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MUTUA'S FORMULA: THE LAND SURVEYING AREA SOLVER

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Abstract. Land surveying has been a challenge for many years both in developed and developing countries. For instance the triangular survey consumes a lot of time as several sides of the field need to be measured. Solutions to this have been reached by the discovery of **Mutua's formula** of finding the area of a triangular shape.

1.0 Introduction

In future the time spent on land surveying will be reduced by half, based on the current work to present a new formula of finding the area of triangular land. We detail the derivation of this new formula as follows:

1.1 A brief description of previous work.

The book named, "Journey through Genius" motivates us very much. Severally, over the past we have tried to come up with new formulae which could ease the complex arithmetic algorithms encountered while executing results for just simple arithmetic problems which mathematicians commonly face in their life.

In June (2012) we tried to extend the reduction formula for evaluating the integrals of the form $\int x^n e^{mx}\,dx$ where $n \in \mathbb{Z}^+$ and m (is a real number other than zero (0)) by expressing it in terms of factorials. This seemed not to bear fruits as the modification made had no much contribution (as advised by peers).

Our first attempt triggered off further efforts; further research was done to solve future surveying problems. It was not until on June 2013 while studying on curvilinear co-ordinates that we were prompted with an idea which enabled us to discover that the area of a triangle could be found by a relation connecting area to length of one side and trigonometric ratios of two angles emanating from the line considered. The knowledge of Vector analysis, equation of straight lines and calculus all combined proved to be helpful in our formula derivation.

2.0 NEW FORMULA OF FINDING THE AREA OF A TRIANGLE

2.1 PRELIMINARIES

Key words and phrases: Mutua's formula, Land surveying

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Consider the triangle $A_1A_2A_3$ on the first quadrant of Cartesian plane with co-ordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively. If the vertices A1(x_1, y_1), A2(x_2, y_2), A3 (x_3, y_3) are arranged in anticlockwise manner as shown in the figure below.

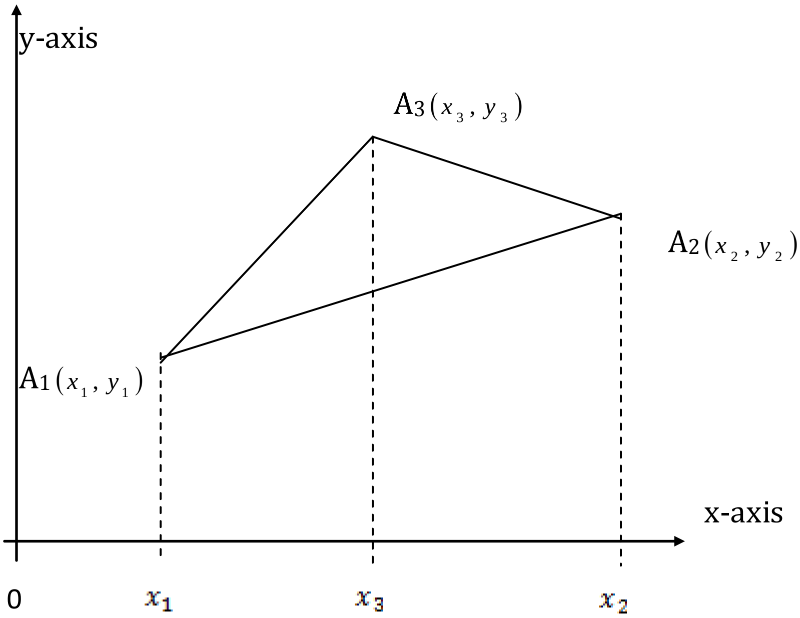


Figure 1: Triangle $A_1A_2A_3$ on the first quadrant of Cartesian plane

$$Area\ of\ \Delta A_1A_2A_3 = Area\ of\ trapezium\ A_1MPA_3 + Area\ of\ trapezium\ A_3PNA_2 - Area\ of\ trapezium\ A_1MNA_2$$

$$= \frac{1}{2}\{(x_3 - x_1)(y_1 + y_3)\} + \frac{1}{2}\{(x_2 - x_3)(y_2 + y_3)\} - \frac{1}{2}\{(x_2 - x_1)(y_1 + y_2)\}$$

