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Differantial Equations of Motion Objects with an Almost Paracontact Metric Structure

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The geometry of almost paracontact manifolds is a natural extension in the odd dimensional case of almost Hermitian geometry. In additional, the paracontact geometry as symplectic geometry has large and comprehensive applications in physics, geometrical optics, classical mechanics, thermodynamics, geometric quantization, differential geometry and applied mathematics. the Euler-Lagrange differential equations one of the common ways of solving problems in classical and analytical mechanics. In the study, we consider Euler-Lagrange differential equations with almost paracontact metric structure for motion objects. Also, implicit solutions of the differential equations found in this study will be solved by Maple computation program and a graphic example will be drawn.

Keywords: Paracontact Manifold, Mechanical System, Dynamic Equation, Lagrangian Formalism.

I. INTRODUCTION

An almost paracontact structure on a differentiable manifold was introduced by *Sato* [1], which is an analogue of an almost contact structure and is closely related to almost product structure. The normal almost paracontact metric manifolds are para-CR. Any para-CR paracontact metric manifold of constant sectional curvature and of dimension greater than 3 must be para-Sasakian. Almost paracontact metric manifolds are the famous examples of almost para-CR manifolds. An almost contact manifold is always odd dimensional but an almost paracontact manifold could be even dimensional as well. *Tripathi et al.* introduced the concept of ε-almost paracontact manifolds, and in particular, of $\varepsilon-$para-Sasakian manifolds [2]. *Kr. Srivastava et al.* submitted the concept of (ε)-almost paracontact manifolds. They showed that some typical identities for curvature tensor and Ricci tensor of (ε)-para Sasakian manifolds are obtained and studied the properties of $\varepsilon - S$ paracontact metric manifold [3]. *Girtu* showed that K induces an almost $2-$paracontact Riemannian structure on T_0^*M whose restriction to the guratrix bundle $K = \{(x,p)\,|K(x,p) = 1\}$ is an almost paracontact structure [4]. *Ahmad and Jun* defined a semi-symmetric non-metric connection in an almost $r-$paracontact Riemannian manifold and they considered submanifolds of an almost r-paracontact Riemannian manifold endowed with a semi-symmetric non-metric connection [5]. *Erken* is to investigate 3-dimensional ξ-projectively flat and φ-projectively flat normal almost paracontact metric manifolds [6]. *Kupeli* studied 3-dimensional normal almost paracontact metric manifolds [7]. *Kupeli Erken and Murathan* completed a study of three-dimensional paracontact metric $(\tilde{\kappa},\tilde{\mu},\tilde{\nu})$-manifolds. They focus on some curvature properties by considering the class of paracontact metric $(\kappa,\mu,\nu)-$manifolds under a condition [8]. *Welyczko* showed that the curvature and torsion of slant Frenet curves in 3-dimensional normal almost paracontact metric manifolds [9]. *Welyczko* is shown that the normal almost paracontact metric manifolds are para-CR [10]. *De and Modal* are to study ξ-projectively flat and φ-projectively flat 3-dimensional normal almost contact metric manifolds. An illustrative example is given [11]. *Ahmad et al* defined a quarter symmetric semi-metric connection in an almost r-paracontact Riemannian manifold and considered invariant, non-invariant and anti-invariant hypersurfaces of an almost r-paracontact Riemannian manifold with that connection [12].

Differential geometry have a lots of different applications in the branches of science. These applications, came into our lives, are used in many areas and the popular science. We can say that differential geometry provides a good working area for studying Lagrangians of classical mechanics and field theory. The dynamic equation for moving bodies is obtained for Lagrangian mechanic. *Kasap and Tekkoyun* obtained Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler manifolds, representing an interesting multidisciplinary field of research [13]. *Kasap* introduced that the Weyl-Euler–Lagrange and Weyl-Hamilton equations on $\mathbb{R}_n^{2n}$ which is a model of tangent manifolds of constant W-sectional curvature [14]. *Tekkoyun* found paracomplex analogue of Euler-Lagrange and Hamiltonian equations [15]. *Tekkoyun and Celik* present a new analogue of Euler–Lagrange and Hamilton equations on an almost Kähler model of a Finsler manifold [16].

