A Common Fixed Point Result for Multi-Valued Mappings in Spherically Complete Ultrametric Spaces

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In this paper, we apply the strong contractive type mappings on the results of Rhodes [10] and prove a common fixed point theorem for a single-valued and the multi-valued mappings in spherically complete ultrametric spaces. The presented results unify, extend and improve several results in the related literature.

Keywords: ultrametric space, spherically complete, Multi-valued maps, fixed point.

1. INTRODUCTION AND PRELIMINARIES

In 1978, A. C. M. van Roovij [1] introduced the concept of ultrametric spaces. Since then, several fixed point and common fixed point theorems in the framework of ultrametric spaces have been investigated in [2]-[9]. In 1977, Rhodes [10] listed contractive type mappings which were generalizations of Banach contraction principle.

Now, we give some basic definitions and results which are used throughout the paper.

Definition 1.1 [2] Let \( (X,d) \) be a metric space. If the metric \( d \) satisfies the strong triangle inequality:

\[
 d(x,y) \leq \max\{d(x,y),d(y,z)\} \quad \text{ for all } x,y,z \in X,
\]

it said to be ultrametric on \( X \). The pair \( (X,d) \) is said to be an ultrametric space.

Example 1.2 The discrete metric \( d \) defined on \( X \neq \emptyset \) by

\[
 d(x,y) = \begin{cases} 
 0, & x = y \\
 1, & x \neq y
\end{cases}
\]

is an ultrametric.

Definition 1.3 [2] An ultra metric space \( (X,d) \) is said to be spherically complete if every shrinking collection of balls in \( X \) has a nonempty intersection.

Definition 1.4 An element \( x \in X \) is called a coincidence point of \( S : X \to X \) and \( T : X \to 2^X \) (where \( 2^X \) is the space of all nonempty compact subsets in \( X \)) if \( Sx \in T x \).

Definition 1.5 Let \( S : X \to X \) and \( T : X \to 2^X \). The mappings \( S \) and \( T \) are called coincidentally commuting at \( x \in X \) if \( STx \subseteq TSx \) whenever \( Sx \in T x \).

Theorem 1.6 (Zorn’s lemma) Let \( S \) be a partially ordered set.

If every totally ordered subset of \( S \) has an upper bound, then \( S \) contains a maximal element.


Theorem 1.7 ([3]) Let \( (X,d) \) be a spherically complete ultrametric space. If \( T : X \to 2^X \) is such that for any \( x,y \in X, x \neq y, \)

\[
 H(Tx,Ty) < \max\{d(x,y),d(x,Tx),d(y,Ty)\},
\]

then \( T \) has a fixed point, that is, there exists \( x \in X \) such that \( x \in Tx \), where \( H \) is the Hausdorff metric induced by the metric \( d \).
In this paper, we establish a unique common fixed point theorem for a single-valued and the multi-valued maps involving some strong contractive type mappings in spherically complete ultrametric spaces.

2. MAIN RESULTS

In this section, we apply strong contractive type mappings on the results of Rhoades [10] and established some new fixed point results in ultrametric spaces for multi-valued maps. Let us prove our main result.

Theorem 2.1 Let \((X, d)\) be an ultrametric space. Let \(S : X \to X\) and \(T : X \to 2^X\) be maps satisfying
\[
H(Tx, Ty) < \max\{d(Sx, Sy), d(Sx, Tx), d(Ty, Ty), d(Sx, Ty), d(Tx, Ty)\}
\]
for all \(x, y \in X\) such that \(x \neq y\).

Suppose that

(i) \(Sx\) is spherically complete.

Then there exists \(w \in X\) such that \(Sw \in Tw\).

Assume in addition that

(ii) \(S\) and \(T\) are coincidentally commuting at \(w\);

(iii) \(d(Sx, Sy) \leq d(y, Tx)\) for all \(x, y \in X\).

Then \(Sw\) is the unique common fixed point of \(S\) and \(T\), that is, \(S(Sw) = Sw \in T(Sw)\).

Proof. Assume that \(d(Sx, Tx) = \inf_{z \in T_{x,x}} d(Sx, z) > 0\) for all \(x \in X\).

