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Fixed Points for Cyclic Contractions via Simulation Functions and Applications

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The main aim of this paper is to prove the existence and uniqueness of a fixed point for a class of cyclic contractions via simulation functions in the context of metric-like spaces. These results are generalizations of the recent results in [M. Jleli, B. Samet, An improvement result concerning fixed point theory for cyclic contractions, Carpathian J. Math., 32 (2016), 339-347]. We also prove a stability and well-posedness of a fixed point problem. Moreover, some examples and an application on the existence and uniqueness of solutions to a class of functional integral equations are given to support the obtained results.

Keywords: metric-like, fixed point, simulation functions, cyclic contraction.

I. INTRODUCTION AND PRELIMINARIES

It is well known that the Banachąŕs contraction principle (BCP) [2] is a fundamental result in the field of fixed point theory. This famous result has a strong applications in proving the existence and uniqueness of solution of integral equations.

Theorem I.1 *[2] Let (X,d) be a complete metric space and $T: X \to X$ a mapping. If there exists a real number $k \in [0,1)$ such that for all $x,y \in X$, the following inequality holds:*

$$d(Tx,Ty) \le kd(x,y), \tag{I.1}$$

then T has a unique fixed point in X.

Notice that, the contractive condition (I.1) is satisfied for all $x,y \in X$, which forces the mapping T to be continuous, and so the principle is not applicable if T is discontinuous. Moreover, Banach contraction principle is dependent on the continuity of usual metric. This also brings us a limitation to utilize this principle. To overcome this difficulty, the (BCP) has been extended and generalized in many various directions (by generalizing (or extending) the condition contraction (I.1) or by changing the topology (via generalized distances)).

The following generalization is due to Kirk et al. [7].

Theorem I.2 *[7] Let (X,d) be a complete metric space and $\{A_i\}_{i=1}^m$ be a finite family of nonempty closed subsets of X. Let $T: \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) $T(A_i) \subseteq A_{i+1}$ for all $i = 1, \cdots, m$, with $A_{m+1} = A_1$;

(ii) The mapping T satisfies a cyclic contraction, that is, there exists some constant $k \in (0,1)$ such that

$$d(Tx,Ty) \le kd(x,y) \quad for\, all\, (x,y) \in A_i \times A_{i+1},\, i = 1, \cdots, m.$$

Then T has a unique fixed point in $\bigcap_{i=1}^m A_i$.

It is know that the above result has been extended in many various directions by several authors, see [12], [10], [9], [6], [16], [8],[11] and [13] and the references therein. In 2016, Jleli and Samet have weakened the closure condition that often used in this context and they proved the following result.

Theorem I.3 *[1] Let (X,d) be a complete metric space and $\{A_i\}_{i=1}^m$ be a finite number of nonempty subsets of X. Let $T: \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) A_1 is closed.

(i) $T(A_i) \subseteq A_{i+1}$ *for all* $i = 1, \cdots, m$, *with* $A_{m+1} = A_1$.

(ii) *There exists a (c)-comparison function* $\varphi : [0, \infty)$ *such that*

$$d(Tx, Ty) \le \varphi(d(x,y)) \quad \text{for all} (x,y) \in A_i \times A_{i+1}, i = 1, \cdots, m.$$

Then T *has a unique fixed point* $z \in \bigcap_{i=1}^m A_i$. *For any* $x_0 \in \bigcup_{i=1}^m A_i$, *the Picard sequence* $\{T^n x_0\}$ *converges to z.*

The notion of simulation functions is introduced by Khojasteh, Shukla and Radenović [28] as a generalization of the Banach's contraction principle. The above notion is slightly modified later by Argoubi et al. [15] by withdrawing a condition. In 2016, the notion of generalized simulation functions is introduced by J. Chen and X. Tang [14]. On the other hand, the notion of metric-like (dislocated) metric spaces was rediscovered by Harandi [27] as a generalization of a metric space. For fixed point results on metric-like spaces, see [23]-[27]. In what follows, we recall some notations and definitions we will need in the sequel.

Definition I.4 *[27] Let X be a nonempty set. A function* $\sigma : X \times X \to \mathbb{R}^+$ *is said to be a metric-like (or a dislocated metric) on X if for any* $x, y, z \in X$, *the following conditions hold:*

(σ_1) $\sigma(x,y) = 0 \Longrightarrow x = y$;

(σ_2) $\sigma(x,y) = \sigma(y,x)$;

(σ_3) $\sigma(x,z) \le \sigma(x,y) + \sigma(y,z)$.

The pair (X, σ) is then called a metric-like space. Note that each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls $B_\sigma(x, \varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Let (X, σ) be a metric-like space. A sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x,x)$.

A sequence $\{x_n\}$ in X is called σ-Cauchy if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite. The metric-like space (X, σ) is called complete if for each σ-Cauchy sequence $\{x_n\}$, there is some $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n, x_m).$$

Lemma I.5 *Let* (X, σ) *be a metric-like space and* $\{x_n\}$ *be a sequence that converges to x with* $\sigma(x,x) = 0$. *Then, for each* $y \in X$ *one has*

$$\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y).$$

Definition I.6 *[15] A simulation function is a mapping* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *satisfying the following conditions:*

(ζ_1) $\zeta(t,s) < s - t$ *for all* $t, s > 0$;

(ζ_2) *if* $\{t_n\}$ *and* $\{s_n\}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0, \infty)$, *then*

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

Let $\mathscr{Z}^*$ be the set of simulation functions in the sense of Argoubi et al. [15].

Example I.7 *[15] Let* $\zeta_\lambda : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *be the function defined by*

$$\zeta_\lambda(t,s) = \begin{cases} 1 & \text{if } (t,s) = (0,0), \\ \lambda s - t & \text{otherwise}, \end{cases}$$

where $\lambda \in (0,1)$. *Then,* $\zeta_\lambda \in \mathscr{Z}^*$.

Example I.8 *Let* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *be the function defined by* $\zeta(t,s) = \psi(s) - \varphi(t)$ *for all* $t, s \ge 0$, *where* $\psi : [0, \infty) \to \mathbb{R}$ *is an upper semi-continuous function and* $\varphi : [0, \infty) \to \mathbb{R}$ *is a lower semi-continuous function such that* $\psi(t) < t \le \varphi(t)$ *for all* $t > 0$. *Then,* $\zeta \in \mathscr{Z}^*$.