II. PRELIMINARIES

Definition 1. Let M be a differentiable manifold of dimension $(2n+1)$, and suppose J is a differentiable vector bundle isomorphism $J : TM \to TM$ such that $J_x : T_xM \to T_xM$ is a (almost) complex structure for T_xM, i.e. $J^2 = -I$ where I is the identity (unit) operator on V. Then J is called an almost complex structure for the differentiable manifold M. A manifold with a fixed (almost) complex structure is called **an (almost) complex manifold.** Where $J^2 = J \circ J$, and I is the identity operator on TM and V is a vector space.

Definition 2. Let M be a differentiable manifold of dimension $(2n+1)$, and suppose J is a differentiable vector bundle isomorphism $J : TM \to TM$ such that $J_x : T_xM \to T_xM$ is a (almost) complex structure for T_xM, i.e. $J^2 = I$ where I is the identity (unit) operator on V. Then J is called an almost paracomplex structure for the differentiable manifold M. A manifold with a fixed almost paracomplex structure is called **an (almost) paracomplex manifold.** Where $J^2 = J \circ J$, and I is the identity operator on TM.

Definition 3. Suppose that ξ is a vector field: that is, a vector-valued function with Cartesian coordinates $(\xi_1, ..., \xi_n)$; and $\mathbf{x}(t)$ a parametric curve with Cartesian coordinates $(x_1(t), ..., x_n(t))$. Then $\mathbf{x}(t)$ is **an integral curve** of ξ if it is a solution of the following autonomous system of ordinary differential equations: $\frac{dx_1}{dt} = \xi_1(x_1, ..., x_n), ..., \frac{dx_n}{dt} = \xi_n(x_1, ..., x_n)$. Such a system may be written as a single vector equation

$$\xi(\mathbf{x}(t)) = \mathbf{x}'(t) = \frac{\partial}{\partial t}(\mathbf{x}(t)). \tag{1}$$

Definition 4. Let M be an almost paracontact manifold and (φ, ξ, η) its **almost paracontact structure** (e.g. [17]). This means that M is an $(2n+1)$-dimensional differentiable manifold and φ, ξ, η are tensor fields on M of type $(1,1)$, $(1,0)$, $(0,1)$, respectively, such that

$$\begin{aligned} &1.\ \varphi^2 X = X - \eta(X)\xi, \\ &2.\ \eta(\xi) = 1,\ \varphi\xi = 0,\ \eta(\varphi X) = 0, \\ &3.\ \eta(X) = g(X, \xi), \\ &4.\ d\eta(\ ,\ ) = g(\ , \varphi). \end{aligned} \tag{2}$$

A pseudo-Riemannian metric g on M satisfying the condition $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$ is said to be compatible with the structure (φ, ξ, η). Then, $(M, \varphi, \xi, \eta, g)$ or M a **paracontact metric manifold** [18]. ■

For such a manifold, we additionally have $\eta(X) = g(X, \xi)$, and we define the (skew-symmetric) fundamental 2-form Φ by $\Phi(X,Y) = g(X, \varphi Y)$, and X, Y, Z are arbitrary vector fields. Let M be an almost paracontact manifold with almost paracontact structure (φ, ξ, η, g) and consider the product manifold $M \times \mathbb{R}$, where $\mathbb{R}$ is the real line. A vector field on $M \times \mathbb{R}$ can be represented by $(X, f(d/dt))$, where X is tangent to M,

f a smooth function on $M\times\mathbb{R}$, and t the coordinates of $\mathbb{R}$. For any two vector fields $(X, f(d/dt))$ and $(Y, h(d/dt))$, it is easy to verify the following:

$$\left[\left(X,f(\frac{d}{dt})\right),\left(Y,h(\frac{d}{dt})\right)\right]=\left([X,Y],(Xh-Yf)\frac{d}{dt}\right). \quad (3)$$

Definition 5. If the induced almost product structure J on $M\times\mathbb{R}$ defined by

$$J\left(X,f(\frac{d}{dt})\right)=\left(\varphi X+f\xi,\eta(X)\frac{d}{dt}\right) \quad (4)$$

is integrable, then we say that the almost paracontact structure (φ,ξ,η,g) is **normal**.

Theorem 1. Any 3-dimensional almost paracontact metric manifold is a para-CR manifold (proof see [9]).

Theorem 2. A paracontact metric manifold is a para-CR manifold if and only if $(\nabla_X\varphi)Y = g(\varphi\nabla_X\xi,Y)\xi - \eta(Y)\varphi\nabla_X\xi$ (proof see [9]).

