end{aligned}$$

Consequently,

$$(K+\overline{L})/\overline{L} \cong K/(K \cap \overline{L}) = K/L$$

is a direct sum of uniserial modules. On the other hand,

$$M/\overline{L} \cong (M/L)/(\overline{L}/L) = (M/L)(M/L)^1$$

is separable, and

$$M/\overline{L}/(K+\overline{L})/\overline{L} \cong M/(K+\overline{L})$$

is countably generated as an epimorphic image of the countably generated quotient M/L. Finally, Corollary 2.1 enables us to infer that $M/\overline{L}$ is a direct sum of uniserial modules, whence the early used necessary and sufficient condition due to Remark 2.1 is a guarantor that M is $(\omega+n)$-projective, as asserted. This completes the proof. □

Remark 2.2. *It is still unknown at this stage whether or not under the required circumstances (a) and (b) the claim remains true for $(\omega+n)$-projective modules of length $\in(\omega,\omega+n]$.*

The next question is of some importance.

Problem. Can the assumptions on K to be h-pure or nice in M as well as on M to be separable be ignored?

In order to do this, we need the following technical lemma.

Lemma 2.1. *Let K be a h-pure submodule in a $QTAG$-module M, then $(K+M^1)/M^1$ is h-pure in M/M^1.*

Proof. By definition, $K\cap H_n(M)=H_n(K)$, $\forall\ n\geq 1$. Therefore, we calculate that

$$\begin{aligned}
[(K+M^1)/M^1]\cap H_n(M/M^1) &= [(K+M^1)/M^1]\cap[H_n(M)/M^1],\\
&= [(K+M^1)\cap H_n(M)]/M^1,\\
&= [M^1+(K\cap H_n(M))]/M^1,\\
&= [M^1+H_n(M)]/M^1,\\
&= H_n((K+M^1)/M^1),
\end{aligned}$$

and thus the desired h-purity follows. The proof is finished. □

And so, we are now prepared to prove the following.

Corollary 2.2. *Let M be a $QTAG$-module and K is a submodule of M such that it satisfies the following conditions:*

(a) K is h-pure in M;
(b) M/K is countably generated.

Then,

(i) M/M^1 is $(\omega+n)$-projective $\Leftrightarrow$ K/K^1 is $(\omega+n)$-projective.
(ii) K being $(\omega+n)$-projective $\Rightarrow$ M/M^1 is $(\omega+n)$-projective.
(iii) K being $(\omega+n)$-projective $\Rightarrow$ M is $(\omega+2n)$-projective, provided the length of M is less than or equal to $(\omega+n)$.

Proof. (i) Notice that $M/M^1\supseteq(K+M^1)/M^1\cong M/M^1$ since $K\cap M^1=K^1$. Now, we observe that $(M/M^1)/((K+M^1)/M^1)\cong M/(K+M^1)$ is at most countably generated because so is M/K. Furthermore, the utilization of Lemma 2.1 along with Theorem 2.3, both applied to the module $(K+M^1)/M^1$, substantiates the equivalence.

(ii) If K is $(\omega+n)$-projective, then so does K/K^1. Indeed, there is $L\subseteq K$ with $H_n(L)=0$ and K/L is a direct sum of uniserial modules. Consequently, $K^1\subseteq L$ and $H_n(L/K^1)=0$

with $(K/K^1)/(L/K^1) \cong K/L$ is a direct sum of uniserial modules. Knowing this, the aforementioned Remark 2.1 works.

Another confirmation that M/M^1 must be $(\omega+n)$-projective is like this. Letting $\overline{L} = \cap_{t<\omega}(L + H_t(M))$ be the closure of L in M with respect to the h-topology of M, we see that $(M/M^1)/(\overline{L}/M^1) \cong M/\overline{L}$ is a direct sum of uniserial modules and $H_n(\overline{L}/M^1) = 0$. That is why, with the aid of Remark 2.1, we are finished.

(iii) Again letting $\overline{L} = \cap_{t<\omega}(L + H_t(M))$ be the closure of L in M with respect to the h-topology of M, we easily find that $H_n(\overline{L}) \subseteq M^1 \subseteq H^n(M)$ whence $\overline{L} \subseteq H^{2n}(M)$. Further, with this in hand, we ascertain at once that same arguments as in Theorem 2.2 are applicable to get the desired implication. This concludes the proof. □

Combining Theorem 2.2 and 2.3, one can state the following.

Theorem 2.4. *Let M be a QTAG-module with a $(\omega+n)$-projective submodule K so that M/K is countably generated and $n \in \mathbb{N} \cup \{0\}$. If*

(c) K is h-pure and nice in M, then M is $(\omega+n)$-projective;

(d) K is h-pure in the separable module M, then M is $(\omega+n)$-projective.

Now, we discuss some questions as those alluded to above concerning when a given separable module is $(\omega+n)$-projective provided that it has a countably generated $(\omega+n)$-projective submodule, but by removing the h-pureness of the submodule in the whole module.

We are now in a position to proceed by proving the next extension of point (d).