Definition I.9 *[14] A generalized simulation function is a mapping* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *satisfying the following condition:*

$$\zeta(t,s) \le s - t, \quad \text{for all } t, s > 0.$$

Let $\mathfrak{I}$ be the set of generalized simulation functions.

Remark I.10 *Each simulation function is a generalized simulation function but the converse is not true in general.*

Example I.11 *Let* $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ *be the function defined by* $\zeta(t,s) = s - t$ *for all* $t, s \ge 0$,. *Then,* ζ *is a generalized simulation function but it is not a simulation function.*

The objective of this paper is to establish some fixed point results for generalized cyclic contractions in the context of metric-like spaces. Presented theorems extend, generalize and improve many existing results in the literature. Our obtained results are supported by some illustrated examples and an application on the existence and uniqueness of solutions to a class of functional integral equations.

II. FIXED POINTS RESULTS

Our results concern two types of cyclic contractions.

A. Cyclic contractions via simulation functions

Our first main result is the following.

Theorem II.1 *Let (X,σ) be a metric-like space. Let $\{A_i\}_{i=1}^m$ be a finite family of nonempty subsets of X. Let $T:\bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) $T(A_i)\subseteq A_{i+1}$ for all $i=1,\cdots,m$, with $A_{m+1}=A_1$;

(ii) there exists $i_0\in\{1,\cdots,m\}$ such that A_{i_0} is closed;

(iii) $\bigcup_{i=1}^m A_i$ is a complete subset of X;

(iv) there exists a simulation function $\zeta\in\mathscr{Z}^*$ such that

$$\zeta\left(\sigma(Tx,Ty),\sigma(x,y)\right)\geq 0 \qquad \text{(II.1)}$$
$$\text{for all}\,(x,y)\in A_i\times A_{i+1},\, i=1,\cdots,m.$$

Then for every $x_0\in\bigcup_{i=1}^m A_i$, the picard sequence $\{T^n x_0\}$ converges to u, the unique fixed point of T in $\bigcap_{i=1}^m A_i$ such that $\sigma(u,u)=0$.

Proof. Let $x_0\in\bigcup_{i=1}^m A_i$. Without loss of generality, let $x_0\in A_1$. Consider the Picard iteration $\{x_n\}$ defined by $x_{n+1}=Tx_n$ for all $n\geq 0$. If $x_n=x_{n+1}$ for some n, then $x_n=x_{n+1}=Tx_n$, that is, x_n is a fixed point of T and so the proof is complete.

Suppose that $x_n\neq x_{n+1}$ for all $n\geq 0$. For any $n\geq 0$, there is $i_n\in\{1,\cdots,m\}$ such that $x_n\in A_{i_n}$ and $x_{n+1}\in A_{i_n+1}$. By (II.1), we have

$$\begin{aligned}&\zeta\left(\sigma(x_{n+1},x_{n+2}),\sigma(x_n,x_{n+1})\right)\\ &=\zeta\left(\sigma(Tx_n,Tx_{n+1}),\sigma(x_n,x_{n+1})\right)\geq 0.\end{aligned} \qquad \text{(II.2)}$$

From the condition (ζ_1),

$$0\leq\zeta\left(\sigma(x_{n+1},x_{n+2}),\sigma(x_n,x_{n+1})\right)<\sigma(x_n,x_{n+1})-\sigma(x_{n+1},x_{n+2}).$$

Necessarily, we have

$$\sigma(x_{n+1},x_{n+2})<\sigma(x_n,x_{n+1}),\quad \text{for all } n\geq 0, \qquad \text{(II.3)}$$

which implies that $\{\sigma(x_n,x_{n+1})\}$ is a decreasing sequence of positive real numbers, so there exists $t\geq 0$ such that

$$\lim_{n\to\infty}\sigma(x_n,x_{n+1})=t. \qquad \text{(II.4)}$$

Suppose that $t>0$. By (II.2), (II.4) and the condition (ζ_2),

$$0\leq\limsup_{n\to\infty}\left(\sigma(x_{n+1},x_{n+2}),\sigma(x_n,x_{n+1})\right)<0,$$

which is a contradiction. Then, we conclude that $t=0$, that is

$$\lim_{n\to\infty}\sigma(x_n,x_{n+1})=0. \qquad \text{(II.5)}$$

Now, we shall prove that

$$\lim_{n,m\to\infty}\sigma(x_n,x_m)=0. \qquad \text{(II.6)}$$

Suppose to the contrary. Then there exists $\varepsilon>0$ for which we can find subsequences $(x_{m(k)})$ and $(x_{n(k)})$ of (x_n) with $n(k)>m(k)>k$ such that

$$\sigma(x_{n(k)},x_{m(k)})\geq\varepsilon. \qquad \text{(II.7)}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k)>m(k)>k$ and satisfying (II.7). Then

$$\sigma(x_{n(k)-1},x_{m(k)})<\varepsilon. \qquad \text{(II.8)}$$

Using (II.8) and the triangular inequality

$$\begin{aligned}\varepsilon\leq\sigma(x_{n(k)},x_{m(k)})&\leq\sigma(x_{n(k)},x_{n(k)-1})+\sigma(x_{n(k)-1},x_{m(k)})\\ &<\varepsilon+\sigma(x_{n(k)},x_{n(k)-1}).\end{aligned} \qquad \text{(II.9)}$$

Letting $k\to+\infty$ in (II.9) and using (II.5), we find

$$\lim_{k\to\infty}\sigma(x_{n(k)},x_{m(k)})=\varepsilon. \qquad \text{(II.10)}$$

On the other hand, for all k, there exists $j(k)$, $0 \leq j(k) \leq m$, such that $n(k) - m(k) + j(k) \equiv 1(m)$. Then $x_{m(k)-j(k)}$ (for k large enough, $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labeled sets A_i and A_{i+1} for certain $i = 1, \cdots, m$. From (II.1), we have

$$\begin{aligned} 0 &\leq \zeta(\sigma(x_{n(k)+1}, x_{m(k)-j(k)+1}), \sigma(x_{n(k)}, x_{m(k)-j(k)})) \\ &= \zeta(\sigma(Tx_{n(k)}, Tx_{m(k)-j(k)}), \sigma(x_{n(k)}, x_{m(k)-j(k)})). \end{aligned} \tag{II.11}$$

If $x_n = x_m$ for some $n < m$, then $x_{n+1} = Tx_n = Tx_m = x_{m+1}$ it follows from (II.3),

$$0 < \sigma(x_n, x_{n+1}) = \sigma(x_m, x_{m+1}) < \sigma(x_{m-1}, x_m) < \cdots < \sigma(x_n, x_{n+1}),$$

which is a contradiction. Then $x_n \neq x_m$ for all $n < m$.