Example 1: Let $\mathbb{R}^3$ be the 3-dimensional real number space with a coordinate system (x,y,z). Define an almost paracontact metric structure (φ,ξ,η,g) on $\mathbb{R}^3$ by assuming

$$\begin{aligned}&(1)\ \varphi\left(\tfrac{\partial}{\partial x}\right)=\cosh(2z)\tfrac{\partial}{\partial z},\\ &(2)\ \varphi\left(\tfrac{\partial}{\partial y}\right)=\sinh(2z)\tfrac{\partial}{\partial z},\\ &(3)\ \varphi\left(\tfrac{\partial}{\partial z}\right)=\cosh(2z)\tfrac{\partial}{\partial x}-\sinh(2z)\tfrac{\partial}{\partial y}.\end{aligned} \quad (5)$$

We define

$$\eta=\sinh(2z)dx+\cosh(2z)dy,$$

$$\xi=-\sinh(2z)\tfrac{\partial}{\partial x}+\cosh(2z)\tfrac{\partial}{\partial y}, \quad (6)$$

$$g=-dx\otimes dx+dy\otimes dy+dz\otimes dz.$$

In fact, this structure is flat, tree-dimensional, para-CR and paracontact metric.

Theorem 3. Every paracontact metric structure is almost paracontact metric structure.

Proof: Let's take a look at (5).

$$\begin{aligned}1.\ \varphi^2\tfrac{\partial}{\partial x}&=\varphi\circ\varphi\left(\tfrac{\partial}{\partial x}\right)=\varphi\left(\cosh(2z)\tfrac{\partial}{\partial z}\right)\\&=\cosh^2(2z)\tfrac{\partial}{\partial x}-\cosh(2z)\sinh(2z)\tfrac{\partial}{\partial y},\end{aligned}$$

$$\begin{aligned}\varphi^2\tfrac{\partial}{\partial y}&=\varphi\left(\sinh(2z)\tfrac{\partial}{\partial z}\right)\\&=\sinh(2z)\cosh(2z)\tfrac{\partial}{\partial x}-\sinh^2(2z)\tfrac{\partial}{\partial y},\end{aligned}$$

$$\begin{aligned}\varphi^2\tfrac{\partial}{\partial z}&=\varphi(\cosh(2z)\tfrac{\partial}{\partial x}-\sinh(2z)\tfrac{\partial}{\partial y})\\&=\cosh^2(2z)\tfrac{\partial}{\partial z}-\sinh(2z)\sinh(2z)\tfrac{\partial}{\partial z}=\tfrac{\partial}{\partial z}\end{aligned}$$

$$2.\ \eta(\xi)=(\sinh(2z)dx+\cosh(2z)dy)\left(-\sinh(2z)\tfrac{\partial}{\partial x}+\cosh(2z)\tfrac{\partial}{\partial y}\right)=1,$$

$$\begin{aligned}3.\ \varphi(\xi)&=\varphi\left(-\sinh(2z)\tfrac{\partial}{\partial x}+\cosh(2z)\tfrac{\partial}{\partial y}\right)\\&=-\sinh(2z)\cosh(2z)\tfrac{\partial}{\partial z}+\cosh(2z)\sinh(2z)\tfrac{\partial}{\partial z}=0,\end{aligned}$$

$$\begin{aligned}&4.\ \eta(\varphi X)=0,\\&(a)\ (\sinh(2z)dx+\cosh(2z)dy)\left[\varphi\left(\tfrac{\partial}{\partial x}\right)\right]\\&\quad=(\sinh(2z)dx+\cosh(2z)dy)\left[\cosh(2z)\tfrac{\partial}{\partial z}\right]=0,\\&(b)\ (\sinh(2z)dx+\cosh(2z)dy)\left[\varphi\left(\tfrac{\partial}{\partial y}\right)\right]\\&\quad=(\sinh(2z)dx+\cosh(2z)dy)\left[\sinh(2z)\tfrac{\partial}{\partial z}\right]=0,\\&(c)\ (\sinh(2z)dx+\cosh(2z)dy)\left[\varphi\left(\tfrac{\partial}{\partial z}\right)\right]\\&\quad=(\sinh(2z)dx+\cosh(2z)dy)\left[\cosh(2z)\tfrac{\partial}{\partial x}-\sinh(2z)\tfrac{\partial}{\partial y}\right]\\&=0,\end{aligned} \quad (7)$$

