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Common Fixed Point Theorems for Some Mappings Satisfying $\varphi-$Type Contractive Conditions in $A-$Metric Spaces

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We prove the existence and the uniqueness of a fixed point for the pair of weakly compatible self-mappings satisfying some $\varphi-$type contractive conditions in $A-$metric spaces. Our results generalize, extend recently fixed point results in the literature.

Keywords: Fixed point, A-metric space, point of coincidence, continuous.

I. INTRODUCTION AND PRELIMINARIES

Fixed point theory plays a major role in many applications, including variational and linear inequalities, optimization and applications in the field of approximation theory and minimum norm problem. In 1922, S. Banach proved the famous and well known Banach contraction principle concerning the fixed of contraction mappings defined on a complete metric space. In recent years, Gahler [1, 2] introduced the notion of 2-metric spaces, while Dhage [3] introduced the concept of D-metric spaces. Later on, Mustafa and Sims [11] introduced a new notion of generalized metric space called G-metric spaces. After then many authors studied fixed and common fixed points in generalized metric spaces see [8, 10–17, 20]. Next, S. Sedghi et al [18] introduced a D^*-metric space, In [9], S. Sedghi, N. shobe and A. Aliouche have introduced the notion of an S-metric space. A generalization of the S-metric space is called the $A-$metric space see [19] Moreover in [13], we find some properties of $A-$metric spaces were represented. In the present paper, we going to prove the existence and the uniqueness of some common fixed point theorems by using a $\varphi-$ contractive mappings on $A-$metric space. The results obtained in this paper extend some recently fixed point in the $S-$metric spaces and in the $G-$metric spaces.

Definition 1 *[4] Let X be a nonempty set. An A-metric on X^n is a function $A : X\times X\times\cdots\times X = X^n \to [0,\infty)$ that satisfies the following conditions, for each $x = (x_1,x_2,\cdots,x_n)\in X^n$*

(A1): $A(x_1,x_2,\cdots,x_n)\geq 0$

(A2): $A(x_1,x_2,\cdots,x_n) = 0$ if and only if $x_1 = x_2 = \cdots = x_n$

(A3): $A(x_1,x_2,\cdots,x_n) \leq A(x_1,x_1,\cdots,x_1,a) + A(x_2,x_2,\cdots,x_2,a)+ \cdots + A(x_n,x_n,\cdots,x_n,a)$ for all $x_i,a\in X$, $i\in\{1,\cdots,n\}$

The pair (X^n,A) is called an $A-$metric space.

Example 2 *[4] Let $X = \mathbb{R}$. Define a function $A : X^n \to [0,\infty)$ by*

$$A(x_1,x_2,\cdots,x_n) = \sum_{i=1}^{n}\sum_{i<j}|x_i - x_j|$$

Then $(\mathbb{R}^n,A)$ is an $A-$metric space.

Example 3 *[4, 13] For a ste*

ndart ordinary metric d on X, we define an $A-$metric A_1on X^n by

$$A_1(x_1,x_2,\cdots,x_n)) = \sum_{i=1}^{n}\sum_{i<j} d(x_i,x_j)$$

called the stendart $A-$metric space.

Lemma 4 *[4], [5] Let (X,A) be an $A-$metric space. then for all $x,y,z\in X$., we have*

$$A(x,x,\cdots,x,y)\leq(n-1)A(x,x,\cdots,x,z)+A(z,z,\cdots,z,y)$$

and

$$A(x,x,\cdots,x,z)\leq(n-1)A(x,x,\cdots,x,y)+A(y,y,\cdots,y,z)$$

Definition 5 *[4] Let (X,A) be an $A-$metric space. For $r>0$ and $x\in X$ we define the open ball $B_A(x,r)$ and closed ball $B_A[x,r]$ with center x and radius r as follows respectively*

$$B_A(x,r)=\{y\in X: A(y,y,\cdots y,x)<r\}$$
$$B_A[x,r]=\{y\in X: A(y,y,\cdots y,x)\leq r\}$$

Definition 6 *[4, 13] Let (X,A) be an $A-$metric space and $\Omega\subset X$.*
If for every $x\in\Omega$ there exists $r>0$ such that $B_A(x,r)\subset\Omega$ then the subset Ω is called open subset of X.

Definition 7 *[4] Subset Ω of X is said to be $A-$bounded if there exists $r>0$ such that $A(y,y,\cdots y,x)<r$ for all $x,y\in\Omega$*

Definition 8 *[4] Let (X,A) be an $A-$metric space. A sequence $\{x_n\}$ in X converges to x if and only if $A(x_n,x_n,\cdots,x_n,x)\to 0$ as $n\to\infty$. That is or each $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that $A(x_n,x_n,\cdots,x_n,x)<\varepsilon$ whenever $n\geq n_0$ and we denote this $\lim_{n\to\infty}x_n=x$*

Lemma 9 *[4] Let (X,A) be an $A-$metric space . If the sequence $\{x_n\}$ in X converges to a point x, then x is unique.*

Definition 10 *[4] Let (X,A) be an $A-$metric space . A sequence $\{x_n\}$ is called Cauchy sequence if $A(x_n,x_n,\cdots,x_n,x)\to 0$ as $n;m\to\infty$. That is or each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $A(x_n,x_n,\cdots,x_n,x_m)\leq\varepsilon$ whenever $n,m\geq n_0$.*

Lemma 11 *[4] Every convergent sequence in $A-$metric space is a Cauchy sequence. The converse does not hold in general.*

Definition 12 *[4] The $A-$metric space (X,A) is said to be complete if every Cauchy sequence in X is convergent.*

Let τ be the set of all $\Omega\subset X$. with $x\in\Omega$ if and only if there exists $r>0$ such that $B_A(x,r)\subset\Omega$. Then τ is a topology on X (induced by the $A-$metric space).