Using the triangular inequality,

$$\begin{aligned} |\sigma(x_{n(k)}, x_{m(k)-j(k)}) - \sigma(x_{n(k)}, x_{m(k)})| &\leq \sigma(x_{m(k)-j(k)}, x_{m(k)}) \\ &\leq \sigma(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}) \\ &\quad + \cdots + \sigma(x_{m(k)-1}, x_{m(k)}) \\ &= \sum_{l=0}^{j(k)-1} \sigma(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \to 0 \\ &\quad \text{as } k \to \infty \text{ (from } (II.5)), \end{aligned}$$

which implies from (II.10) that

$$\lim_{k \to \infty} \sigma(x_{n(k)}, x_{m(k)-j(k)}) = \varepsilon. \tag{II.12}$$

Also

$$\begin{aligned} \sigma(x_{n(k)}, x_{m(k)-j(k)}) &\leq \sigma(x_{n(k)}, x_{n(k)+1}) + \sigma(x_{n(k)+1}, x_{m(k)-j(k)+1}) \\ &\quad + \sigma(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}), \end{aligned}$$

$$\begin{aligned} \sigma(x_{n(k)+1}, x_{m(k)-j(k)+1}) &\leq \sigma(x_{n(k)+1}, x_{n(k)}) + \sigma(x_{n(k)}, x_{m(k)-j(k)}) \\ &\quad + \sigma(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the two above inequalities and using (II.5) and (II.12), we find

$$\lim_{k \to \infty} \sigma(x_{n(k)+1}, x_{m(k)-j(k)+1}) = \varepsilon. \tag{II.13}$$

Now, using (II.11), (II.12), (II.13) and the condition (ζ_2), we get that

$$0 \leq \limsup_{k \to \infty} \zeta\left(\sigma(x_{n(k)+1}, x_{m(k)-j(k)+1}), \sigma(x_{n(k)}, x_{m(k)-j(k)})\right) < 0,$$

which is a contradiction. Then (II.6) holds. This shows that (x_n) is a σ-Cauchy sequence in $\bigcup_{i=1}^m A_i$.

Since $(\bigcup_{i=1}^m A_i, \sigma)$ is complete, hence there exists $u \in \bigcup_{i=1}^m A_i$ such that

$$\lim_{n \to \infty} \sigma(x_n, u) = \sigma(u, u) = \lim_{n,m \to \infty} \sigma(x_n, x_m) = 0. \tag{II.14}$$

We claim that u is a fixed point of T. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = u$ or $x_{n_k+1} = Tu$ for all k, then, $\sigma(u, Tu) = \sigma(u, x_{n_k+1})$. So by letting $k \to \infty$, we get $\sigma(u, Tu) = \sigma(u, u) = 0$. Then $Tu = u$ and so the proof is complete. So, without loss of generality, we may suppose that $x_n \neq u$ and $x_n \neq Tu$ for all nonnegative integer n.

Without loss of generality, we suppose that A_1 is closed. Since $x_0 \in A_1$, we have $(x_{nm})_{n \geq 0} \in A_1$. The fact that A_1 is closed together with (II.14) yield that $u \in A_1$.

Since $u \in A_1$ and $(x_{nm+1} = Tx_{nm})_{n \geq 0} \in A_2$, so applying (II.1) for $x = u$ and $y = x_{nm+1}$, we get that

$$0 \leq \zeta(\sigma(Tu, Tx_{nm+1}), \sigma(u, x_{nm+1})).$$

From the condition (ζ_1), we have

$$0 \leq \zeta(\sigma(Tu, x_{nm+2}), \sigma(u, x_{nm+1})) < \sigma(u, x_{nm+1}) - \sigma(Tu, x_{nm+2}).$$

It follows that

$$\sigma(Tu, x_{nm+2}) < \sigma(u, x_{nm+1}).$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\sigma(Tu, u) \leq \sigma(u, u) = 0,$$

which implies that $\sigma(Tu, u) = 0$. Thus $Tu = u$, that is, u is a fixed point of T. We shall prove that $u \in \bigcap_{i=1}^m A_i$.

Since $u \in A_1$ and $Tu = u$, so by condition (i), we get $u \in \bigcap_{i=1}^m A_i$. Now, we prove that u is the unique fixed point of T in $\bigcup_{i=1}^m A_i$. Assume that v is another fixed point of T in $\bigcup_{i=1}^m A_i$ with $u \neq v$. Taking $x = u$ and $y = v$ in (II.1), we get that

$$0 \leq \zeta(\sigma(Tu, Tv), \sigma(u, v)) = \zeta(\sigma(u, v), \sigma(u, v)) < \sigma(u, v) - \sigma(u, v) = 0,$$

which is a contradiction. Hence $u = v$.

■

Example II.2 *Take $X = (-2, +\infty)$ and $\sigma(x, y) = |x - y| + |x| + |y|$ for all $x, y \in X$. Clearly, (X, σ) is a metric-like space.*

Set $A_1 = [-1, \frac{1}{2}], A_2 = (-\frac{1}{2}, 1]$. *Notice that* $A_1 \cup A_2 = [-1, 1]$ *is a complete metric-like subset of* X. *However,* (X, σ) *is not. Consider the mapping* $T : A_1 \cup A_2 \to A_1 \cup A_2$ *given by* $Tx = -\frac{1}{2}x$ *for all* $x \in A_1 \cup A_2$. *Note that* $TA_1 = [-\frac{1}{4}, \frac{1}{2}] \subseteq A_2$ *and* $TA_2 = [-\frac{1}{2}, \frac{1}{4}) \subseteq A_1$. *Also,* A_1 *is closed. Take* $\zeta(t,s) = s - \frac{2+t}{1+t}t$ *for all* $s,t \geq 0$.

Now, we show that the contraction condition (II.1) is verified for all $(x,y) \in A_1 \times A_2$. *We have*

$$\sigma(Tx,Ty) = \frac{1}{2}(|x-y| + |x| + |y|) = \frac{1}{2}\sigma(x,y).$$

It follows that

$$\zeta(\sigma(Tx,Ty), \sigma(x,y)) = \sigma(x,y)[1 - \frac{4+\sigma(x,y)}{4+2\sigma(x,y)}] = \frac{(\sigma(x,y))^2}{4+2\sigma(x,y)} \geq 0.$$

Hence, all hypotheses of Theorem II.1 are verified. Here $u = 0$ *is the unique fixed point of* T. *Also* $0 \in A_1 \cap A_2$ *and* $\sigma(0,0) = 0$.