Let \(B_a = B[Sa, d(Sa, Ta)] \cap SX\) denote the closed ball centred at \(Sa\) with radius \(d(Sa, Ta) > 0\) for all \(a \in X\) and let \(F\) be the collection of these balls. We define on \(F\) the following partial order
\[
B_a \subseteq B_b \iff B_b \subseteq B_a.
\]

Let \(F_1\) be a totally ordered subfamily of \(F\). We shall prove that \(F_1\) has an upper bound. By condition (i), \(SX\) is spherically complete, it follows that
\[
\bigcap_{B_a \in F_1} B_a = B \neq \emptyset.
\]

Let \(Sb \in B\). This implies that \(Sb \in B_a\), as \(B_a \subseteq F_1\). So \(d(Sb, Sa) \leq d(Sa, Ta)\). Since \(Ta\) is nonempty compact set, then there exists \(u \in Ta\) such that \(d(Sa, u) = d(Sa, Ta)\). From (2.1) and by the strong triangle inequality, we get
\[
d(Sb, Tb) \leq \max\{d(Sb, Sa), d(Sb, u), d(u, Tb)\} \\
\leq \max\{d(Sa, Ta), H(Ta, Tb)\} \\
< \max\{d(Sa, Ta), d(Sa, Sw), d(Sa, Tb), d(Sa, Ta), d(Sb, Tb), d(Sb, Tb)\} \\
= \max\{d(Sa, Ta), d(Sb, Tb), d(Sa, Tb), d(Sb, Ta)\}.
\]

As \(d(Sa, Tb) \leq \max\{d(Sa, Sa), d(Sb, Tb)\}\) and \(d(Sb, Ta) \leq \max\{d(Sa, Sa), d(Sa, Ta)\}\), then
\[
d(Sb, Tb) < \max\{d(Sa, Ta), d(Sb, Tb)\}.
\]

Necessarily, we have \(d(Sb, Tb) < d(Sa, Ta)\).

For \(x \in B_b\), we have
\[
d(Sb, x) \leq d(Sb, Tb) < d(Sa, Ta).
\]

Then
\[
d(Sa, x) \leq \max\{d(Sb, Sa), d(Sb, x)\} \leq d(Sa, Ta).
\]

It follows that \(x \in B_a\) and so \(B_b \subseteq B_a\). Thus \(B_a \subseteq B_b\) for all \(B_a \in F_1\). Hence \(B_b\) is an upper bound in \(F\) for the family \(F_1\).

By Zorn’s lemma, there exists a maximal element in \(F\), say \(B_w\). We claim that \(Sw \in Tw\). We argue by contradiction, that is, \(Sw \notin Tw\). Since \(Tw\) is a nonempty compact set, there exists \(Sv \in Tw\) such that \(d(Sv, Sw) = d(Sv, Tw)\) and \(Sv \neq Sw\). We shall prove that \(B_v \subseteq B_w\).

We have
\[
d(Sv, Tu) \leq H(Tu, Tu) \\
< \max\{d(Sv, Sw), d(Sv, Tw), d(Sv, Tv), d(Sv, v), d(Sv, Sw), d(Sv, Sw), d(Sv, Sw), d(Sv, Sw)\} \\
= \max\{d(Sw, Tu), d(Sw, Tv)\}.
\]

Then \(d(Sv, Tu) < d(Sv, Tw)\). Now, for \(x \in B_v\), we have
\[
d(Sv, x) \leq d(Sv, Tv) < d(Sv, Tw).
\]

It follows that
\[
d(Sv, x) \leq \max\{d(Sv, Sw), d(Sv, x)\} = d(Sv, Tw).
\]

Hence \(x \in B_w\) and so \(B_v \subseteq B_w\). Moreover, \(Sw \in B_w\) but \(Sw \in B_v\), because \(d(Sv, Sw) = d(Sv, Tw) > d(Sv, Tv)\). Then \(B_v \subseteq B_w\), which is a contradiction to the maximality of \(B_w\). Hence
Sw ∈ Tw. Let z = Sw. We claim that z is a common fixed point of S and T.

By condition (iv), we have

\[ d(z, Sz) = d(Sw, S(Sw)) \leq d(Sw, Tw) = 0, \]

because Sw ∈ Tw, which implies that z = Sz. Further, as Sw ∈ Tw, by condition (ii), we get S(Sw) ∈ STw ⊆ TSw. Then, z = Sz ∈ Tz. Hence, z is a common fixed point of S and T.