Example 13 *Any open ball $B_A(x,r)$, $x\in X$, $r>0$ is an open set. Indeed, Let $y\in B_A(x,r)$, we prove that $B_A(y,\rho)\subset B_A(x,r)$, let z be an element of $B_A(y,\rho)$ then by definition, we have $A(z,z,\cdots,z,y)<\rho=\frac{r-A(y,y,\cdots,y,x)}{n-1}$, Then, it follows from lemma 4, $A(z,z,\cdots,z,x)\leq(n-1)A(z,z,\cdots,z,y)+A(y,y,\cdots,y,x)<r$. So, we have $A(z,z,\cdots,z,y)<r$, and $z\in B_A(x,r)$.*

Example 14 *Let x_0, $y_0\in X$, considering the sets*

$$B_1=\{x\in X: A(x,x,\cdots,x,x_0)<A(x,x,\cdots,x,y_0)\}$$

and

$$B_2=\{x\in X: A(x,x,\cdots,x,x_0)>A(x,x,\cdots,x,y_0)\}\}.$$

$B_1$and B_2 are two open disjoint sets. We prove that the sets B_1 and B_2 are opens. Let $z\in B_1$ then $A(z,z,\cdots,z,x_0)<A(z,z,\cdots,z,y_0)$ which implies $0<A(z,z,\cdots,z,y_0)-A(z,z,\cdots,z,x_0)$ Setting $\rho=\frac{A(z,z,\cdots,z,y_0)-A(z,z,\cdots,z,x_0)}{2(n-1)}$. We show that $B_A(z,\rho)\subset B_1$. Indeed, let $a\in B_A(z,\rho)$ then

$$A(a,a,\cdots,a,z)<\rho=\frac{A(z,z,\cdots,z,y_0)-A(z,z,\cdots,z,x_0)}{2(n-1)}$$

therefore

$$2(n-1)A(a,a,\cdots,a,z)+A(z,z,\cdots,z,x_0)<A(z,z,\cdots,z,y_0),$$

so,

$$(n-1)A(a,a,\cdots,a,z)+A(z,z,\cdots,z,x_0)<A(z,z,\cdots,z,y_0)-(n-1)A(a,a,\cdots,a,z),$$

Hence by lemma 4 ,we get

$$\begin{aligned}A(a,a,\cdots,a,x_0)&\leq(n-1)A(a,a,\cdots,a,z)+(n-1)A(a,a,\cdots,a,z)\\&<A(z,z,\cdots,z,y_0)-(n-1)A(a,a,\cdots,a,z)\\&\leq A(a,a,\cdots,a,y_0)\end{aligned}$$

This means that $A(a,a,\cdots,a,x_0)<A(a,a,\cdots,a,y_0)$; the desired result follows. With the same way we prove that B_2 is an open set.

Theorem 15 *The $A-$metric space is a T_2space.*

Proof. It follows from the example14 ■

Definition 16 *[8] Let f and g be singled-valued self mappings on a set X. If $\omega = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g .*

Definition 17 *[8] Let f and g be singled-valued self mappings on a set X. Mappings f and g are said to be weakly compatible if $fx = gx$ implies $fgx = gfx$, $x \in X$.*

Proposition 18 *[8] Let f and g be weakly compatible self mappings on a set X. If f and g have a unique point of coincidence $\omega = fx = gx$, then ω is the unique common fixed point of f and g.*

II. MAIN RESULTS

Let Φ denotes the class of the functions$\varphi : [0,\infty) \longrightarrow [0,\infty)$ which satisfies the following conditions:

1. φ is nondecreasing
2. $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for all $t \in [0,\infty)$
3. $\varphi(0) = 0$
4. $\varphi(t) < t$ for all $t \in [0,\infty)$

The elements of Φ are called $\Phi-$map.

Lemma 19 *Let (X,A) be a $A-$metric space and let $\{x_n\}$ be a sequence in it such that*

$$\lim_{n\to\infty} A(x_{n+1},x_{n+1},\cdots,x_{n+1},x_n) = 0$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε

$A(x_{m_k},x_{m_k},\cdots,x_{m_k},x_{n_k}), A(x_{m_k},x_{m_k},\cdots,x_{m_k},x_{n_k+1}$

$A(x_{m_k-1},x_{m_k-1},\cdots,x_{m_k-1},x_{n_k}), A(x_{m_k-1},x_{m_k-1},\cdots,x_{m_k-1},x_{n_k+1})$

$A(x_{m_k-1},x_{m_k-1},\cdots,x_{m_k-1},x_{n_k+1}), \ldots$

when $k \to \infty$.