Example II.3 *Take* $X = (-5, +\infty)$ *and* $\sigma(x,y) = |x| + |y|$ *for all* $x,y \in X$. *Clearly,* (X, σ) *is a metric-like space. Set* $A_1 = [-1, \frac{1}{2}], A_2 = (-\frac{1}{2}, 1]$. *Notice that* $A_1 \cup A_2 = [-1, 1]$ *is a complete metric-like subset of* X. *However,* (X, σ) *is not. Consider the mapping* $T : A_1 \cup A_2 \to A_1 \cup A_2$ *given by*

$$Tx = \begin{cases} -\frac{x}{2}, & x \in [-1, 1), \\ \frac{1}{4}, & x = 1 \end{cases}$$

Note that $TA_1 = [-\frac{1}{4}, \frac{1}{2}] \subseteq A_2$ *and* $TA_2 = [-\frac{1}{2}, \frac{1}{4}) \subseteq A_1$. *Also,* A_1 *is closed. Take* $\zeta(t,s) = \frac{3}{4}s - t$ *for all* $s,t \geq 0$.

Now, we show that the contraction condition (II.1) is verified for all $(x,y) \in A_1 \times A_2$. *To check this we distinguish the following cases:*

Case 1. If $x \in A_1$ *and* $y \in (-\frac{1}{2}, 1)$. *Here, we have*

$$\begin{aligned} \zeta(\sigma(Tx,Ty), \sigma(x,y)) &= \frac{3}{4}\sigma(x,y) - \sigma(Tx,Ty) \\ &= \frac{3}{4}\sigma(x,y) - \frac{1}{2}\sigma(x,y) = \frac{1}{4}\sigma(x,y) \geq 0. \end{aligned}$$

Case 2. If $x \in A_1$ *and* $y = 1$. *Then, we have*

$$\zeta(\sigma(Tx,Ty), \sigma(x,y)) = \frac{3}{4}(|x|+1) - \frac{1}{2}|x| - \frac{1}{4} = \frac{1}{4}|x| + \frac{1}{2} \geq 0.$$

Thus, (II.1) holds. All hypotheses of Theorem II.1 are satisfied, and $u = 0$ *is the unique fixed point of* T. *Also* $0 \in A_1 \cap A_2$ *and* $\sigma(0,0) = 0$.

Notice that T *is not a contraction in the usual metric space* $X = ([-1,1], |.|)$ *because it is not continuous.*

Example II.4 *Take* $X = [0, 5)$ *and* $\sigma(x,y) = \max\{x,y\}$ *for all* $x,y \in X$. *Clearly,* (X, σ) *is a metric-like space. Set* $A_1 = [0, 2], A_2 = [0, 1)$. *Notice that* $A_1 \cup A_2 = [0, 2]$ *is a complete metric-like subset of* X. *However,* (X, σ) *is not. Consider the mapping* $T : A_1 \cup A_2 \to A_1 \cup A_2$ *given by*

$$Tx = \begin{cases} 0, & x \in [0, 2), \\ \frac{1}{2}, & x = 2 \end{cases}$$

Note that $TA_1 = \{0, \frac{1}{2}\} \subseteq A_2$ *and* $TA_2 = \{0\} \subseteq A_1$. *Also,* A_1 *is closed. Take* $\zeta(t,s) = \frac{1}{2}s - t$ *for all* $s,t \geq 0$.

Now, we show that the contraction condition (II.1) is verified for all $(x,y) \in A_1 \times A_2$. *To check this we distinguish the following cases:*

Case 1. If $x \in [0, 2)$ *and* $y \in A_2$. *Here, we have* $\sigma(Tx,Ty) = 0$. *Then*

$$\zeta(\sigma(Tx,Ty), \sigma(x,y)) = \frac{1}{2}\sigma(x,y) - \sigma(Tx,Ty) = \frac{1}{2}\sigma(x,y) \geq 0.$$

Case 2. If $x = 2$ *and* $y \in A_2$. *Here, we have* $\sigma(Tx,Ty) = \frac{1}{2}$. *Then*

$$\zeta(\sigma(Tx,Ty), \sigma(x,y)) = \frac{1}{2}\max\{2,y\} - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} \geq 0.$$

Thus, (II.1) holds. All hypotheses of Theorem II.1 are satisfied, and $u = 0$ *is the unique fixed point of* T. *Also* $0 \in A_1 \cap A_2$ *and* $\sigma(0,0) = 0$.

Notice that T *is not a contraction in the usual metric space* $X = ([0,2], |.|)$ *because it is not continuous.*

Example II.5 *Let* $X = \{0, 1, 2\}$ *and define* $\sigma : X \times X \to [0, \infty)$ *as follows*

$$\sigma(0,0) = \sigma(1,1) = 0,\ \sigma(2,2) = \frac{11}{20},\ \sigma(0,2) = \sigma(2,0) = \frac{4}{5},$$
$$\sigma(1,2) = \sigma(2,1) = \frac{3}{5},\ \sigma(1,0) = \sigma(0,1) = \frac{1}{2}.$$

Note that (X, σ) *is a complete metric-like space. Consider* $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$ *and* $A_1 \cup A_2 = X$. *It is obvious that* A_1 *is closed subset of* (X, σ). *We define* $T : X \to X$ *by*

$$T0 = T1 = 0 \ \text{ and } \ T2 = 1.$$

We have $T(A_1)=\{0\}\subseteq A_2$ and $T(A_2)=A_1$. Define $\zeta(t,s)=\frac{5}{6}s-t$ for all $s,t\geq 0$. We shall prove that (II.1) holds for all $(x,y)\in A_1\times A_2$. To check this we distinguish the following cases:

Case 1. If $x=y=0$. Here, we have $\sigma(Tx,Ty)=\sigma(0,0)=0$. Then,

$$\zeta(\sigma(Tx,Ty),\sigma(x,y))=\frac{5}{6}\sigma(x,y)-\sigma(Tx,Ty)=\frac{5}{6}\sigma(0,0)=0.$$

Case 2. If $x=0$ and $y=2$. Here, we have $\sigma(Tx,Ty)=\sigma(0,1)=\frac{1}{2}$. Then,

$$\begin{aligned}\zeta(\sigma(Tx,Ty),\sigma(x,y))&=\frac{5}{6}\sigma(0,2)-\sigma(0,1)=\frac{5}{6}\times\frac{4}{5}-\frac{1}{2}\\&=\frac{1}{6}>0.\end{aligned}$$