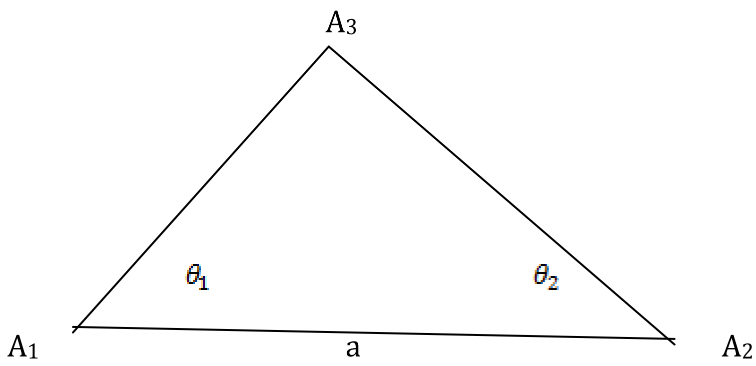
$$= \frac{1}{2}\{x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 + x_2y_2 + x_2y_3 - x_3y_2 - x_3y_3 - x_2y_1 - x_2y_2 + x_1y_1 + x_2y_2\}$$

$$= \frac{1}{2}\{(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)\} \qquad (1)$$

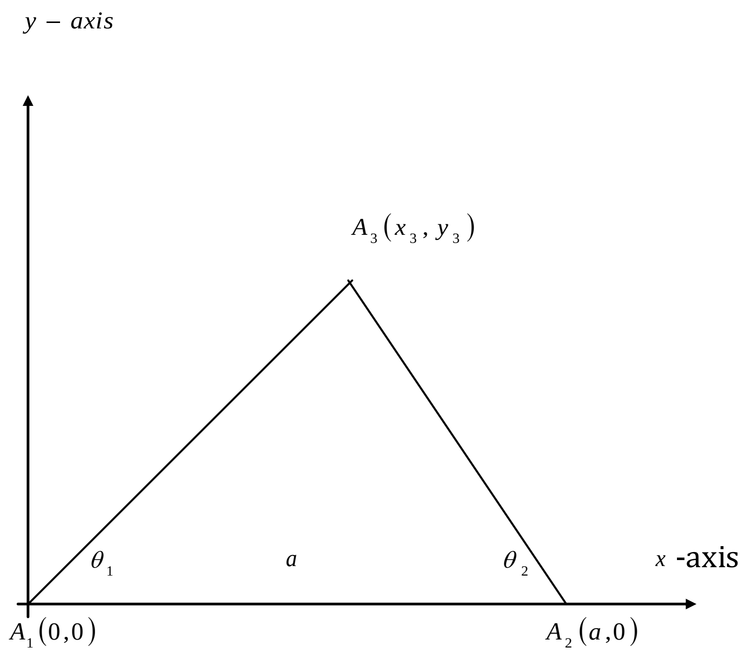
$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

This gives a positive determinant if the points A_1, A_2, A_3 are taken in anticlockwise manner.

Using this result we consider a general case of triangle $A_1A_2A_3$ and given that length $A_1A_2 = a\ units$, angles $A_3A_1A_2 = \theta_1$, $A_1A_2A_3 = \theta_2$ as shown below.



If we let line A_1A_2 lie on the positive x-axis of the Cartesian plane with point A_1 at the origin i.e. $A_1(0,0)$ then A_2 will be at $A_2(a,0)$ which can be illustrated as:



The gradient of the line A_1A_3 is $\tan\theta_1$ and therefore its equation becomes

$y = (\tan\theta_1)x$ .... (... A_1A_3) (y-intercept is zero).

Now the angle made by the positive x-axis and line A_2A_3 is $(180 - \theta_2)$. Thus the gradient of line A_2A_3 is $\tan(180 - \theta_2)$, but from the inter-relationship between signs of trigonometric ratios in different quadrants, we get $\tan(180 - \theta_2) = \tan(-\theta_2)$, since tangent functions are odd functions,

$\tan(-\theta_2) = -\tan(\theta_2)$.

Thus the equation of line A_2A_3 is $y = -(\tan\theta_2)x + c$. Here we are posed with a challenge of finding c (y-intercept).Since the x-intercept is known i.e. $= a$, the equation $y = -(\tan\theta_2)x + c$ can be written in the form

$x = \frac{-1}{(\tan\theta_2)y} + \frac{c}{\tan(\theta_2)}$. Therefore $\frac{c}{\tan(\theta_2)} = a$ which implies that $c = a\tan(\theta_2)$.

We can now write the equation of A_2A_3 as $y = -(\tan\theta_2)x + a\tan\theta_2$ .... (A_2A_3)

To get the co-ordinates of A_3 we solve equations of lines A_1A_3 and A_2A_3 simultaneously.