Proof. Suppose that the sequence $\{x_n\}$is not a Cauchy sequence, then, there exists $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that $n_k > m_k > k$ and

$A(x_{m_k},x_{m_k},\cdots,x_{m_k},x_{n_k}) \geq \varepsilon$ and $A(x_{m_k},x_{m_k},\cdots,x_{m_k},x_{n_k-1}) < \varepsilon$

Then using lemma 4 and (A_3) we have

$$\begin{aligned}
\varepsilon &\leq A(x_{m_k},x_{m_k},\cdots,x_{m_k},x_{n_k}) = A(x_{n_k},x_{n_k},\cdots,x_{n_k},x_{m_k}) \\
&\leq 2(n-1)A(x_{n_k},x_{n_k},\cdots,x_{n_k},x_{n_k-1}) + A(x_{m_k},x_{m_k},\cdots,x_{m_k},x_{n_k-1}) \\
&< 2(n-1)A(x_{n_k},x_{n_k},\cdots,x_{n_k},x_{n_k-1}) + \varepsilon
\end{aligned}$$

Hence by going to the limit, we get

$$\varepsilon \leq \lim_{k\to\infty} A(x_{m_k},x_{m_k},\cdots,x_{m_k},x_{n_k}) \leq 0 + \varepsilon$$

Therefore $\lim\limits_{k\to\infty} A(x_{n_k},x_{n_k},\cdots,x_{n_k},x_{m_k}) = \lim\limits_{k\to\infty} A(x_{m_k},x_{m_k},\cdots,x_{m_k},x_{n_k}) = \varepsilon$. Further, as

$$\begin{aligned}
&\left|A(x_{n_k},x_{n_k},\cdots,x_{n_k},x_{m_k}) - A(x_{n_k+1},x_{n_k+1},\cdots,x_{n_k+1},x_{m_k})\right| \\
&\leq 2(n-1)A(x_{n_k+1},x_{n_k+1},\cdots,x_{n_k+1},x_{n_k})
\end{aligned}$$

we obtain that

$$\lim_{k\to\infty} A(x_{n_k},x_{n_k},\cdots,x_{n_k},x_{m_k}) = \lim_{k\to\infty} A(x_{n_k+1},x_{n_k+1},\cdots,x_{n_k+1},x_{m_k}) = \varepsilon$$

Analogous, it can be proved that

$A(x_{m_k-1},x_{m_k-1},\cdots,x_{m_k-1},x_{n_{k-1}}), A(x_{m_k-1},x_{m_k-1},\cdots,x_{m_k-1},x_{n_k+1}),$
$A(x_{m_k+1},x_{m_k+1},\cdots,x_{m_k+1},x_{n_k+1}),\cdots$

tends to ε. ■

Theorem 20 *Let (X,A) be an $A-$metric space. Suppose that the mapping $f,g : X \to X$ satisfy*

$$A(fx_1,fx_2,\cdots,fx_p) \leq \varphi\left(\max\left\{\begin{array}{c} A(gx_1,gx_1,\cdots,fx_1), \\ A(gx_2,gx_2,\cdots,fx_2), \\ \vdots \\ A(gx_p,gx_p,\cdots,fx_p) \end{array}\right\}\right) \quad (1)$$

for all $x = (x_1,x_2,\cdots,x_p) \in X^p$. Where the function $\varphi \in \Phi$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X.Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Assume that f and g satisfy the condition (1). Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is x_1 such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_p\}$ such that $gx_p = fx_{p-1}$ for all $p \in \mathbb{N}$. If there is $p \in \mathbb{N}$ such that $gx_p = fx_{p-1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_{p+1} \neq gx_p$ for all $p \in \mathbb{N}$. Therefore for each $p \in \mathbb{N}$, we obtain that

$$\begin{aligned} A(gx_p, gx_p, \cdots gx_{p+1}) = A(fx_{p-1}, fx_{p-1}, \cdots, fx_p) &\leq \varphi(\max\{A(gx_{p-1}, gx_{p-1}, \cdots, fx_{p-1}), \\ &\quad A(gx_{p-1}, gx_{p-1}, \cdots, fx_{p-1}), \cdots, \\ &\quad A(gx_p, gx_p, \cdots, fx_p)\}) \\ &\leq \varphi(\max\{A(gx_{p-1}, gx_{p-1}, \cdots, fx_{p-1}), \\ &\quad A(gx_p, gx_p, \cdots, fx_p)\}) \\ &\leq \varphi(\max\{A(gx_{p-1}, gx_{p-1}, \cdots, gx_p), \\ &\quad A(gx_p, gx_p, \cdots, gx_{p+1})\}) \end{aligned}$$

If

$$\max\{A(gx_{p-1}, gx_{p-1}, \cdots, gx_p), A(gx_p, gx_p, \cdots, gx_{p+1})] = A(gx_p, gx_p, \cdots, gx_{p+1})$$

then

$$\begin{aligned} A(gx_p, gx_p, \cdots gx_{p+1}) &\leq \varphi(A(gx_p, gx_p, \cdots, gx_{p+1})) \\ &< A(gx_p, gx_p, \cdots, gx_{p+1}) \end{aligned}$$

which leads to a contradiction. This implies that

$$A(gx_p, gx_p, \cdots gx_{p+1}) \leq A(gx_{p-1}, gx_{p-1}, \cdots, gx_p)$$

That is, for each $p \in \mathbb{N}$, we have

$$\begin{aligned} A(gx_p, gx_p, \cdots gx_{p+1}) = A(fx_{p-1}, fx_{p-1}, \cdots, fx_p) &\leq \varphi(A(gx_{p-1}, gx_{p-1}, \cdots, gx_p)) \\ &\leq \varphi^2(A(gx_{p-2}, gx_{p-2}, \cdots, gx_{p-1}) \\ &\quad \vdots \\ &\leq \varphi^p(A(gx_0, gx_0, \cdots, gx_1) \end{aligned}$$