Case 3. If $x=1$ and $y=0$. We have $\sigma(Tx,Ty)=\sigma(0,0)=0$. Then,

$$\zeta(\sigma(Tx,Ty),\sigma(x,y))=\frac{5}{6}\sigma(1,0)>0.$$

Case 4. If $x=1$ and $y=2$. In this case, we have $\sigma(Tx,Ty)=\sigma(0,1)=\frac{1}{2}$. Then

$$\zeta(\sigma(Tx,Ty),\sigma(x,y))=\frac{5}{6}\sigma(1,2)-\sigma(0,1)=\frac{5}{6}\times\frac{3}{5}-\frac{1}{2}=0.$$

Thus, (II.1) holds. All hypotheses of Theorem II.1 are satisfied, and $u=0$ is the unique fixed point of T. Here $u=0\in A_1\cap A_2$ and $\sigma(0,0)=0$.

Using the same techniques we obtain the following result.

Theorem II.6 *Let (X,σ) be a metric-like space. Let $\{A_i\}_{i=1}^m$ be a finite family of nonempty subsets of X. Let $T:X\to X$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) $T(A_i)\subseteq A_{i+1}$ for all $i=1,\cdots,m$, with $A_{m+1}=A_1$;

(ii) there exists $i_0\in\{1,\cdots,m\}$ such that A_{i_0} is closed;

(iii) $\bigcup_{i=1}^m A_i$ is a complete subset of X;

(iv) there exists a simulation function $\zeta\in\mathscr{Z}^*$ such that

$$\zeta(\sigma(Tx,Ty),\sigma(x,y))\geq 0 \qquad \text{(II.15)}$$

for all $(x,y)\in A_i\times A_{i+1}$, $i=1,\cdots,m$.

Then for every $x_0\in\bigcup_{i=1}^m A_i$, the picard sequence $\{T^nx_0\}$ converges to u, the unique fixed point of T in $\bigcup_{i=1}^m A_i$ such that $\sigma(u,u)=0$.

We give the following example to illustrate Theorem II.6.

Example II.7 *Take $X=\mathbb{R}$ and $\sigma(x,y)=|x|+|y|$ for all $x,y\in X$. Clearly, (X,σ) is a metric-like space. Set $A_1=[-1,\frac{1}{2}]$, $A_2=(-\frac{1}{2},1]$. Notice that $A_1\cup A_2=[-1,1]$ is a complete metric-like subset of X. Consider the mapping $T:X\to X$ given by*

$$Tx=\begin{cases}-\frac{x}{2} & \text{if } x\in[-1,1],\\ 2 & \text{if not}\end{cases}$$

Note that $TA_1\subseteq A_2$ and $TA_2\subseteq A_1$. Also, A_1 is closed. Take $\zeta(t,s)=\frac{3s}{4}-t$ for all $s,t\geq 0$.

Now, we have for all $(x,y)\in A_1\times A_2$

$$\sigma(Tx,Ty)=\frac{1}{2}(|x|+|y|)=\frac{1}{2}\sigma(x,y).$$

It follows that

$$\zeta(\sigma(Tx,Ty),\sigma(x,y))=\frac{3}{4}\sigma(x,y)-\frac{1}{2}\sigma(x,y)=\frac{1}{4}\sigma(x,y)\geq 0.$$

Hence, all hypotheses of Theorem II.6 are verified. Here $u=0$ is the unique fixed point of T in $A_1\cap A_2$ with $\sigma(0,0)=0$. However T has another fixed point in X, which is 2.

B. Cyclic contractions via generalized simulation functions

Denote by Φ the set of functions $\phi:[0,\infty)\to[0,\infty)$ satisfying:

(ϕ_1) ϕ is non-decreasing;

(ϕ_2) there exist $k_0\in\mathbb{N}$, $a\in(0,1)$ and convergent series of nonnegative terms $\sum_{k=1}^{\infty}v_k$ such that

$$\phi^{k+1}(t)\leq a\phi^k(t)+v_k, \qquad \text{(II.16)}$$

for $k\geq k_0$ and any $t>0$. Following [3], a $\phi\in\Phi$ is called a (c)-comparison function.

Again, From [3] we have

Lemma II.8 (see [3]). *If $\phi\in\Phi$, then the following properties hold:*

(i) $(\phi^n(t))_{n\in\mathbb{N}}$ *converges to 0 as* $n\to\infty$*, for all* $t>0$*,*

(ii) $\phi(t)<t$ *for any* $t>0$*,*

(iii) ϕ *is continuous at 0,*

(iv) *the series* $\sum_{k=0}^{\infty}\phi^k(t)$ *converge for any* $t>0$*.*

Lemma II.9 (see [3]). *If* $\phi\in\Phi$*, then the function* $s:(0,\infty)\to(0,\infty)$ *defined by*

$$s(t)=\sum_{k=0}^{\infty}\phi^k(t),\ t>0, \tag{II.17}$$

is non-decreasing and is continuous at 0.

Next, we state and prove the following result.

Theorem II.10 *Let* (X,σ) *be a metric-like space. Let* $\{A_i\}_{i=1}^m$ *be a finite family of nonempty subsets of* X*. Let* $T:\bigcup_{i=1}^m A_i\to\bigcup_{i=1}^m A_i$ *be a given mapping. Suppose that the following conditions are satisfied:*

(i) $T(A_i)\subseteq A_{i+1}$ *for all* $i=1,\cdots,m,$ *with* $A_{m+1}=A_1$;

(ii) *there exists* $i_0\in\{1,\cdots,m\}$ *such that* A_{i_0} *is closed;*

(iii) $\bigcup_{i=1}^m A_i$ *is a complete subset of* X*;*

(iv) *there exists a generalized simulation function* $\zeta\in\mathfrak{I}$ *and* $\phi\in\Phi$ *such that*

$$\zeta\left(\sigma(Tx,Ty),\phi(\sigma(x,y))\right)\geq 0 \tag{II.18}$$

$$\forall(x,y)\in A_i\times A_{i+1},\ i=1,\cdots,m.$$

Then

(I) For every $x_0\in\bigcup_{i=1}^m A_i$*, the picard sequence* $\{T^n x_0\}$ *converges to u, the unique fixed point of* T *in* $\bigcap_{i=1}^m A_i$ *such that* $\sigma(u,u)=0$ *and the following estimates hold:*

$$\sigma(x_n,u)\leq s(\phi^n(\sigma(x_0,Tx_0))),\ n\geq 1, \tag{II.19}$$

$$\sigma(x_n,u)\leq s(\sigma(x_n,x_{n+1})),\ n\geq 1, \tag{II.20}$$

(II) for any $x\in\bigcup_{i=1}^m A_i$

$$\sigma(x,u)\leq s(\sigma(x,Tx)), \tag{II.21}$$

where s is given by (II.17) in Lemma II.9.