Conditions (2) are provided then (5) are holomorphic structures

III. NIJENHUIS TENSOR

A celebrated theorem of Newlander and Nirenberg [19] says that an almost (para) complex structure is a (para) complex structure if and only if its Nijenhuis tensor or torsion vanishes. An almost paracontact structure (φ,ξ,η) is said to be normal, if the Nijenhuis tensor N_J of almost paracontact structure J is defined as

$$N_J(X,Y) = [J,J](X,Y)$$
$$= [JX,JY] + J^2[X,Y] - J[JX,Y] - J[X,JY]. \tag{8}$$

The almost paracontact structure J on M is integrable if and only if the tensor N_J vanishes identically, where N_J is defined on two vector fields X and Y. The tensor $(2,1)$ is called the Nijenhuis tensor (8). If $N_J = 0$ then the almost paracontact structure is called paracontact or integrable.

IV. (EULER)-LAGRANGE DYNAMICS EQUATIONS

A dynamic system with a finite number n degrees of freedom can be described by real functions of time $q^i(t)$ (i = 1, 2, ..., n) which, together with the derivatives image $\dot{q}^i$, uniquely determine its state at any moment of time t. The collection of all values of q^i is called the configuration space M of the system. In the simplest case, M is a Euclidean space $\mathbb{R}^n$. Let M be an n-dimensional manifold and TM its tangent bundle with canonical projection $\tau_M : TM \to M$. TM is called the phase space of velocities of the base manifold M. Let $L : TM \to \mathbb{R}$ be a differentiable function on TM and is called the **Lagrangian function**.

Klein (1962) submitted that the closed 2-form on a vector field and $1-$form reduction function on the phase space defined of a mechanical system is equal to the differential of the energy function 1-form of the Lagrangian mechanical systems [20, 21].

We consider closed 2-form on TM such that $\Phi_L = -d(\mathbf{d}_J L)$ and $\mathbf{i}_\xi$ is 2-form reduction function that reduces the 1-form. Consider the equation

$$\mathbf{i}_\Psi \Phi_L = dE_L. \tag{9}$$

Where the semispray Ψ is a vector field. We shall see that for motion in a potential, $E_L = V(L) - L$ is an energy function and $V = \varphi(\Psi)$ a Liouville vector field. Here dE_L denotes the differential of E. We shall see that (9) under a certain condition on Ψ is the intrinsical expression of the Euler-Lagrange equations of motion. This equation is named as **Euler-Lagrange dynamical equation**. The triple (TM, Φ_L, Ψ) is known as **Euler-Lagrangian system** on the tangent bundle TM. Consider the following combination of the kinetic (T) and potential energies (V), $L = T - V$. This is called the **Lagrangian**. There is a minus sign in the definition (a plus sign would simply give the total energy). In the problem of a mass on the end of a spring $T = m\dot{x}^2/2$, $V = kx^2/2, L = m\dot{x}^2/2 - kx^2/2$. Now, we can be write

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0,\ \dot{x} = \frac{\partial x}{\partial t}. \tag{10}$$

This equation is called the **Euler-Lagrange (E-L) equation** [22, 23].

V. EULER-LAGRANGIAN EQUATIONS

We, using (9),can be obtained Euler-Lagrange equations for classical and analytical mechanics on almost paracontact metric manifold and its shown that by (TM, Ψ, η, g).

Proposition: Let (x, y, z) be coordinate functions and let Ψ be the vector field determined on (TM, Ψ, η, g) by

$$\Psi = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z},\ X = \dot{x} = y, Y = \dot{y} = \ddot{x}, .Z = \dot{z}, \tag{11}$$

the following partial differential equations are obtained on the (5) system:

1. $$-\frac{\partial}{\partial t}\left(\cosh(2z)\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial x} = 0,$$
2. $$-\frac{\partial}{\partial t}\left(\sinh(2z)\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial y} = 0, \tag{12}$$
3. $$-\frac{\partial}{\partial t}\left(\cosh(2z)\frac{\partial L}{\partial x}\right) + \frac{\partial}{\partial t}\left(\sinh(2z)\frac{\partial L}{\partial y}\right) + \frac{\partial L}{\partial z} = 0,$$

