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Kochen-Specker Theorem and the Two Quantum Measurement Theories

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We consider the two quantum measurement theories for measuring a single Pauli observable. We assume also the existence of a classical probability space for the two measurement theories. We cannot avoid the Kochen-Specker (KS) contradiction when we measure the Pauli observable by using the projective measurement theory if we introduce a classical probability space. The results of measurement are either $+1$ or -1 (in $\hbar/2$ unit) when we consider a spin-1/2 system. The projective measurement theory does not accept a classical probability space when we measure the Pauli observable. We propose a new measurement theory based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurement are either $+1$ or 0 (in $\hbar/2$ unit). We avoid the KS contradiction when we measure the Pauli observable by using the new measurement theory if we introduce a classical probability space. The new measurement theory accepts a classical probability space when we measure the Pauli observable.

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I. INTRODUCTION

The quantum theory (cf. [1–6]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data fits to the quantum predictions for long time.

On the other hand, from the incompleteness argument of Einstein, Podolsky, and Rosen (EPR) [7], a hidden-variable interpretation of the quantum theory has been an attractive topic of research [3, 4]. One is the Bell-EPR theorem [8]. This theorem says that the quantum predictions violate the inequality following from the EPR-locality condition. The condition tells that a result of measurement pertaining to one system is independent of any measurement performed simultaneously at a distance on another system.

The other is the no-hidden-variables theorem of Kochen and Specker (the KS theorem) [9]. The original KS theorem says the non-existence of a real-valued function which is multiplicative and linear on commuting operators. The quantum theory does not accept the KS type of hidden-variable theory. The proof of the original KS theorem relies on intricate geometric argument. Greenberger, Horne, and Zeilinger discover [10, 11] the so-called GHZ theorem for four-partite GHZ state. And, the KS theorem becomes very simple form (see also Refs. [12–18]). Especially, the KS theorem can be derived by only two results of measurement [17, 18].

Mermin considers the Bell-EPR theorem in a multipartite state. He derives multipartite Bell inequality [19]. The quantum predictions by n-partite GHZ state violate the Bell-Mermin inequality by an amount that grows exponentially with n. And, several multipartite Bell inequalities are reported [20–28]. They also say that the quantum predictions violate local hidden-variable theories by an amount that grows exponentially with n.

It is begun to research the validity of the KS theorem by using inequalities (see Refs. [29–32]). To find such inequalities to test the validity of the KS theorem is particularly useful for experimental investigation [33]. One of authors derives an inequality [32] as tests for the validity of the KS theorem. The quantum predictions violate the inequality when the system is in an uncorrelated state. An uncorrelated state is defined in Ref. [34]. The quantum predictions by n-partite uncorrelated state violate the inequality by an amount that grows exponentially with n.

Leggett-type nonlocal hidden-variable theory [35] is experimentally investigated [36–38]. The experiments report that the quantum theory does not accept Leggett-type nonlocal hidden-variable theory. These experiments are done in four-dimensional space (two parties) in order to study nonlocality of hidden-variable theories. However there are debates for the conclusions of the experiments. See Refs. [39–41].

Meanwhile, as for application of the quantum theory, implementation of a quantum algorithm to solve Deutsch's problem [42–44] on a nuclear magnetic resonance quantum computer is reported firstly [45]. An implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer is also reported [46]. There are several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implements Deutsch's algorithm with polarization and transverse spatial modes of the electromagnetic field as qubits [47]. Single-photon Bell states are prepared and measured [48]. Also the decoherence-free implementation of Deutsch's algorithm is reported by using such single-photon and by using two logical qubits [49]. More recently, a one-way based experimental implementation of Deutsch's algorithm is reported [50].