Proof. Let $x_0\in\bigcup_{i=1}^m A_i$. Without loss of generality, let $x_0\in A_1$. Consider the Picard iteration $\{x_n\}$ defined by $x_{n+1}=Tx_n$ for all $n\geq 0$.

If $x_n=x_{n+1}$ for some n, then $x_n=x_{n+1}=Tx_n$, that is, x_n is a fixed point of T and so the proof is complete.

Suppose that $x_n\neq x_{n+1}$ for all $n\geq 0$. For any $n\geq 0$, there is $i_n\in\{1,\cdots,m\}$ such that $x_n\in A_{i_n}$ and $x_{n+1}\in A_{i_n+1}$. By (II.31), we have

$$\begin{aligned}&\zeta\left(\sigma(x_{n+1},x_{n+2}),\phi(\sigma(x_n,x_{n+1}))\right)\\&=\zeta\left(\sigma(Tx_n,Tx_{n+1}),\phi(\sigma(x_n,x_{n+1}))\right)\geq 0.\end{aligned} \tag{II.22}$$

From the definition of $\zeta\in\mathfrak{I}$, we have

$$\begin{aligned}0&\leq\zeta\left(\sigma(x_{n+1},x_{n+2}),\sigma(x_n,x_{n+1})\right)\\&\leq\phi(\sigma(x_n,x_{n+1}))-\sigma(x_{n+1},x_{n+2}).\end{aligned}$$

Then

$$\sigma(x_{n+1},x_{n+2})\leq\phi(\sigma(x_n,x_{n+1})),\quad\text{for all } n\geq 0, \tag{II.23}$$

The function ϕ is non-decreasing, so by induction

$$\sigma(x_n,x_{n+1})\leq\phi^n(\sigma(x_0,x_1))\quad\text{for all } n\geq 0. \tag{II.24}$$

By triangle inequality and (II.24), for $p\geq 1$

$$\sigma(x_n,x_{n+p})\leq\sum_{k=n}^{n+p-1}\phi^k(\sigma(x_0,x_1))\leq\sum_{k=n}^{\infty}\phi^k(\sigma(x_0,x_1)). \tag{II.25}$$

Since the function $\phi\in\Phi$ and $\sigma(x_0,x_1)>0$, so by Lemma II.8, (iv), we get that

$$\sum_{k=0}^{\infty}\phi^k(\sigma(x_0,x_1))<\infty.$$

Thus, from (II.25), we have

$$\lim_{n\to\infty}\sigma(x_n,x_{n+p})=0.$$

This yields that $\{x_n\}$ is a σ-Cauchy sequence in $\bigcup_{i=1}^m A_i$. Since $(\bigcup_{i=1}^m A_i,\sigma)$ is complete, hence there exists $u\in\bigcup_{i=1}^m A_i$ such that

$$\lim_{n\to\infty}\sigma(x_n,u)=\sigma(u,u)=\lim_{n,m\to\infty}\sigma(x_n,x_m)=0. \tag{II.26}$$

We claim that u is a fixed point of T. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}=u$ or $x_{n_k+1}=Tu$ for all k, then, by letting $k \to \infty$, we get $Tu = u$ and so the proof is complete. So, without loss of generality, we may suppose that $x_n \neq u$ and $x_n \neq Tu$ for all nonnegative integer n. Without loss of generality, we suppose that A_1 is closed. Since $x_0 \in A_1$, we have $(x_{nm})_{n\geq 0} \in A_1$. The fact that A_1 is closed together with (II.26) yield that $u \in A_1$. Since $u \in A_1$ and $(x_{nm+1} = Tx_{nm})_{n\geq 0} \in A_2$, so applying (II.31) for $x = u$ and $y = x_{nm+1}$, we get that

$$\begin{aligned} 0 &\leq \zeta(\sigma(Tu, Tx_{nm+1}), \phi(\sigma(u, x_{nm+1}))) \\ &= \zeta(\sigma(Tu, x_{nm+2}), \phi(\sigma(u, x_{nm+1}))) \\ &\leq \phi(\sigma(u, x_{nm+1})) - \sigma(Tu, x_{nm+2}). \end{aligned}$$

It means that

$$\sigma(Tu, x_{nm+2}) \leq \phi(\sigma(u, x_{nm+1})). \tag{II.27}$$

Since ϕ is continuous at 0 and $\lim_{n\to\infty} \sigma(x_n, u) = 0$, so

$$\lim_{n\to\infty} \sigma(Tu, x_{nm+2}) \leq \phi(0) = 0,$$

because, since $\phi(t) < t$ for all $t > 0$ and ϕ is continuous at 0, hence we get that $\phi(0) = 0$. Thus we deduce that $\sigma(u, Tu) = 0$ and so $Tu = u$. Since $u \in A_1$, so by condition (i) we get $u \in \bigcap_{i=1}^m A_i$.

Now, we prove that u is the unique fixed point of T. Assume that v is another fixed point of T, that is, $Tv = v$. We have $v \in \bigcup_{i=1}^m A_i$. There exists $i_0 \in \{1, \cdots, m\}$ such that $v \in A_{i_0}$. Suppose that $u \neq v$, so $\sigma(u, v) > 0$. Taking $x = v$ and $y = u$ in (II.31), we get that

$$\begin{aligned} 0 &\leq \zeta(\sigma(Tv, Tu), \phi(\sigma(v, u))) = \zeta(\sigma(u, v), \phi(\sigma(u, v))) \\ &\leq \phi(\sigma(u, v)) - \sigma(u, v) < \sigma(u, v) - \sigma(u, v) = 0, \end{aligned}$$

which is a contradiction. We deduce u is the unique fixed point of T. This completes the proof of (I).

We shall prove (II). From (II.25), we have

$$\sigma(x_n, x_{n+p}) \leq \sum_{k=n}^{n+p-1} \phi^k(\sigma(x_0, x_1)).$$

Letting $p \to \infty$ in above inequality, we get the estimate (II.32).