So we have $\lim_{p\to\infty} A(gx_p, gx_p, \cdots gx_{p+1}) = 0$. Now we prove that $\{gx_p\} = \{fx_{p-1}\}$ is a Cauchy sequence. Suppose that $\{gx_p\} = \{fx_{p-1}\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$A(gx_{m_k+1}, gx_{m_k+1}, \cdots, gx_{m_k+1}, gx_{n_k+1}), A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{n_k}) \tag{2}$$

Putting now in (1) $x_p = x_{m_k}$ for $p = 1, 2, \cdots, n-1$ and $x_p = x_{n_k}$ we obtain

$$\begin{aligned} &A(gx_{m_k+1}, gx_{m_k+1}, \cdots, gx_{m_k+1}, gx_{n_k+1}) \\ &= A(fx_{m_k}, fx_{m_k}, \cdots, fx_{m_k}, fx_{n_k}) \\ &\leq \varphi(\max\{A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, fx_{m_k}), \cdots, \\ &\quad A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, fx_{m_k}), A(gx_{n_k}, gx_{n_k}, \cdots, gx_{n_k}, fx_{n_k})\}) \\ &\leq \varphi(\max\{A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{m_k+1}), \\ &\quad A(gx_{n_k}, gx_{n_k}, \cdots, gx_{n_k}, gx_{n_k+1})\}) \end{aligned}$$

If

$$\max\left\{\begin{array}{c} A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{m_k+1}), \\ A(gx_{n_k}, gx_{n_k}, \cdots, gx_{n_k}, gx_{n_k+1}) \end{array}\right\} = A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{m_k+1})$$

and since $A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{m_k+1}) > 0$ we have

$$\begin{aligned} &A(gx_{m_k+1}, gx_{m_k+1}, \cdots, gx_{m_k+1}, gx_{n_k+1}) \\ &\leq \varphi(A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{m_k+1})) \\ &< A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{m_k+1}) \end{aligned}$$

by going to the limit as $k \to \infty$ we obtain

$$\varepsilon \leq \lim_{k\to\infty} \varphi(A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{m_k+1}) \leq 0$$

Contradiction.

Analogous, If

$$\max\left\{\begin{array}{c} A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{m_k+1}), \\ A(gx_{n_k}, gx_{n_k}, \cdots, gx_{n_k}, gx_{n_k+1}) \end{array}\right\} = A(gx_{n_k}, gx_{n_k}, \cdots, gx_{n_k}, gx_{n_k+1})$$

we got a contradiction.

So, it follows that $\{gx_p\} = \{fx_{p-1}\}$ is a Cauchy sequence. By the completeness of $g(X)$ or $(f(X))$, we obtain that $\{gx_p\}$ is convergent to some $q \in g(X)$. So there exists $a \in X$ such that $ga = q$. We will show that $ga = fq$. Suppose the contrary, $ga \neq fa$. By (1), we have

$$\begin{aligned} & A(gx_p, gx_p, \ldots, gx_p, fa) \\ &= A(fx_{p-1}, fx_{p-1}, \ldots, fx_{p-1}, fa) \\ &\leq \varphi\left(\max\left\{\begin{array}{c} A(gx_{p-1}, gx_{p-1}, \ldots, gx_{p-1}, gx_p), \\ A(gx_{p-1}, gx_{p-1}, \ldots, gx_{p-1}, gx_p), \\ A(ga, ga, \cdots, ga, fa) \end{array}\right\}\right) \\ &\leq \varphi\left(\max\left\{\begin{array}{c} A(gx_{p-1}, gx_{p-1}, \ldots, gx_{p-1}, gx_p), \\ A(ga, ga, \cdots, ga, fa) \end{array}\right\}\right) \end{aligned}$$

If

$$\begin{aligned} & \varphi\left(\max\left\{\begin{array}{c} A(gx_{p-1}, gx_{p-1}, \ldots, gx_{p-1}, gx_p), \\ A(ga, ga, \cdots, ga, fa) \end{array}\right\}\right) \\ &= A(gx_{p-1}, gx_{p-1}, \ldots, gx_{p-1}, gx_p) \end{aligned}$$

we obtain

$$\begin{aligned} A(gx_p, gx_p, \ldots, gx_p, fa) &\leq \varphi(A(gx_{p-1}, gx_{p-1}, \ldots, gx_{p-1}, gx_p)) \\ &< A(gx_{p-1}, gx_{p-1}, \ldots, gx_{p-1}, gx_p) \end{aligned}$$

by taking $p \to \infty$, we have $A(ga, ga, \cdots, ga, fa) = 0$ and so $ga = fa$.
If

$$\begin{aligned} & \varphi\left(\max\left\{\begin{array}{c} A(gx_{p-1}, gx_{p-1}, \ldots, gx_{p-1}, gx_p), \\ A(ga, ga, \cdots, ga, fa \end{array}\right\}\right) \\ &= A(ga, ga, \cdots, ga, fa) \end{aligned}$$