Let \( z' \) another common fixed point of S and T. Suppose that \( z \neq z' \). Using the condition (iii), from (2.1), we have

\[ 0 < d(z, z') = d(Sz, Sz') \]
\[ \leq d(z', Tz) \]
\[ \leq H(Tz', Tz) \]
\[ < \max\{d(Sz, Sz'), d(Sz, Tz), d(Sz', Tz')\} = \max\{d(z, z'), d(z, Tz), d(z', Tz')\} \]
\[ \leq \max\{d(z, z'), d(z, z'), d(z', Tz'), d(z', z), d(z, Tz), d(z', Tz')\} = d(z, z'), \]

which is a contradiction. Hence \( z = z' \).

If there exists \( x \in X \) such that \( d(Sx, Tx) = 0 \), then \( Sx \in Tx \). Similarly, we prove that \( Sx \) is the unique common fixed point of \( S \) and \( T \) and this completes the proof. ■

**Corollary 2.2** Let \( (X, d) \) be a spherically complete ultrametric space. Let \( T : X \to 2^X \) be a multi-valued map satisfying

\[ H(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\} \]

for all \( x, y \in X \) such that \( x \neq y \). Then \( T \) has a fixed point. Assume in addition that \( d(x, y) \leq d(y, Tx) \) for all \( x, y \in X \). Then, the fixed point of \( T \) is unique.

**Corollary 2.3** Let \( (X, d) \) be an ultrametric space. Let \( S : X \to X \) and \( T : X \to X \) be maps satisfying

\[ d(Tx, Ty) < \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sy, Tx)\} \]

for all \( x, y \in X, x \neq y \).

Suppose that

(i) \( SX \) is spherically complete.

Then there exists \( w \in X \) such that \( Sw = Tw \). Assume in addition that

(ii) \( S \) and \( T \) are coincidentally commuting at \( w \).

Then \( Sw \) is the unique common fixed point of \( S \) and \( T \).

**Corollary 2.4** Let \( (X, d) \) be a spherically complete ultrametric space. Let \( T : X \to X \) be a map satisfying

\[ d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx)\} \]

for all \( x, y \in X \) such that \( x \neq y \). Then \( T \) has a unique fixed point.

**Corollary 2.5** Let \( (X, d) \) be an ultrametric space. Let \( S : X \to X \) and \( T : X \to 2^X \) such that

\[ H(Tx, Ty) < \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty)\} \]

for all \( x, y \in X \) such that \( x \neq y \). Suppose that

(i) \( SX \) is spherically complete.

Then there exists \( w \in X \) such that \( Sw \in Tw \). Assume in addition that

(ii) \( S \) and \( T \) are coincidentally commuting at \( w \);

(iii) \( d(Sx, Sy) \leq d(y, Tx) \) for all \( x, y \in X \).

Then \( Sw \) is the unique common fixed point of \( S \) and \( T \), that is, \( S(Sw) = Sw \in T(Sw) \).

**Corollary 2.6** ([3], Theorem ) Let \( (X, d) \) be a spherically complete ultrametric space. If \( T : X \to 2^X \) is such that for any \( x, y \in X, x \neq y \),

\[ H(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\}, \]

then \( T \) has a fixed point.

**Corollary 2.7** Let \( (X, d) \) be an ultrametric space. Let \( S : X \to X \) and \( T : X \to 2^X \) such that

\[ H(Tx, Ty) < \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty)\} \]

for all \( x, y \in X \) such that \( x \neq y \). Suppose that

(i) \( SX \) is spherically complete.

Then there exists \( w \in X \) such that \( Sw \in Tw \). Assume in addition that
(ii) $S$ and $T$ are coincidentally commuting at $w$;

(iii) $d(Sx, Sy) \leq d(y, Tx)$ for all $x, y \in X$.

Then $Sw$ is the unique common fixed point of $S$ and $T$, that is, $S(Sw) = Sw = T(Sw)$.

**Corollary 2.8** Let $(X, d)$ be a spherically complete ultrametric space. If $T : X \to 2^X$ is such that for any $x, y \in X$, $x \neq y$,

$$H(Tx, Ty) < \max \{d(x, y), d(y, Tx), d(x, Ty)\},$$

then $T$ has a fixed point. Assume in addition that $d(x, y) \leq d(y, Tx)$ for all $x, y \in X$. Then, the fixed point of $T$ is unique.

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