Theorem 2.5. *Suppose that M is a QTAG-module of length at most ω which contains a submodule K such that M/K is countably generated. Then M is $(\omega+n)$-projective if and only if K is $(\omega+n)$-projective, whenever $n \in \mathbb{N} \cup \{0\}$.*

Proof. The necessity is immediate because $(\omega+n)$-projectives are closed with respect to submodules. As for the sufficiency, consulting with Remark 2.1, there exists $L \subseteq H^n(K)$ with K/L a direct sum of uniserial modules. But observing that

$$(M/L)_r/((M/L)_r \cap K/L) \cong ((M/L)_r + K/L)/K/L \subseteq (M/L)/(K/L) \cong M/K$$

is countably generated with $(M/L)_r \cap K/L \subseteq K/L$ a direct sum of uniserial modules. Thus, we see that all conditions from Theorem 2.1 are satisfied, hence it may be exploited to get the desired claim that $(M/L)_r$ is totally projective. Letting $\overline{L} = \cap_{t<\omega}(L + H_t(M))$ be the closure of L in M with respect to the h-topology of M, we clearly observe that $(M/L)(M/L)^1 = (M/L)/(\overline{L}/L) \cong M/\overline{L}$ is a direct sum of uniserial modules. It is straightforward argument that $H_n(\overline{L}) \subseteq H_\omega(M)$. Since M is separable, that is $H_\omega(M) = 0$, we derive that $H_n(\overline{L}) = 0$, so employing once again Remark 2.1 we are done.

The condition on separability may be avoided if the following strategy is attainable: Since L is bounded, one can write $L = \cup_{i<\omega} L_i$ where $L_i \subseteq L_{i+1} \leq L$ with $H_s(L_i) = 0$ for each $i < \omega$ and some $s \in \mathbb{N}$. It is easily seen that $\overline{L} = \cup_{i<\omega} P_i$ where $P_i = \cap_{t<\omega}(L_i + H_t(M))$. The crucial moment is whether we may choose a nice submodule Q of M such that $Q \subseteq Li$ and such that $L_i \cap H_i(M) \subseteq Q$ for each integer $i \geq 1$; thus L/Q is bounded in M/Q. Consequently, we have

$$\begin{aligned} P_i \cap H_i(M) &= \cap_{i\leq t<\omega}(L_i + H_t(M)) \cap H_i(M), \\ &= \cap_{i\leq t<\omega}(L_i \cap H_i(M) + H_t(M)), \\ &\subseteq \cap_{i\leq t<\omega}(Q + H_t(M)), \\ &= Q + \cap_{i\leq t<\omega} H_t(M), \\ &= Q + H_\omega(M), \\ &\leq H^n(\overline{L}). \end{aligned}$$

Furthermore, we observe that $\overline{L}/(Q + H_\omega(M)) = \cup_{i<\omega}[P_i/(Q + H_\omega(M))]$ where, for each $i < \omega$, we compute with the aid of forgoing calculation that

$$\begin{aligned} (P_i/(Q + H_\omega(M))) \cap H_i(M/(Q + H_\omega(M))) &= [P_i \cap (H_i(M) + Q)]/(Q + H_\omega(M)), \\ &= (Q + P_i \cap H_i(M))/(Q + H_\omega(M)), \\ &\subseteq (Q + H_\omega(M))/(Q + H_\omega(M)), \\ &= \{0\}. \end{aligned}$$

Besides, we have already shown above, $[M/(Q + H_\omega(M))]/[\overline{L}/(Q + H_\omega(M))] \cong M/\overline{L}$ is a direct sum of uniserial modules. Knowing this, we deduce that $M/(Q + H_\omega(M))$ is, in fact, a direct sum of uniserial modules. Hence and from Remark 2.1, we conclude that M is $(\omega + n)$-projective, as claimed. This completes our conclusions in all generality. □

Remark 2.3. *Actually,* $M/L = (M/L)_r$ *since* $H_{\omega+n}(M/L) = 0$ *by setting that* $(M/L)^1 = \cap_{t<\omega}(H_t(M) + L)/L \subseteq H^n(M)/L$ *with* $L \subseteq H^n(M)$ *and* $M^1 = 0$. *However, our approaches in the proof gives a more general strategy even for inseparable modules. Nevertheless, this general case is still in question.*

3. Open Problem

In conclusion, we pose one question that we find interesting.

Following [14], for $QTAG$-modules M and M', a homomorphism $f : M \to M'$ is said to be small if $ker f$ contains a large submodule of M. The set of all small homomorphisms from M to M', denoted by $Hom_s(M, M')$ is a submodule of $Hom(M, M')$. Moreover, for a family $\mathcal{F}$ of $QTAG$-modules, M is a HT-module with respect to $\mathcal{F}$ if $Hom(M, K) = Hom_s(M, K)$, for every $K \in \mathcal{F}$. M is said to be a HT-module if $Hom(M,K) = Hom_s(M,K)$, for every $QTAG$-module K which is a direct sum of uniserial modules. These modules are also defined with a different perspective in [12], that is, a $QTAG$-module M is a HT -module if and only if there exists some $n \in \mathbb{N}$, such that $Soc(H_n(M)) \subseteq K \subseteq M$ if M/K is a direct sum of uniserial modules.

So, we are ready to state the following.

Problem. Suppose M is a $QTAG$-module with a submodule K which belongs the the family $\mathcal{F}$ of $QTAG$-modules. If $Soc(M/K)$ is finitely generated, then whether or not M belongs to $\mathcal{F}$.

Investigate with a priority when $\mathcal{F}$ coincide with the family of HT-modules, closed modules, quasi-complete modules or h-pure-complete modules, respectively.

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