we obtain

$$A(gx_p, gx_p, \ldots, gx_p, fa) \leq \varphi(A(ga, ga, \cdots, ga, fa))$$

by taking $p \to \infty$, we have

$$\begin{aligned} A(ga, ga, \cdots, ga, fa) &\leq \varphi(A(ga, ga, \cdots, ga, fa)) \\ &< A(ga, ga, \cdots, ga, fa) \end{aligned}$$

which leads to a contradiction. Therefore $ga = fa$.
We now show that f and g have a unique point of coincidence. Suppose that $fl = gl$ for some $l \in X$. By applying (1), it follows that

$$\begin{aligned} A(ga, ga, \ldots, ga, gl) = A(fa, fa, \ldots, fa, fl) \\ &\leq \varphi(\max\{A(ga, ga, \ldots, ga, fa), \\ &\quad A(ga, ga, \ldots, ga, fa), \\ &\quad A(gl, gl, \cdots, gl, fl)\}) \\ &= 0 \end{aligned}$$

Therefore $ga = gl$. This implies that f and$g$ have a unique point of coincidence. By proposition 18, we can conclude that f and$g$ have a unique fixed point. ■

Corollary 21 *Let (X, A) be an $A-$metric space. Suppose that the mapping $f, g : X \to X$ satisfy*

$$A(fx_1, fx_2, \cdots, fx_p) \leq k\left(\max\left\{\begin{array}{c} A(gx_1, gx_1, \cdots, fx_1), \\ A(gx_2, gx_2, \cdots, fx_2), \\ \vdots \\ A(gx_p, gx_p, \cdots, fx_p) \end{array}\right\}\right)$$

for all $x = (x_1, x_2, \cdots, x_p) \in X^p$. Where the constant $k \in [0, 1)$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X.Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Putting $\varphi(t) = kt, t \geq 0, t \in [0, 1)$ in (1), the result follows. ■

Theorem 22 *Let (X, A) be an $A-$metric space. Suppose that the mapping $f, g : X \to X$ satisfy*

$$A(fx_1, fx_2, \cdots, fx_p) \leq \max\left\{\begin{array}{c} \varphi(A(gx_1, gx_1, \cdots, fx_1)), \\ \varphi(A(gx_2, gx_2, \cdots, fx_2)), \\ \vdots \\ \varphi(A(gx_p, gx_p, \cdots, fx_p)) \end{array}\right\}$$

for all $x = (x_1, x_2, \cdots, x_p) \in X^p$. Where the function $\varphi \in \Phi$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X.Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. The proof is very similar to the proof of theorem 20, so, we omitted it. ■

Theorem 23 *Let (X,A) be an $A-$metric space. Suppose that the mapping $f,g:X\to X$ satisfy*

$$A(fx_1,fx_2,\cdots,fx_p)\le\varphi\left(A(gx_1,gx_2,\cdots,gx_{p-1},gx_p)\right)\quad(3)$$

for all $x=(x_1,x_2,\cdots,x_p)\in X^p$. Where the function φ satisfies $\lim_{s\to t_+}\varphi(t)<t$ for all $t>0$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X.Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Assume that f and g satisfy the condition (3). Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is x_1 such that $gx_1=fx_0$. By continuing the process as before, we can construct a sequence $\{gx_p\}$ such that $gx_p=fx_{p-1}$ for all $p\in\mathbb{N}$. If there is $p\in\mathbb{N}$ such that $gx_p=fx_{p-1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_{p+1}\ne gx_p$ for all $p\in\mathbb{N}$. Therefore for each $p\in\mathbb{N}$, we obtain that

$$\begin{aligned}A(gx_p,gx_p,\cdots gx_{p+1})&=A(fx_{p-1},fx_{p-1},\cdots,fx_p)\\&\le\varphi\left(A(gx_{p-1},gx_{p-1},\cdots,gx_p)\right)\\&\le\varphi^2\left(A(gx_{p-2},gx_{p-2},\cdots,gx_{p-1})\right)\\&\ \ \vdots\\&\le\varphi^n\left(A(gx_0,gx_0,\cdots,gx_1)\right)\end{aligned}$$

this implies that

$$\lim_{p\to\infty}A(gx_p,gx_p,\cdots gx_{p+1})=0.$$

If $\{gx_p\}=\{fx_{p-1}\}$ is not a Cauchy sequence in $A-$metric space, then their exist an $\varepsilon>0$ and two sequences$\{m_k\}$ and $\{n_k\}$, $n_k>m_k>k$ of positive integers such that the following sequences tend to ε when $k\to\infty$:

$$A(gx_{m_k+1},gx_{m_k+1},\cdots,gx_{m_k+1},gx_{n_k+1}),A(gx_{m_k},gx_{m_k},\cdots,gx_{m_k},gx_{n_k})\quad(4)$$

Putting now in (3) $x_p=x_{m_k}$ for $p=1,2,\cdots,n-1$ and $x_p=x_{n_k}$ we obtain

$$\begin{aligned}&A(gx_{m_k+1},gx_{m_k+1},\cdots,gx_{m_k+1},gx_{n_k+1})\\&=A(fx_{m_k},fx_{m_k},\cdots,fx_{m_k},fx_{n_k})\\&\le\varphi(A(gx_{m_k},gx_{m_k},\cdots,gx_{m_k},gx_{n_k}))\end{aligned}$$