Equating the RHS terms we get

$$(\tan\theta_1)x = -(\tan\theta_2)x + a\tan\theta_2$$

Or

$$(\tan\theta_1 + \tan\theta_2)x = a\tan\theta_2$$

Implying that

$$x = \frac{a\tan\theta_2}{(\tan\theta_1 + \tan\theta_2)} \quad\text{———————}(2)$$

Substituting the value of x in $y = (\tan\theta_1)x$ we get

$y = \frac{(a\tan\theta_1\tan\theta_2)}{(\tan\theta_1 + \tan\theta_2)}$. So that A_3 is the point.

$$A_3\left(\frac{\tan\theta_2}{(\tan\theta_1 + \tan\theta_2)}, \frac{a\tan\theta_1\tan\theta_2}{(\tan\theta_1 + \tan\theta_2)}\right) \quad\text{———————}(3)$$

At this point we have the three co-ordinates as;

$$A_1(0,0),\ A_2(a,0),\ A_3\left(\frac{\tan\theta_2}{(\tan\theta_1 + \tan\theta_2)}, \frac{a\tan\theta_1\tan\theta_2}{(\tan\theta_1 + \tan\theta_2)}\right) \quad\text{———————}(4)$$

which corresponds to $A_1(x_1, y_1), A_2(x_2, y_2), A_3(x_3, y_3)$ respectively.

Now $\textit{Area of Triangle}\ A_1A_2A_3 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$ ———————(5)

Or

$$Area = \frac{1}{2}\{(x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)\}$$ from equation (1)

$$A = \frac{1}{2}\left\{\frac{a(a\tan\theta_1\tan\theta_2)}{(\tan\theta_1 + \tan\theta_2)} - 0\left(\frac{a\tan\theta_2}{(\tan\theta_1+\tan\theta_2)}\right) + (0-0) + (0-0)\right\} \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(6)}$$

$$= \frac{1}{2}\left\{\frac{a^2(\tan\theta_1\tan\theta_2)}{(\tan\theta_1 + \tan\theta_2)}\right\}$$

There are three possible cases:

(i) When the angles θ_1 and θ_2 are both not right angles. i.e.

$$\theta_1 \neq \frac{\pi}{2}, \theta_2 \neq \frac{\pi}{2}$$

$$Area = \frac{1}{2}\left\{\frac{a^2(\tan\theta_1\tan\theta_2)}{(\tan\theta_1 + \tan\theta_2)}\right\}$$. Thus

$$Area = \frac{1}{2}(a^2)\left(\frac{\tan\theta_1\tan\theta_2}{\tan\theta_1 + \tan\theta_2}\right) \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(7)}$$

(ii) When one of the angles is a right angle, say $\theta_1 = \frac{\pi}{2}$, we take the limiting value of $\frac{(\tan\theta_1\tan\theta_2)}{(\tan\theta_1+\tan\theta_2)}$ as $\theta_1 \to \frac{\pi}{2}$

We know that as $\theta_1 \to \frac{\pi}{2}$, $\tan\theta_1 \to \infty$

$$\lim_{\theta_1 \to \frac{\pi}{2}}\left(\frac{\tan\theta_1\tan\theta_2}{\tan\theta_1 + \tan\theta_2}\right) = \lim_{\theta_1 \to \frac{\pi}{2}}\left(\frac{\tan\theta_2}{1 + \frac{\tan\theta_2}{\tan\theta_1}}\right) = \tan\theta_2$$

Since $\frac{\tan\theta_2}{\tan\theta_1}$ is very small i.e. $\lim_{\theta_1 \to \frac{\pi}{2}}\left(\frac{\tan\theta_2}{\tan\theta_1}\right) = 0$

$$Area \quad A = \frac{1}{2}(a^2)\tan\theta_2 \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (8)}$$

(iii) Ambiguous case when $\theta_1 = \theta_2 = \frac{\pi}{2}$ we have A_1A_3, A_2A_3 being parallel lines, possibly they meet at infinity and the height blows up i.e. The limiting value of $\frac{(\tan\theta_1 \tan\theta_2)}{(\tan\theta_1 + \tan\theta_2)}$ as $\theta_1 \to \frac{\pi}{2}$ is

$$\lim_{\theta_1 \to \frac{\pi}{2}} \left(\frac{\tan\theta_1 \tan\theta_2}{\tan\theta_1 + \tan\theta_2} \right) = \lim_{\theta_1 \to \frac{\pi}{2}} \left(\frac{\tan\theta_2}{1 + \frac{\tan\theta_2}{\tan\theta_1}} \right) = \infty$$

Since $\frac{\tan\theta_2}{\tan\theta_1} << \tan\theta_2$.

Therefore the area is **undefined.**____________________________________ (9)

Note that, if $y_3 = 0$ $Area = 0$ we have $Area = 0$. (This is not worth discussing).We have no triangle and three lines lie on top of one another.