In 1993, the Bernstein-Vazirani algorithm was reported [51, 52]. It can be considered as an extended Deutsch-Jozsa algorithm. In 1994, Simon's algorithm was reported [53]. Implementation of a quantum algorithm to solve the Bernstein-Vazirani parity problem without entanglement on an ensemble quantum computer is reported [54]. Fiber-optics implementation of the Deutsch-Jozsa and Bernstein-Vazirani quantum algorithms with three qubits is discussed [55]. Quantum learning robust against noise is studied [56]. A quantum algorithm for approximating the influences of Boolean functions and its applications is recently reported [57]. Quantum computation with coherent spin states and the close Hadamard problem is also discussed [58]. Transport implementation of the Bernstein-Vazirani algorithm with ion qubits is more recently reported [59]. Quantum Gauss-Jordan elimination and simulation of accounting principles on quantum computers are discussed [60]. Finally, we mention that the dynamical analysis of Grover's search algorithm in arbitrarily high-dimensional search spaces is studied [61].

On the other hand, the earliest quantum algorithm, the Deutsch-Jozsa algorithm, is representative to show that quantum computation is faster than classical counterpart with a magnitude that grows exponentially with the number of qubits. In 2015, it was discussed that the Deutsch-Jozsa algorithm can be used for quantum key distribution [62]. In 2017, it was discussed that secure quantum key distribution based on Deutsch's algorithm using an entangled state [63]. Subsequently, a highly speedy secure quantum cryptography based on the Deutsch-Jozsa algorithm is proposed [64].

Recently, we discover [65] a new measurement theory based on the truth values. The results of measurement are either $+1$ or 0 (in $\hbar/2$ unit). We consider the significance of the new measurement theory based on the truth values [66, 67]. Especially, we investigate the relation between the new measurement theory and the KS theorem.

We assume an implementation of the double-slit experiment [68]. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. This is an easy detector model to a Pauli observable.

In this paper, we consider the two quantum measurement theories for measuring a single Pauli observable. We assume also the existence of a classical probability space for the two measurement theories. We cannot avoid the Kochen-Specker (KS) contradiction when we measure the Pauli observable by using the projective measurement theory if we introduce a classical probability space. The results of measurement are either $+1$ or -1 (in $\hbar/2$ unit) when we consider a spin-1/2 system. The projective measurement theory does not accept a classical probability space when we measure the Pauli observable. We propose a new measurement theory based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurement are either $+1$ or 0 (in $\hbar/2$ unit). We avoid the KS contradiction when we measure the Pauli observable by using the new measurement theory if we introduce a classical probability space. The new measurement theory accepts a classical probability space when we measure the Pauli observable.

II. PROJECTIVE MEASUREMENT THEORY

In this section, we discuss the relation between the projective measurement theory and the KS theorem. The results of measurement are either -1 or $+1$ (in $\hbar/2$ unit). We measure σ_x that is a single Pauli observable. Then, we cannot avoid the KS contradiction when we introduce a classical probability space into the projective measurement theory. Especially, we systematically describe our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

A. Projective measurement theory cannot avoid the KS contradiction

We assume an implementation of the double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the results of measurement are either $+1$ or -1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes another slit, then the value of the result of measurement is -1. This is an easy detector model of a single Pauli observable.

1. A wave function analysis

Let σ_x be a single Pauli observable. We assume that a source of a spin-carrying particle emits themselves in a state ρ. We consider a quantum expected value $\mathrm{Tr}[\rho\sigma_x]$. If we consider only a wave function analysis, the possible values of the square of the quantum expected value are

$$0 \leq (\mathrm{Tr}[\rho\sigma_x])^2 \leq 1. \tag{1}$$

We define $\|E_{\mathrm{QM}}\|^2$ as

$$\|E_{\mathrm{QM}}\|^2 = (\mathrm{Tr}[\rho\sigma_x])^2. \tag{2}$$

Then we have

$$\|E_{\mathrm{QM}}\|^2_{\mathrm{min}} = 0 \text{ and } \|E_{\mathrm{QM}}\|^2_{\mathrm{max}} = 1. \tag{3}$$

$\|E_{\mathrm{QM}}\|^2_{\mathrm{max}}$ and $\|E_{\mathrm{QM}}\|^2_{\mathrm{min}}$ are the maximal and minimal possible values of the product, respectively. We get $\|E_{\mathrm{QM}}\|^2_{\mathrm{min}} = 0$ or $\|E_{\mathrm{QM}}\|^2_{\mathrm{max}} = 1$ if the system is a pure state lying in either the z-axis or the x-axis, respectively.