If

$$\begin{aligned}&\max\{A(gx_{m_k},gx_{m_k},\cdots,gx_{m_k},gx_{m_k+1}),\\&A(gx_{n_k},gx_{n_k},\cdots,gx_{n_k},gx_{n_k+1})\}\\&=A(gx_{m_k},gx_{m_k},\cdots,gx_{m_k},gx_{m_k+1})\end{aligned}$$

Letting $k\to\infty$ and using the assumption of the mapping φ we obtain

$$\begin{aligned}\varepsilon&\le\lim_{t\to\varepsilon^+}\varphi(\underbrace{A(gx_{m_k},gx_{m_k},\cdots,gx_{m_k},gx_{n_k})}_{t})\\&=\lim_{t\to\varepsilon^+}\varphi(t)\\&<\varepsilon\end{aligned}$$

Contradiction. Therefore the sequences $\{gx_p\}=\{fx_{p-1}\}$ are Cauchy sequence. By the completeness of $g(X)$ or $(f(X))$, we obtain that $\{gx_p\}$is convergent to some $q\in g(X)$. so there is $a\in X$ such that $ga=q$. We will show that $ga=fa$. By (3) we have

$$\begin{aligned}A(ga,ga,\cdots,ga,fa)&\le(n-1)A(ga,ga,\cdots,ga,gx_{p+1})\\&\quad+A(gx_{p+1},gx_{p+1},\cdots,gx_{p+1},fa)\\&\le(n-1)A(ga,ga,\cdots,ga,gx_{p+1})\\&\quad+\varphi(A(gx_p,gx_p,\cdots,gx_p,ga))\\&\le(n-1)A(ga,ga,\cdots,ga,gx_{p+1})\\&\quad+A(gx_p,gx_p,\cdots,gx_p,ga)\end{aligned}$$

By taking $p\to\infty$, we have $A(ga,ga,\cdots,ga,fa)=0$ and so $ga=fa$. We now show that f and g have a unique point of coincidence. Suppose that $fq=gq$ for some $q\in X$. Assume that $ga\ne gq$. By applying (3), it follows that

$$\begin{aligned}A(ga,ga,\cdots,ga,gq)&=A(fa,fa,\cdots,fa,fq)\\&\le\varphi(A(ga,ga,\cdots,ga,gq))\\&<A(ga,ga,\cdots,ga,gq)\end{aligned}$$

which leads to a contradiction. Therefore $ga = gq$. This implies that f and g have a unique common fixed point. ■

Corollary 24 *Let (X,A) be a complete $A-$metric space. Suppose that the mapping $f : X \to X$ satisfy*

$$A(fx_1, fx_2, \cdots, fx_p) \leq \varphi\left(A(gx_1, gx_2, \cdots, gx_{p-1}, gx_p)\right)$$

for all $x = (x_1, x_2, \cdots, x_p) \in X^p$. Where the function φ satisfies $\lim\limits_{s\to t_+} \varphi(t) < t$ for all $t > 0$. Then f has a unique fixed point.

Proof. By setting g to be the identity function on X. ■

Theorem 25 *Let (X,A) be an $A-$metric space. Suppose that the mapping $f,g : X \to X$ satisfy*

$$\begin{aligned} A(fx_1, fx_2, \cdots, fx_p) \leq\ & k_1\varphi(A(gx_1, gx_1, \cdots, fx_1)) \\ & + k_2\varphi(A(gx_2, gx_2, \cdots, fx_2)) \\ & + \vdots \\ & + k_p\varphi(A(gx_p, gx_p, \cdots, fx_p)) \end{aligned} \quad (5)$$

for all $x = (x_1, x_2, \cdots, x_p) \in X^p, k_i \geq 0$ for all $i \in \{1, 2, \cdots, p\}$ and $\sum\limits_{i=1}^{p} k_i < 1$. The function $\varphi \in \Phi$. If the range of g contains the range of f and one of $f(X)$ or $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Assume that f and g satisfy the condition (5). Let x_0 be an arbitrary point in X. Since the range of g contains the range of f, there is x_1 such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_p\}$ such that $gx_p = fx_{p-1}$ for all $p \in \mathbb{N}$. If there is $p \in \mathbb{N}$ such that $gx_p = fx_{p-1}$, then f and g have a point of coincidence. Thus we can suppose that $gx_{p+1} \neq gx_p$ for all $p \in \mathbb{N}$. Therefore for each $p \in \mathbb{N}$, we obtain that

$$\begin{aligned} A(gx_p, gx_p, \cdots gx_{p+1}) &= A(fx_{p-1}, fx_{p-1}, \cdots, fx_p) \\ &\leq k_1\varphi\left(A(gx_{p-1}, gx_{p-1}, \cdots, fx_{p-1})\right) \\ &\quad + k_2\varphi\left(A(gx_{p-1}, gx_{p-1}, \cdots, fx_{p-1})\right) \\ &\quad \vdots \\ &\quad + k_{p-1}\left(A(gx_{p-1}, gx_{p-1}, \cdots, fx_{p-1})\right) \\ &\quad + k_p\left(A(gx_p, gx_p, \cdots, fx_p)\right) \end{aligned}$$