Consider triangle ABC as shown below

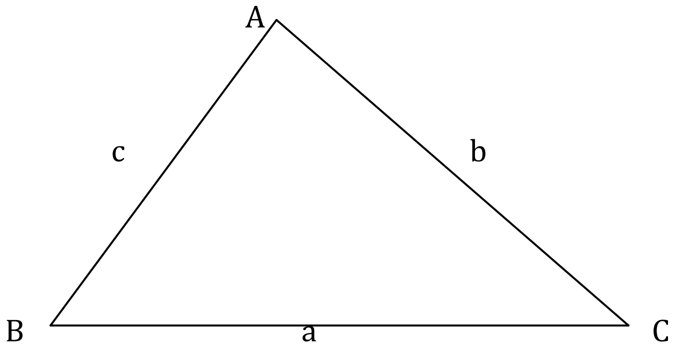


Figure 2: General Triangle ABC

$$Area = \frac{1}{2}(a^2)\left(\frac{\tan B \tan C}{\tan B + \tan C}\right) = \frac{1}{2}(b^2)\left(\frac{\tan A \tan C}{\tan A + \tan C}\right) = \frac{1}{2}(c^2)\left(\frac{\tan A \tan B}{\tan A + \tan B}\right) \_\_\_\_\_\_\_\_ (10)$$

With the three cases considered (for validity):

Equation (10) above gives ***Mutua's Formula*** of finding the area of a triangle.

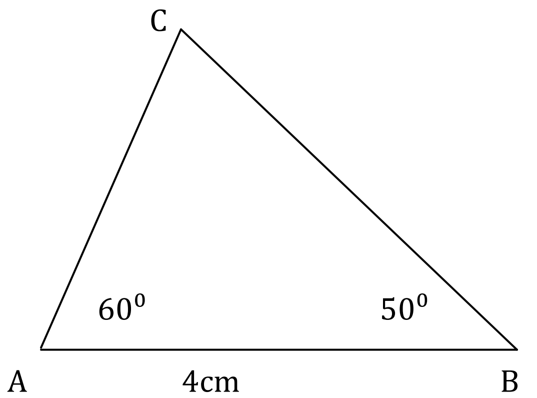
3.0 Application

The above formula can be applied in approximating the area(s) of piece(s) of land where Triangular Survey is the Technique adopted. It saves time for Surveyors as only one side of the field requires to be measured.

3.1 Case examples

EXAMPLE 1

Calculate the area of the triangle ABC such that AB=4cm, <CAB=60^0, <ABC=50^0.



Solution

< ACB =180^0-(60^0+50^0) =70^0

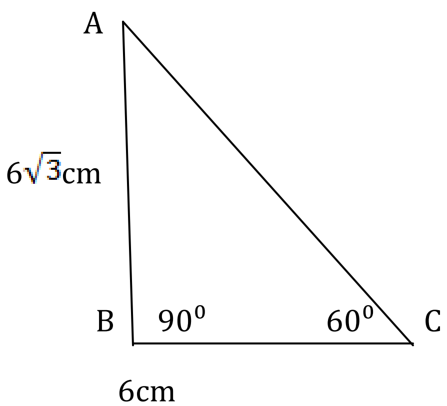
Using sine rule; $\frac{4}{\sin 70^0} = \frac{AC}{\sin 50^0}$ implying $AC = \frac{4 \sin 50^0}{\sin 70^0} = 3.261 \ (3d.p)$. Thus,

$$Area \ of \ triangle \ ABC = \frac{1}{2} \times 4 \times 3.261 \sin 60^0 = 5.648 \ cm^2 \ (correct \ to \ 3d.ps).$$

Using Felix's and Nicholas's formula;

$$Area = \frac{1}{2} \times 4^2 \left(\frac{\tan 60^0 \tan 50^0}{\tan 60^0 + \tan 50^0} \right) = 8(0.7060) = 5.648 \ cm^2 \ (correct \ to \ 3 \ d.ps)$$

Example 2



Solution

$$Area = \frac{1}{2} \times 6 \times 6\sqrt{3} = 18\sqrt{3}cm^2 = 31.177\ cm^2\ (correct\ \ to\ 3d.ps).$$

Using Felix's and Nicholas's formula;

$$Area = \frac{1}{2} \times 6^2 \tan 60^0 = 18\sqrt{3}cm^2 = 31.177\ cm^2\ (correct\ \ to\ 3d.ps).$$

Hence ***Mutua's*** formula yields the same solutions as the already existing formulas and can be considered simpler as requires only one side to be known, thus it can we recommend for its adoption by surveyors in their field work.

3.2 Conclusions

Several formulae exist that are useful in computing the area of a triangle. The best method will definitely depend on its ease of use, stability and computational efficiency. In this regard, land surveying being a pertinent issue in the world, an easy, stable and computationally efficient solver is imperative. For this reason, Felix's and Nicholas's formula find use in such a field.

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