2. Projective measurement theory says the KS theorem

A mean value E satisfies the projective measurement theory if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m}, \tag{4}$$

where l denotes a notation of the lth measurement and r is the result of the projective measurement of the Pauli observable σ_x. We assume the values of r are either -1 or $+1$ (in $\hbar/2$ unit). Assume the quantum mean value with the system in a state admits the projective measurement theory. One has the following proposition concerning the projective measurement theory

$$\mathrm{Tr}[\rho\sigma_x](m) = \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m}. \tag{5}$$

We can assume the following by Strong Law of Large Numbers,

$$\mathrm{Tr}[\rho\sigma_x](+\infty) = \mathrm{Tr}[\rho\sigma_x]. \quad (6)$$

We define $\|E_{\mathrm{QM}}\|^2(m)$ as

$$\|E_{\mathrm{QM}}\|^2(m) = (\mathrm{Tr}[\rho\sigma_x](m))^2. \quad (7)$$

We can assume the following by Strong Law of Large Numbers,

$$\|E_{\mathrm{QM}}\|^2(+\infty) = \|E_{\mathrm{QM}}\|^2 = (\mathrm{Tr}[\rho\sigma_x])^2. \quad (8)$$

In what follows, we show that we cannot accept the relation (5) concerning the projective measurement theory. Assume the proposition (5) is true. By changing the notation l into l', we have same quantum mean value as follows

$$\mathrm{Tr}[\rho\sigma_x](m) = \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m}. \quad (9)$$

We assume Sum rule and Product rule commute with each other [18]. We have the following when the system is in a pure state lying in the x-axis,

$$\begin{aligned}
&\|E_{\mathrm{QM}}\|^2(m) \\
&= \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m} \\
&\le \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} |r_l(\sigma_x) r_{l'}(\sigma_x)| \\
&= \frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} = 1. \quad (10)
\end{aligned}$$

Clearly, the above inequality can have the upper limit since the following case is possible:

$$\|\{l | l \in \mathbf{N} \wedge r_l(\sigma_x) = 1\}\| = \|\{l' | l' \in \mathbf{N} \wedge r_{l'}(\sigma_x) = 1\}\|, \quad (11)$$

and

$$\|\{l | l \in \mathbf{N} \wedge r_l(\sigma_x) = -1\}\| = \|\{l' | l' \in \mathbf{N} \wedge r_{l'}(\sigma_x) = -1\}\|. \quad (12)$$

And we have the following when the system is in a pure state lying in the z-axis,

$$\begin{aligned}
&\|E_{\mathrm{QM}}\|^2(m) \\
&= \frac{\sum_{l=1}^{m} r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_x)}{m} \\
&\ge \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} (-1) \\
&= (-1) \left(\frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} \right) = -1. \quad (13)
\end{aligned}$$

Clearly, the above inequality can have the lower limit since the following case is possible:

$$\|\{l | l \in \mathbf{N} \wedge r_l(\sigma_x) = 1\}\| = \|\{l' | l' \in \mathbf{N} \wedge r_{l'}(\sigma_x) = -1\}\|, \quad (14)$$

and

$$\|\{l | l \in \mathbf{N} \wedge r_l(\sigma_x) = -1\}\| = \|\{l' | l' \in \mathbf{N} \wedge r_{l'}(\sigma_x) = 1\}\|. \quad (15)$$

Thus we derive a proposition concerning the quantum mean value under the assumption that the projective measurement theory is used (in a spin-1/2 system), that is

$$-1 \le \|E_{\mathrm{QM}}\|^2(m) \le 1. \quad (16)$$

From Strong Law of Large Numbers, we have

$$-1 \le \|E_{\mathrm{QM}}\|^2 \le 1. \quad (17)$$

Hence we derive the following proposition concerning the projective measurement theory

$$\|E_{\mathrm{QM}}\|^2_{\min} = -1 \text{ and } \|E_{\mathrm{QM}}\|^2_{\max} = 1. \quad (18)$$

3. *The KS contradiction*

We cannot accept the two relations (3) (concerning a wave function analysis) and (18) (concerning the projective measurement theory), simultaneously. Thus, we are in the KS contradiction.