Proof: Then the vector field defined by

$$V = \varphi(\Psi) = \varphi(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}), \tag{13}$$

is thought to be *Liouville vector field* on almost paracontact metric manifold (TM, Ψ, η, g). $\Phi_L = -d(\mathbf{d}_\varphi L)$ is the closed 2-form given by (9) such that

$$\mathbf{d} = \frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz,\ \mathbf{d} : F(M) \to \wedge^1 M, \tag{14}$$

and $\mathbf{d}_\varphi = i_\varphi \mathbf{d} - \mathbf{d} i_\varphi$,

$$\begin{aligned}\mathbf{d}_\varphi = \varphi(\mathbf{d}) &= \varphi\left(\tfrac{\partial}{\partial x}dx + \tfrac{\partial}{\partial y}dy + \tfrac{\partial}{\partial z}dz\right)\\ &= \varphi\left(\tfrac{\partial}{\partial x}\right)dx + \varphi\left(\tfrac{\partial}{\partial y}\right)dy + \varphi\left(\tfrac{\partial}{\partial z}\right)dz\\ &= \left(\cosh(2z)\tfrac{\partial}{\partial z}\right)dx + \left(\sinh(2z)\tfrac{\partial}{\partial z}\right)dy\\ &+ \left(\cosh(2z)\tfrac{\partial}{\partial x} - \sinh(2z)\tfrac{\partial}{\partial y}\right)dz,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\mathbf{d}_\varphi L &= \varphi(\Psi)(L\\ &= \left(\cosh(2z)\tfrac{\partial L}{\partial z}\right)dx + \left(\sinh(2z)\tfrac{\partial L}{\partial z}\right)dy\\ &+ \left(\cosh(2z)\tfrac{\partial LX}{\partial x} - \sinh(2z)\tfrac{\partial L}{\partial y}\right)dz.\end{aligned} \tag{16}$$

Also, the *vertical differentiation* $\mathbf{d}_\varphi$ is given by d is the usual exterior derivation. Then there is the following result. Here, we can be account Euler-Lagrange equations for classical and quantum mechanics on almost paracontact metric manifold $(TM, \varphi, \xi, \eta, g)$. We get the equations given by

$$\begin{aligned}\mathbf{d} &= \tfrac{\partial}{\partial x}dx + \tfrac{\partial}{\partial y}dy + \tfrac{\partial}{\partial z}dz,\\ \mathbf{d}_\varphi &= \varphi(\tfrac{\partial}{\partial x})dx + \varphi(\tfrac{\partial}{\partial y})dy + \varphi(\tfrac{\partial}{\partial z})dz.\end{aligned} \tag{17}$$

Let we account Φ_L

$$\begin{aligned}\Phi_L &= -d(\mathbf{d}_\varphi L)\\ &= \left[-\cosh(2z)\tfrac{\partial^2 L}{\partial x \partial z}\right]dx \wedge dx\\ &+ \left[-\cosh(2z)\tfrac{\partial^2 L}{\partial z \partial y}\right]dy \wedge dx + \left[-2\sinh(2z)\tfrac{\partial L}{\partial z}\right]dz \wedge dx\\ &+ \left[-\sinh(2z)\tfrac{\partial^2 L}{\partial x \partial z}\right]dx \wedge dy\\ &+ \left[-\sinh(2z)\tfrac{\partial^2 L}{\partial y \partial z}\right]dy \wedge dy - \left[2\cosh(2z)\tfrac{\partial L}{\partial z}\right]dz \wedge dy\\ &+ \left[-\cosh(2z)\tfrac{\partial^2 L}{\partial x \partial x} + \sinh(2z)\tfrac{\partial^2 L}{\partial x \partial y}\right]dx \wedge dz\\ &+ \left[\sinh(2z)\tfrac{\partial^2 L}{\partial y \partial y} - \cosh(2z)\tfrac{\partial^2 L}{\partial x \partial y}\right]dy \wedge dz\\ &+ \left[-2\sinh(2z)\tfrac{\partial L}{\partial x} + 2\cosh(2z)\tfrac{\partial L}{\partial y}\right]dz \wedge dz.\end{aligned} \tag{18}$$