Then

$$\begin{aligned} A(gx_p, gx_p, \cdots gx_{p+1}) &\leq k_1\varphi\left(A(gx_{p-1}, gx_{p-1}, \cdots, gx_p)\right) \\ &\quad + k_2\varphi\left(A(gx_{p-1}, gx_{p-1}, \cdots, x_p)\right) \\ &\quad \vdots \\ &\quad + k_{p-1}\varphi\left(A(gx_{p-1}, gx_{p-1}, \cdots, gx_p)\right) \\ &\quad + k_p\varphi\left(A(gx_p, gx_p, \cdots, gx_{p+1})\right) \\ &< \left(\sum_{i=1}^{p-1} k_i\right) A(gx_{p-1}, gx_{p-1}, \cdots, gx_p) \\ &\quad + k_pA(gx_p, gx_p, \cdots, gx_{p+1}) \end{aligned}$$

Now, we have

$$A(gx_p, gx_p, \cdots gx_{p+1}) < \frac{\sum\limits_{i=1}^{p-1} k_i}{1-k_p} A(gx_{p-1}, gx_{p-1}, \cdots, gx_p)$$

Let $r = \frac{\sum_{i=1}^{p-1} k_i}{1-k_p}$, then

$$\begin{aligned} A(gx_p, gx_p, \cdots gx_{p+1}) &< rA(gx_{p-1}, gx_{p-1}, \cdots, gx_p) \\ &< r^2A(gx_{p-2}, gx_{p-2}, \cdots, gx_{p-1}) \\ &\quad \vdots \\ &< r^pA(gx_0, gx_0, \cdots, gx_1) \end{aligned}$$

this implies that

$$\lim_{p\to\infty} A(gx_p, gx_p, \cdots gx_{p+1}) = 0.$$

If $\{gx_p\} = \{fx_{p-1}\}$ is not a Cauchy sequence in $A-$metric space, then their exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$, $n_k > m_k > k$ of positive integers such that the following sequences tend to ε when $k \to \infty$:

$$A(gx_{m_k+1}, gx_{m_k+1}, \cdots, gx_{m_k+1}, gx_{n_k+1}), A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{n_k})$$

Putting now in (5) $x_p = x_{m_k}$ for $p = 1, 2, \cdots, n-1$ and $x_p = x_{n_k}$ and using the fact that

$A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k}, gx_{m_k+1}) > 0$ and $A(gx_{n_k}, gx_{n_k}, \cdots, gx_{n_k}, gx_{n_k+1}) > 0$

we obtain

$$\begin{aligned}
&A(gx_{m_k+1}, gx_{m_k+1}, \cdots, gx_{m_k+1}, gx_{n_k+1}) \\
&= A(fx_{m_k}, fx_{m_k}, \cdots, fx_{m_k}, fx_{n_k}) \\
&\le k_1\varphi\left(A(gx_{m_k}, gx_{m_k}, \cdots, fx_{m_k})\right) \\
&\quad + k_2\varphi\left(A(gx_{m_k}, gx_{m_k}, \cdots, fx_{m_k})\right) \\
&\quad \vdots \\
&\quad + k_p\varphi\left(A(gx_{n_k}, gx_{n_k}, \cdots, fx_{n_k})\right)
\end{aligned}$$

hence

$$\begin{aligned}
&A(gx_{m_k+1}, gx_{m_k+1}, \cdots, gx_{m_k+1}, gx_{n_k+1}) \\
&= A(fx_{m_k}, fx_{m_k}, \cdots, fx_{m_k}, fx_{n_k}) \\
&\le k_1 A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k+1}) \\
&\quad + k_2 A(gx_{m_k}, gx_{m_k}, \cdots, gx_{m_k+1}) \\
&\quad \vdots \\
&\quad + k_p A(gx_{n_k}, gx_{n_k}, \cdots, gx_{n_k+1})
\end{aligned}$$

Letting $k \to \infty$, we obtain $\varepsilon \le 0$. A contradiction. So the sequences $\{gx_p\} = \{fx_{p-1}\}$ is Cauchy sequence. By the completeness of $g(X)$ (or $f(X)$), we obtain that $\{gx_p\}$ converges to some $q \in g(X)$. So there exists $a \in X$ such that $ga = q$. We will show that $ga = fa$. Suppose the contrary $ga \neq fa$. By (5), we have

$$\begin{aligned}
&A(gx_p, gx_p, \cdots, gx_p, fa) \\
&= A(fx_{p-1}, fx_{p-1}, \cdots, fx_{p-1}, fa) \\
&\le k_1\varphi(A(gx_{p-1}, gx_{p-1}, \cdots, gx_{p-1}, gx_p)) \\
&\quad + k_2\varphi(A(gx_{p-1}, gx_{p-1}, \cdots, gx_{p-1}, gx_p)) \\
&\quad \vdots \\
&\quad + k_p\varphi(A(ga, ga, \cdots, ga, fa))
\end{aligned}$$

Letting $p \to \infty$, we have

$$\begin{aligned}
A(ga, ga, \cdots, ga, fa) &\le k_p\varphi(A(ga, ga, \cdots, ga, fa)) \\
&< k_p A(ga, ga, \cdots, ga, fa) \\
&< A(ga, ga, \cdots, ga, fa)
\end{aligned}$$

we got a contradiction. So $ga = fa$. The proof that f and g have a unique point of coincidence is similar to the proof of theorem 20, so we omitted it. ■