III. MEASUREMENT THEORY BASED ON THE TRUTH VALUES

In this section, we propose a new measurement theory, in qubits handling, based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurement are either $+1$ or 0 (in $\hbar/2$ unit).

We discuss the relation between the new measurement theory and the KS theorem. The results of measurement are either $+1$ or 0 (in $\hbar/2$ unit). We measure σ_x that is a single Pauli observable. Then, surprisingly, we can avoid the KS contradiction when we introduce a classical probability space into the new measurement theory. Especially, we systematically describe our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

A. The new measurement theory can avoid the KS contradiction

We assume an implementation of the double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurement are either $+1$ or 0 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes another slit, then the value of the result of measurement is 0. This is an easy detector model of a single Pauli observable.

1. A wave function analysis

Let σ_x be a single Pauli observable. We assume that a source of a spin-carrying particle emits themselves in a state ρ. We consider a quantum expected value $\mathrm{Tr}[\rho\sigma_x]$. If we consider only a wave function analysis, the possible values of the square of the quantum expected value are

$$0 \leq (\mathrm{Tr}[\rho\sigma_x])^2 \leq 1. \tag{19}$$

We define $\|E_{\mathrm{QM}}\|^2$ as

$$\|E_{\mathrm{QM}}\|^2 = (\mathrm{Tr}[\rho\sigma_x])^2. \tag{20}$$

Then we have

$$\|E_{\mathrm{QM}}\|^2_{\min} = 0 \text{ and } \|E_{\mathrm{QM}}\|^2_{\max} = 1. \tag{21}$$

$\|E_{\mathrm{QM}}\|^2_{\max}$ and $\|E_{\mathrm{QM}}\|^2_{\min}$ are the maximal and minimal possible values of the product, respectively. We get $\|E_{\mathrm{QM}}\|^2_{\min} = 0$ or $\|E_{\mathrm{QM}}\|^2_{\max} = 1$ if the system is a pure state lying in either the z-axis or the x-axis, respectively.

2. The new measurement theory does not say the KS theorem

A mean value E satisfies the new measurement theory if it can be written as

$$E = \frac{\sum_{l=1}^m r_l(\sigma_x)}{m}, \tag{22}$$

where l denotes a notation of the lth measurement and r is the result of the new measurement of the Pauli observable σ_x. We assume the values of r are either $+1$ or 0 (in $\hbar/2$ unit). Assume the quantum mean value with the system in a state admits the new measurement theory. One has the following proposition concerning the new measurement theory

$$\mathrm{Tr}[\rho\sigma_x](m) = \frac{\sum_{l=1}^m r_l(\sigma_x)}{m}. \tag{23}$$

We can assume the following by Strong Law of Large Numbers,

$$\mathrm{Tr}[\rho\sigma_x](+\infty) = \mathrm{Tr}[\rho\sigma_x]. \tag{24}$$

We define $\|E_{\mathrm{QM}}\|^2(m)$ as

$$\|E_{\mathrm{QM}}\|^2(m) = (\mathrm{Tr}[\rho\sigma_x](m))^2. \tag{25}$$

We can assume the following by Strong Law of Large Numbers,

$$\|E_{\mathrm{QM}}\|^2(+\infty) = \|E_{\mathrm{QM}}\|^2 = (\mathrm{Tr}[\rho\sigma_x])^2. \tag{26}$$