Thus, $\Phi_L(\Psi)$ is found using (18).as follows:

$$\begin{aligned}&\Phi_L(\Psi) =\\ &-X\sinh(2z)\tfrac{\partial^2 L}{\partial x \partial z}dy - X\cosh(2z)\tfrac{\partial^2 L}{\partial x \partial x}dz + X\sinh(2z)\tfrac{\partial^2 L}{\partial x \partial y}dz\\ &+X\cosh(2z)\tfrac{\partial^2 L}{\partial z \partial y}dy + X2\sinh(2z)\tfrac{\partial L}{\partial z}dz + Y\sinh(2z)\tfrac{\partial^2 L}{\partial x \partial z}dx\\ &-Y\cosh(2z)\tfrac{\partial^2 L}{\partial z \partial y}dx - Y\cosh(2z)\tfrac{\partial^2 L}{\partial y \partial x}dz + Y\sinh(2z)\tfrac{\partial^2 L}{\partial y \partial y}dz\\ &+Y2\cosh(2z)\tfrac{\partial L}{\partial z}dz + Z\cosh(2z)\tfrac{\partial^2 L}{\partial x \partial x}dx - Z\sinh(2z)\tfrac{\partial^2 L}{\partial x \partial y}dx\\ &+Z\cosh(2z)\tfrac{\partial^2 L}{\partial x \partial y}dy - Z\sinh(2z)\tfrac{\partial^2 L}{\partial y \partial y}dy - Z2\sinh(2z)\tfrac{\partial L}{\partial z}dx\\ &-Z2\cosh(2z)\tfrac{\partial L}{\partial z}dy.\end{aligned} \tag{19}$$

Also, $V = \varphi(\Psi)$ a Liouville vector field. for $E_L = V(L) - L$

the energy function of system is

$$E_L = \varphi(\Psi)(L) - L$$
$$= \varphi\left(X\frac{\partial L}{\partial x} + Y\frac{\partial L}{\partial y} + Z\frac{\partial L}{\partial z}\right) - L$$
$$= X\cosh(2z)\frac{\partial L}{\partial z} + Y\sinh(2z)\frac{\partial L}{\partial z}$$
$$+ Z\cosh(2z)\frac{\partial L}{\partial x} - Z\sinh(2z)\frac{\partial L}{\partial y} - L, \quad (20)$$

and the differential of E_L is

$$dE_L = X\cosh(2z)\frac{\partial^2 L}{\partial x\partial z}dx + X\cosh(2z)\frac{\partial^2 L}{\partial y\partial z}dy + 2X\sinh(2z)\frac{\partial L}{\partial z}dz$$
$$+ Y\sinh(2z)\frac{\partial^2 L}{\partial x\partial z}dx + Y\sinh(2z)\frac{\partial^2 L}{\partial y\partial z}dy + 2Y\cosh(2z)\frac{\partial L}{\partial z}dz$$
$$+ Z\cosh(2z)\frac{\partial^2 L}{\partial x\partial x}dx + Z\cosh(2z)\frac{\partial^2 L}{\partial y\partial x}dy + 2Z\sinh(2z)\frac{\partial L}{\partial x}dz$$
$$- Z\sinh(2z)\frac{\partial^2 L}{\partial y\partial x}dx - Z\sinh(2z)\frac{\partial^2 L}{\partial y\partial x}dy - 2Z\cosh(2z)\frac{\partial L}{\partial y}dz$$
$$- \frac{\partial L}{\partial y}dy - \frac{\partial L}{\partial x}dx - \frac{\partial L}{\partial z}dz. \quad (21)$$

Using (9), we get first and second equations as follows:

$$-X\left[\cosh(2z)\frac{\partial^2 L}{\partial x\partial x} + \sinh(2z)\frac{\partial^2 L}{\partial x\partial y}\right]dx$$
$$-X\left[-\sinh(2z)\frac{\partial^2 L}{\partial x\partial x} - \cosh(2z)\frac{\partial^2 L}{\partial x\partial y}\right]dy$$
$$-X\frac{\partial^2 L}{\partial x\partial z}dz - Y\left[\cosh(2z)\frac{\partial^2 L}{\partial y\partial x} + \sinh(2z)\frac{\partial^2 L}{\partial y\partial y}\right]dx$$
$$-Y\left[-\sinh(2z)\frac{\partial^2 L}{\partial y\partial x} - \cosh(2z)\frac{\partial^2 L}{\partial y\partial y}\right]dy - Y\frac{\partial^2 L}{\partial y\partial z}dz$$
$$-Z\left[2\sinh(2z)\frac{\partial L}{\partial x} + \cosh(2z)\frac{\partial^2 L}{\partial z\partial x} + 2\cosh(2z)\frac{\partial L}{\partial y} + \sinh(2z)\frac{\partial^2 L}{\partial z\partial y}\right]dx$$
$$-Z\left[-2\cosh(2z)\frac{L}{\partial x} - \sinh(2z)\frac{\partial^2 L}{\partial z\partial x} - 2\sinh(2z)\frac{\partial L}{\partial y} - \cosh(2z)\frac{\partial^2 L}{\partial z\partial y}\right]dy$$
$$-Z\frac{\partial^2 L}{\partial z\partial z}dz = -\frac{\partial L}{\partial x}dx - \frac{\partial L}{\partial y}dy - \frac{\partial L}{\partial z}dz. \quad (22)$$