III. APPLICATION

Later, from the previous obtained results, we deduce some coincidence point results for mappings satisfying a contraction of an integral type as an application of Theorem 14 above. For this purpose, let

$$Y = \left\{ \begin{array}{c} \chi, \chi : \mathbb{R}^+ \to \mathbb{R}^+, \text{ satisfies that } \chi \text{ is a Lebesgue integrable,} \\ \text{summable on each compact of subset of } \mathbb{R}^+ \\ \text{and } \int_0^\varepsilon \chi(t)dt > 0 \text{ for each } \varepsilon > 0 \text{ and subadditive, that is} \\ \int\limits_0^{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p} \chi(t)dt \le \sum\limits_{i=1}^{p} \int\limits_0^{\varepsilon_i} \chi(t)dt \;\text{ for each } \varepsilon_i > 0, i = 1, \cdots, p \end{array} \right\}$$

Example 26 *We consider* $\chi(x) = \frac{1}{1+x}, \chi : \mathbb{R}^+ \to \mathbb{R}^+, \chi$ *is Lebesgue integrable which is nonnegative, satisfies*

$$\begin{aligned}
\int\limits_0^{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p} \chi(t)dt &= \ln(1+\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_p) \\
&\le \ln\left[(1+\varepsilon_1)(1+\varepsilon_2)\cdots(1+\varepsilon_p)\right] \\
&= \ln(1+\varepsilon_1) + \ln(1+\varepsilon_2) + \cdots + \ln(1+\varepsilon_p) \\
&= \sum_{i=1}^{p} \int\limits_0^{\varepsilon_i} \chi(t)dt
\end{aligned}$$

This shows that χ *is an example of subadditive, nonnegative, Lebesgue integrable function.*

Theorem 27 *Let* (X, A) *be an* $A-$*metric space. Suppose that the mapping* $f, g : X \to X$ *satisfy*

$$\int_0^{A(fx_1, fx_2, \cdots, fx_p)} \chi(t)dt \tag{6}$$
$$\le \int_0^{\varphi\left(\max\left\{\begin{array}{c} A(gx_1, gx_1, \cdots, fx_1), A(gx_2, gx_2, \cdots, fx_2), \\ \cdots, A(gx_p, gx_p, \cdots, fx_p) \end{array}\right\}\right)} \chi(t)dt$$

for all $x = (x_1, x_2, \cdots, x_p) \in X^p$. *Where the function* $\varphi \in \Phi$. *If the range of* g *contains the range of* f *and one of* $f(X)$ *or* $g(X)$ *is a complete subspace of* X. *Then* f *and* g *have a unique point of coincidence in* X. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. For $\chi \in Y$, We consider the function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$defined by $\Lambda(x) = \int_0^x \chi(t)dt$. we note that $\Lambda \in \Psi$. Thus the inequality (**??**)becomes

$$\Lambda(A(fx_1, fx_2, \cdots, fx_p)) \le$$
$$\Lambda\left(\varphi\left(\max\left\{\begin{array}{c} A(gx_1, gx_1, \cdots, fx_1), A(gx_2, gx_2, \cdots, fx_2), \\ \cdots, A(gx_p, gx_p, \cdots, fx_p) \end{array}\right\}\right)\right) \tag{7}$$

Setting in (6), and $\Lambda \circ \varphi = \varphi_1$ and $\Lambda \circ A = A_1$ we obtain

$$A_1(fx_1, fx_2, \cdots, fx_p) \leq$$
$$\varphi_1 \left(\max \left\{ \begin{array}{c} A(gx_1, gx_1, \cdots, fx_1), A(gx_2, gx_2, \cdots, fx_2), \\ \cdots, A(gx_p, gx_p, \cdots, fx_p) \end{array} \right\} \right)$$

therefore from Theorem 14, the desired result follows. ■

Example 28 *Let* $X = [0,2]^p = [0,2] \times [0,2] \times \cdots \times [0,2]$, $A(x_1, x_2, \cdots, x_p) = \max\{|x_1 - x_2|, |x_2 - x_3|, \cdots, |x_1 - x_p|\}$ *and* $\varphi \in \Phi$. *Define* $fx = 1$ *and* $gx = 2 - x$, $x \in [0,2]$, *we obtain that* f *and* g *satisfy* (1) *in theorem* 20 . *Indeed, we have* $A(fx_1, fx_2, \cdots, fx_p) = 0$,

$$\begin{aligned} &\varphi(\max\{A(gx_1, gx_1, \cdots, fx_1), A(gx_2, gx_2, \cdots, fx_2), \\ &\cdots, A(gx_p, gx_p, \cdots, fx_p)\}) \\ = &\varphi(\max\{|x_1 - x_2|, |x_2 - x_3|, \cdots, |x_1 - x_p|\}) \end{aligned}$$

Hence

$$0 \leq \varphi(\max\{|x_1 - x_2|, |x_2 - x_3|, \cdots, |x_1 - x_p|\}) \text{ for all } x = (x_1, x_2, \cdots, x_p) \in X^p$$

It is obvious that the range of g *contains the range of* f *and* $g(X)$ *is a complete subspace of* (X, A). *Furthermore,* f *and* g *are weakly compatible. Thus all assumptions in Theorem* 14 *are satisfied. This implies that* f *and* g *have a unique common fixed point which is* $x = 1$.

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