In what follows, we show that we can accept the relation (23) concerning the new measurement theory. Assume the proposition (23) is true. By changing the notation l into l', we have same quantum mean value as follows

$$\mathrm{Tr}[\rho\sigma_x](m) = \frac{\sum_{l'=1}^m r_{l'}(\sigma_x)}{m}. \tag{27}$$

We assume Sum rule and Product rule commute with each other [18]. We have the following when the system is in a pure state lying in the x-axis,

$$\begin{aligned}
&\|E_{\mathrm{QM}}\|^2(m)\\
&= \frac{\sum_{l=1}^m r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^m r_{l'}(\sigma_x)}{m}\\
&\leq \frac{\sum_{l=1}^m}{m} \cdot \frac{\sum_{l'=1}^m}{m} |r_l(\sigma_x) r_{l'}(\sigma_x)|\\
&= \frac{\sum_{l=1}^m}{m} \times \frac{\sum_{l'=1}^m}{m} = 1.
\end{aligned} \tag{28}$$

Clearly, the above inequality can have the upper limit since the following case is possible:

$$\|\{l|l\in\mathbf{N}\wedge r_l(\sigma_x)=1\}\|=\|\{l'|l'\in\mathbf{N}\wedge r_{l'}(\sigma_x)=1\}\|, \tag{29}$$

and

$$\|\{l|l\in\mathbf{N}\wedge r_l(\sigma_x)=0\}\|=\|\{l'|l'\in\mathbf{N}\wedge r_{l'}(\sigma_x)=0\}\|. \tag{30}$$

And we have the following when the system is in a pure state lying in the z-axis,

$$\begin{aligned}
&\|E_{\rm QM}\|^2(m)\\
&=\frac{\sum_{l=1}^m r_l(\sigma_x)}{m}\times\frac{\sum_{l'=1}^m r_{l'}(\sigma_x)}{m}\\
&\geq\frac{\sum_{l=1}^m}{m}\cdot\frac{\sum_{l'=1}^m}{m}(0)\\
&=(0)\left(\frac{\sum_{l=1}^m}{m}\times\frac{\sum_{l'=1}^m}{m}\right)=0.
\end{aligned} \tag{31}$$

Clearly, the above inequality can have the lower limit since the following case is possible:

$$\|\{l|l\in\mathbf{N}\wedge r_l(\sigma_x)=1\}\|=\|\{l'|l'\in\mathbf{N}\wedge r_{l'}(\sigma_x)=0\}\|, \tag{32}$$

and

$$\|\{l|l\in\mathbf{N}\wedge r_l(\sigma_x)=0\}\|=\|\{l'|l'\in\mathbf{N}\wedge r_{l'}(\sigma_x)=1\}\|. \tag{33}$$

Thus we derive a proposition concerning the quantum mean value under the assumption that the new measurement theory is used (in a spin-1/2 system), that is

$$0\leq\|E_{\rm QM}\|^2(m)\leq 1. \tag{34}$$

From Strong Law of Large Numbers, we have

$$0\leq\|E_{\rm QM}\|^2\leq 1. \tag{35}$$

Hence we derive the following proposition concerning the new measurement theory

$$\|E_{\rm QM}\|^2_{\rm min}=0 \text{ and } \|E_{\rm QM}\|^2_{\rm max}=1. \tag{36}$$

3. Consistency

We can accept the two relations (21) (concerning a wave function analysis) and (36) (concerning the new measurement theory), simultaneously. Thus, we avoid the KS contradiction.

IV. CONCLUSIONS

In conclusion, we have considered the two quantum measurement theories for measuring a single Pauli observable. We have assumed also the existence of a classical probability space for the two measurement theories. We cannot have avoided the Kochen-Specker (KS) contradiction when we measure the Pauli observable by using the projective measurement theory if we introduce a classical probability space. The results of measurement have been either $+1$ or -1 (in $\hbar/2$ unit) when we consider a spin-1/2 system. The projective measurement theory does not have accepted a classical probability space when we measure the Pauli observable. We have proposed a new measurement theory based on the truth values, i.e., the truth T (1) for true and the falsity F (0) for false. The results of measurement have been either $+1$ or 0 (in $\hbar/2$ unit). We have avoided the KS contradiction when we measure the Pauli observable by using the new measurement theory if we introduce a classical probability space. The new measurement theory has accepted a classical probability space when we measure the Pauli observable.