For $n \geq 0$ and $k \geq 1$, we obtain from (II.23)

$$\sigma(x_{n+k}, x_{n+k+1}) \leq \phi(\sigma(x_{n+k-1}, x_{n+k})). \tag{II.28}$$

By induction and by monotonicity of ϕ, we get that

$$\sigma(x_{n+k}, x_{n+k+1}) \leq \phi^k(\sigma(x_n, x_{n+1})),\ n \geq 0, k \geq 0. \tag{II.29}$$

Hence, by triangle inequality and from (II.29), we have

$$\sigma(x_n, x_{n+p}) \leq \sum_{k=0}^{n+p-1} \phi^k(\sigma(x_n, x_{n+1})).$$

Letting $p \to \infty$ in above inequality, we get that

$$\sigma(x_n, u) \leq \sum_{k=0}^{\infty} \phi^k(\sigma(x_n, x_{n+1})) = s(\sigma(x_n, x_{n+1})). \tag{II.30}$$

This yields (II).

Now we will prove (III). Let $x \in \bigcup_{i=1}^m A_i$. From (II.30), for $x_0 = x$, we have

$$\sigma(x, u) \leq \sum_{k=0}^{\infty} \phi^k(\sigma(x, Tx)) = s(\sigma(x, Tx)),$$

which is the estimate (II.34). ■

Theorem II.11 *Let (X, σ) be a metric-like space. Let $\{A_i\}_{i=1}^m$ be a finite family of nonempty subsets of X. Let $T : X \to X$ be a given mapping. Suppose that the following conditions are satisfied:*

- (i) $T(A_i) \subseteq A_{i+1}$ *for all* $i = 1, \cdots, m$, *with* $A_{m+1} = A_1$;
- (ii) *there exists* $i_0 \in \{1, \cdots, m\}$ *such that* A_{i_0} *is closed;*
- (iii) $\bigcup_{i=1}^m A_i$ *is a complete subset of* X*;*
- (iv) *there exists a generalized simulation function* $\zeta \in \mathfrak{I}$ *and* $\phi \in \Phi$ *such that*

$$\zeta(\sigma(Tx, Ty), \phi(\sigma(x, y))) \geq 0 \tag{II.31}$$
$$\forall (x, y) \in A_i \times A_{i+1},\ i = 1, \cdots, m.$$

Then

(I) For every $x_0 \in \bigcup_{i=1}^m A_i$, the picard sequence $\{T^n x_0\}$ converges to u, the unique fixed point of T in $\bigcap_{i=1}^m A_i$ such that $\sigma(u, u) = 0$ and the following estimates hold:

$$\sigma(x_n, u) \leq s(\phi^n(\sigma(x_0, Tx_0))),\ n \geq 1, \tag{II.32}$$

$$\sigma(x_n, u) \leq s(\sigma(x_n, x_{n+1})),\ n \geq 1, \tag{II.33}$$

(II) for any $x \in \bigcup_{i=1}^{m} A_i$

$$\sigma(x,u) \leq s(\sigma(x,Tx)), \tag{II.34}$$

where s is given by (II.17) in Lemma II.9.

The notion of well-posedness of a fixed point has evoked much interest to several mathematicians. Recently, Karapinar [9] studied a well-posed problem for a cyclic weak $\phi-$contraction mapping on a complete metric space (see also, [10, 17]). Let $Fix(T)$ denote the set of all fixed points of a self map T on a nonempty set X. We introduce the following definition.

Definition II.12 *Let X be a nonempty set. A fixed point problem of a given mapping $T : X \to X$ on X is called well-posed if $Fix(T)$ is a singleton and for any sequence $\{x_n\}$ in X with $x^* \in Fix(T)$ and $\lim_{n\to\infty} \sigma(x_n,Tx_n) = 0$ implies $\{x_n\}$ converges to x^*.*

Theorem II.13 *Let $T : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ be defined as in Theorem II.10. Then the fixed point problem for T is well posed, that is, assuming that there exists $\{x_n\} \subseteq \bigcup_{i=1}^{m} A_i$ such that $\lim_{n\to\infty} \sigma(x_n,Tx_n) = 0$ implies $\{x_n\}$ converges to u.*

Proof. Let $\{x_n\} \subseteq \bigcup_{i=1}^{m} A_i$ such that $\lim_{n\to\infty} \sigma(x_n,Tx_n) = 0$. Applying (II.34) for $x = x_n$, we have

$$\sigma(x_n,u) \leq s(\sigma(x_n,Tx_n)), \quad \forall n \geq 0. \tag{II.35}$$

Having in mind from Lemma II.9 that s is continuous at 0 and $s(0) = 0$, so letting $n \to \infty$ in (II.35), we have

$$\lim_{n\to\infty} \sigma(x_n,u) = 0.$$

Thus

$$\lim_{n\to\infty} \sigma(x_n,u) = \sigma(u,u) = 0.$$

So $\{x_n\}$ converges to u. Hence the fixed point problem for T is well posed. ■

Now, we state and prove the following stability result.

Theorem II.14 *Let $T : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ be defined as in Theorem II.10. Let $f : \bigcup_{i=1}^{m} A_i \to \bigcup_{i=1}^{m} A_i$ such that*

(1) $Fix(f) \neq \emptyset$;

(2) $\sup_{x\in\bigcup_{i=1}^{m} A_i} \sigma(fx,Tx) < \infty$.

Then

$$\sup_{x\in Fix(f)} \sigma(x,Fix(T)) \leq s(\sup_{x\in\bigcup_{i=1}^{m} A_i} \sigma(fx,Tx)), \tag{II.36}$$

where $Fix(T) = x_T$.