[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1955).

[2] R. P. Feynman, R. B. Leighton, and M. Sands, *Lectures on Physics, Volume III, Quantum mechanics* (Addison-Wesley Publishing Company, 1965).

[3] M. Redhead, *Incompleteness, Nonlocality, and Realism* (Clarendon Press, Oxford, 1989), 2nd ed.

[4] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, The Netherlands, 1993).

[5] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley Publishing Company, 1995), Revised ed.

[6] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2000).

[7] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[8] J. S. Bell, Physics **1**, 195 (1964).

[9] S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).

[10] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos (Kluwer Academic, Dordrecht, The Netherlands, 1989), pp. 69-72.

[11] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Am. J. Phys. **58**, 1131 (1990).

[12] C. Pagonis, M. L. G. Redhead, and R. K. Clifton, Phys. Lett. A **155**, 441 (1991).

[13] N. D. Mermin, Phys. Today **43**(6), 9 (1990).

[14] N. D. Mermin, Am. J. Phys. **58**, 731 (1990).

[15] A. Peres, Phys. Lett. A **151**, 107 (1990).

[16] N. D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).

[17] K. Nagata, T. Nakamura, and A. Farouk, Asian J. Math. Phys. Volume 1, Issue 1, 15 (2017).

[18] K. Nagata and T. Nakamura, Physics Journal, Volume 1, Issue 3, 183 (2015).

[19] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).

[20] S. M. Roy and V. Singh, Phys. Rev. Lett. **67**, 2761 (1991).

[21] M. Ardehali, Phys. Rev. A **46**, 5375 (1992).

[22] A. V. Belinskii and D. N. Klyshko, Phys. Usp. **36**, 653 (1993).

[23] R. F. Werner and M. M. Wolf, Phys. Rev. A **61**, 062102 (2000).

[24] M. Żukowski, Phys. Lett. A **177**, 290 (1993).

[25] M. Żukowski and D. Kaszlikowski, Phys. Rev. A **56**, R1682 (1997).

[26] M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).

[27] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001).

[28] R. F. Werner and M. M. Wolf, Quantum Inf. Comput. **1**, 1 (2001).

[29] C. Simon, Č. Brukner, and A. Zeilinger, Phys. Rev. Lett. **86**, 4427 (2001).

[30] J.-Å. Larsson, Europhys. Lett. **58**, 799 (2002).

[31] A. Cabello, Phys. Rev. A **65**, 052101 (2002).

[32] K. Nagata, J. Math. Phys. **46**, 102101 (2005).

[33] Y. -F Huang, C. -F. Li, Y. -S. Zhang, J. -W. Pan, and G. -C. Guo, Phys. Rev. Lett. **90**, 250401 (2003).

[34] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).

[35] A. J. Leggett, Found. Phys. **33**, 1469 (2003).

[36] S. Gröblacher, T. Paterek, R. Kaltenbaek, Č. Brukner, M. Żukowski, M. Aspelmeyer, and A. Zeilinger, Nature (London) **446**, 871 (2007).

[37] T. Paterek, A. Fedrizzi, S. Gröblacher, T. Jennewein, M. Żukowski, M. Aspelmeyer, and A. Zeilinger, Phys. Rev. Lett. **99**, 210406 (2007).

[38] C. Branciard, A. Ling, N. Gisin, C. Kurtsiefer, A. Lamas-Linares, and V. Scarani, Phys. Rev. Lett. **99**, 210407 (2007).

[39] A. Suarez, Found. Phys. **38**, 583 (2008).

[40] M. Żukowski, Found. Phys. **38**, 1070 (2008).

[41] A. Suarez, Found. Phys. **39**, 156 (2009).

[42] D. Deutsch, *Proc. Roy. Soc. London Ser. A* **400**, 97 (1985).

[43] D. Deutsch and R. Jozsa, *Proc. Roy. Soc. London Ser. A* **439**, 553 (1992).

[44] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca, *Proc. Roy. Soc. London Ser. A* **454**, 339 (1998).

[45] J. A. Jones and M. Mosca, J. Chem. Phys. **109**, 1648 (1998).

[46] S. Gulde, M. Riebe, G. P. T. Lancaster, C. Becher, J. Eschner, H. Häffner, F. Schmidt-Kaler, I. L. Chuang, and R. Blatt, Nature (London) **421**, 48 (2003).

[47] A. N. de Oliveira, S. P. Walborn, and C. H. Monken, J. Opt. B: Quantum Semiclass. Opt. **7**, 288-292 (2005).

[48] Y.-H. Kim, Phys. Rev. A **67**, 040301(R) (2003).

[49] M. Mohseni, J. S. Lundeen, K. J. Resch, and A. M. Steinberg, Phys. Rev. Lett. **91**, 187903 (2003).

[50] M. S. Tame, R. Prevedel, M. Paternostro, P. Böhi, M. S. Kim, and A. Zeilinger, Phys. Rev. Lett. **98**, 140501 (2007).

[51] E. Bernstein and U. Vazirani, Proceedings of the Twenty-Fifth Annual ACM Symposium on Theory of Computing (STOC '93), pp. 11-20 (1993), doi:10.1145/167088.167097.

[52] E. Bernstein and U. Vazirani, SIAM J. Comput. 26-5, pp. 1411-1473 (1997).

[53] D. R. Simon, Foundations of Computer Science, (1994) Proceedings., 35th Annual Symposium on: 116-123, retrieved 2011-06-06.

[54] J. Du, M. Shi, X. Zhou, Y. Fan, B. J. Ye, R. Han, and J. Wu, Phys. Rev. A **64**, 042306 (2001).

[55] E. Brainis, L.-P. Lamoureux, N. J. Cerf, Ph. Emplit, M. Haelterman, and S. Massar, Phys. Rev. Lett. **90**, 157902 (2003).

[56] A. W. Cross, G. Smith, and J. A. Smolin, Phys. Rev. A **92**, 012327 (2015).

[57] H. Li and L. Yang, Quantum Inf. Process. **14**, 1787 (2015).

[58] M. R. A. Adcock, P. Hoyer, and B. C. Sanders, Quantum Inf. Process. **15**, 1361 (2016).

[59] S. D. Fallek, C. D. Herold, B. J. McMahon, K. M. Maller, K. R. Brown, and J. M. Amini. New J. Phys. **18**, 083030 (2016).

[60] D. N. Diep, D. H. Giang, and N. Van Minh, Int J Theor Phys (2017) 56: 1948. doi:10.1007/s10773-017-3340-8.

[61] W. Jin, Quantum Inf. Process. **15**, 65 (2016).

[62] K. Nagata and T. Nakamura, Open Access Library Journal, 2: e1798 (2015), http://dx.doi.org/10.4236/oalib.1101798.

[63] K. Nagata and T. Nakamura, Int J Theor Phys (2017) 56: 2086. doi:10.1007/s10773-017-3352-4.

[64] K. Nagata, T. Nakamura, and A. Farouk, International Journal of Theoretical Physics, **56**, 2887 (2017).

[65] K. Nagata and T. Nakamura, Int. J. Theor. Phys. **55**, 3616 (2016).

[66] K. Nagata, T. Nakamura, J. Batle, S. Abdalla, and A. Farouk, Journal of the Korean Physical Society, Volume 70, Issue 3, (2017), Page 229–235.

[67] K. Nagata and T. Nakamura, Quantum Studies: Mathematics and Foundations, Volume 4, Issue 1, (2017) Page 7–11.

[68] C. Schon and A. Beige, Phys. Rev. A **64**, 023806 (2001).