Proof. Let $x_f \in Fix(f)$. Assume $x_f \neq x_T$. Otherwise the proof is completed. We apply (II.34) from Theorem II.10 for $x = x_f$ to have,

$$\sigma(x_f,x_T) \leq s(\sigma(x_f,Tx_f) = s(\sigma(fx_f,Tx_f)) \leq s(\sup_{x\in\bigcup_{i=1}^{m} A_i} \sigma(fx,Tx)),$$

because, by Lemma II.9, the function s is non-decreasing. Thus

$$\sup_{x\in Fix(f)} \sigma(x,x_T) \leq s(\sup_{x\in\bigcup_{i=1}^{m} A_i} \sigma(fx,Tx)).$$

■

Example II.15 *Consider $X = (-3,\infty)$ and $\sigma(x,y) = |x|+|y|$ for all $x,y \in X$. Clearly, (X,σ) is a metric-like space. Take $A_1 = [0,1]$, $A_2 = [-1,0]$. Consider the mapping $T : A_1 \cup A_2 \to A_1 \cup A_2$ defined by $Tx = -\frac{1}{2}x$ for all $x \in A_1 \cup A_2$. We have $T(A_1) \subset A_2$ and $T(A_2) \subset A_1$. Take $\zeta(t,s) = s-t$, $\phi(t) = \frac{3}{4}t$ for all $t,s \geq 0$. For $x \in A_1$, $y \in A_2$, we have*

$$\zeta(\sigma(Tx,Ty),\phi(\sigma(x,y))) = \phi(\sigma(x,y)) - \sigma(Tx,Ty) = \frac{1}{4}\sigma(x,y) \geq 0.$$

Therefore, all hypotheses of Theorem II.10 are satisfied, so $u = 0 \in A_1 \cap A_2$ is the unique fixed point of T. Also $\sigma(0,0) = 0$.

III. APPLICATION

In this section, we present the following application concerning the existence and uniqueness of solutions to a class of nonlinear integral equations.

We consider the nonlinear integral equation

$$u(t) = f(t) + \int_0^t k(t,s,u(s))\,ds \quad \text{for all } t \in [0,1], \tag{III.1}$$

where f is a given continuous function and $k : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X=\mathscr{C}([0,1])$ be the set of real continuous functions on $[0,1]$. Consider on X the metric-like σ given by

$$\sigma(u,v)=\max_{t\in[0,1]}|u(t)-v(t)|$$

for all $u,v\in X$. It is clear that (X,σ) is a complete metric-like space. Consider the mapping $T:X\to X$ defined as

$$Tu(t)=f(t)+\int_0^t k(t,s,u(s))\,ds \quad \text{for all } t\in[0,1]. \tag{III.2}$$

Note that u is a solution of (III.1) if and only if u is a fixed point of T.

Let $(\alpha,\beta)\in X^2$ and $(\alpha_0,\beta_0)\in\mathbb{R}^2$ such that

$$\alpha_0\le\alpha(t)\le\beta(t)\le\beta_0 \quad \text{for all } t\in[a,b]. \tag{III.3}$$

Assume that, for all $t\in[0,1]$,

$$\alpha(t)\le f(t)+\int_0^t k(t,s,\beta(s))\,ds \tag{III.4}$$

and

$$\beta(t)\ge f(t)+\int_0^t k(t,s,\alpha(s))\,ds. \tag{III.5}$$

We also suppose that for all $t,s\in[0,1]$, $k(t,s,.)$ is a decreasing function, that is,

$$x,y\in\mathbb{R},\quad x\le y\Longrightarrow k(t,s,x)\ge k(t,s,y). \tag{III.6}$$

Finally, let $t,s\in[0,1]$, $x,y\in\mathbb{R}$ such that for ($x\le\beta_0$ and $y\ge\alpha_0$) or ($x\ge\alpha_0$ and $y\le\beta_0$) or ($x\ge\alpha_0$ and $y\ge\alpha_0$)

$$|k(t,s,x)-k(t,s,y)|\le g(t,s)|x-y|, \tag{III.7}$$

where $g:[0,1]\times[0,1]\to[0,\infty)$ is continuous functions such that

$$\lambda:=\sup_{t\in[0,1]}\int_0^t g(t,s)ds<1. \tag{III.8}$$

We take

$$\mathscr{W}=\{u\in X,\ \alpha<u\le\beta\}.$$

Theorem III.1 *Under the assumptions (III.3)-(III.8), Problem (III.1) has one and only one solution $u\in\mathscr{W}$.*

Proof. Take

$$A_1=\{u\in X,\ u\le\beta\}\quad \text{and } A_2=\{u\in X,\ u>\alpha\}.$$

Remark that A_1 is closed. First, we shall check that

$$T(A_1)\subset A_2\quad \text{and } T(A_2)\subset A_1.$$

For all $u\in A_1$, we have $u(s)\le\beta(s)$. Using assumption (III.6), we get

$$k(t,s,u(s))\ge k(t,s,\beta(s))$$

for all $t\in[0,1]$. Thus, from (III.4)

$$Tu(t)=f(t)+\int_0^t k(t,s,u(s))ds\ge f(t)+\int_0^t k(t,s,\beta(s))\ge\alpha(t),$$

so $Tu\in A_2$.

Similarly, let $u\in A_2$, we have $u(s)\ge\alpha(s)$. Using again assumption (III.6), we get

$$k(t,s,u(s))\le k(t,s,\alpha(s))$$

for all $t\in[0,1]$. Thus, from (III.4)

$$Tu(t)=f(t)+\int_0^t k(t,s,u(s))ds\le f(t)+\int_0^t k(t,s,\alpha(s))\le\beta(t),$$

so $Tu\in A_1$.

Now, let $(u,v)\in A_1\times A_2$, that is, for all $t\in[0,1]$

$$u(t)\le\beta(t),\ v(t)\ge\alpha(t).$$

This implies from condition (III.3) that for all $t\in[0,1]$,

$$u(t)\le\beta_0,\ v(t)\ge\alpha_0.$$

In view of (III.7) and above inequalities, we have

$$\begin{aligned}|Tu(t)-Tv(t)|&\le\int_0^t|k(t,s,u(s)-k(t,s,v(s)|ds\\&\le\int_0^t g(t,s)|u(s)-v(s)|ds\\&\le\max_{t\in[0,1]}|u(t)-v(t)|\sup_{t\in[0,1]}\int_0^t g(t,s)ds\\&=\lambda\max_{t\in[0,1]}|u(t)-v(t)|.\end{aligned}$$

Therefore

$$\max_{t\in[0,1]}|Tu(t)-Tv(t)|\le\lambda\max_{t\in[0,1]}|u(t)-v(t)|. \tag{III.9}$$

So, we get

$$\sigma(Tu,Tv) \leq \lambda\sigma(u,v). \qquad \text{(III.10)}$$

Then

$$\zeta(\sigma(Tu,Tv),\phi(\sigma(u,v))) \geq 0,$$

where $\zeta(t,s) = s - t$ for all $t,s \geq 0$ and $\phi(t) = \lambda t$ for all $t \geq 0$. All hypotheses of Theorem II.10 are satisfied and so T has a unique fixed point $u \in A_1 \cap A_2 = \mathscr{W}$, that is u is the unique solution of the problem (III.1). ■

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