As a result of this process, the following equation is reached:

1. $-\left(X\cosh(2z)\frac{\partial^2 L}{\partial x\partial z} + Y\cosh(2z)\frac{\partial^2 L}{\partial y\partial z} + 2Z\sinh(2z)\frac{\partial L}{\partial z}\right)dx = -\frac{\partial L}{\partial x},$
2. $-\left(X\sinh(2z)\frac{\partial^2 L}{\partial x\partial z} + Y\sinh(2z)\frac{\partial^2 L}{\partial y\partial z} + 2Z\cosh(2z)\frac{\partial L}{\partial z}\right)dy = -\frac{\partial L}{\partial y},$
3. $-\left(X\cosh(2z)\frac{\partial^2 L}{\partial x\partial x} + Y\cosh(2z)\frac{\partial^2 L}{\partial y\partial x} + 2Z\sinh(2z)\frac{\partial L}{\partial x}\right)dz + \left(X\sinh(2z)\frac{\partial^2 L}{\partial x\partial y} + Y\sinh(2z)\frac{\partial^2 L}{\partial y\partial y} + 2Z\cosh(2z)\frac{\partial L}{\partial y}\right)dz = -\frac{\partial L}{\partial z}.$ (23)

If we take of the curve α, for all equations, as an integral curve of Ψ such that it is $\Psi(\alpha) = \frac{\partial}{\partial t}(\alpha)$ and we find the following equations:

1. $-\frac{\partial}{\partial t}\left(\cosh(2z)\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial x} = 0,$
2. $-\frac{\partial}{\partial t}\left(\sinh(2z)\frac{\partial L}{\partial z}\right) + \frac{\partial L}{\partial y} = 0,$ (24)
3. $-\frac{\partial}{\partial t}\left(\cosh(2z)\frac{\partial L}{\partial x}\right) + \frac{\partial}{\partial t}\left(\sinh(2z)\frac{\partial L}{\partial y}\right) + \frac{\partial L}{\partial z} = 0,$

such that the differential equations (24) are named **Euler-Lagrange equations** on almost paracontact metric manifold such that this is shown in the form of $(TM, \varphi, \Psi, \eta, g)$. Additionally, therefore the triple (TM, Φ_L, Ψ) is called a **Euler-Lagrangian mechanical system** on $(TM, \varphi, \Psi, \eta, g)$.

VI. EQUATIONS SOLVING WITH COMPUTER AND GRAPH OF SYSTEM

The location of each object in space is represented by three dimensions in physical space. These three dimensions can be labeled by a combination of three chosen from the terms time, length, width, height, depth, mass, density and breadth. (12) are partial differential equations. We, using Maple program of the equation system (12) solution is

$$L(x, y, z, t) = F_1(t). \quad (25)$$

It is well known that paracontact geometry is in many ways an odd-dimensional counterpart of symplectic geometry such that it belongs to the even-dimensional world. Both paracontact and an almost paracontact metric structure motivated by the mathematical formalism of classical, analytical and dynamical mechanics. Additionally, one can consider either the even-dimensional phase space of a mechanical system or the odd-dimensional extended phase space that includes the time variable. Also, classical field theory utilizes traditionally the language of Euler-Lagrangian dynamics. This theory was extended to time-dependent classical mechanics. A Euler-Lagrange space has been certified as an excellent model for some important problems in relativity, gauge theory and electromagnetism. Euler-Lagrangian gives a model for both the gravitational and electromagnetic fields in a very natural blending of the geometrical structures of the space with the characteristic properties of these physical fields.

VII. CONCLUSION

Our universe is three-dimensional such that Einstein added time as the fourth dimension. By this study the above mentioned forms, we were transferred 3-dimensional real number space on an almost paracontact manifold for the mechanical system. Euler-Lagrangian dynamics is used as a model for field theory, quantum physics, optimal control, biology and fluid dynamics [24].

The obtained time-dependent equations system (12) are very important to explain the rotational spatial mechanical-physical problems. In this study, the Euler-Lagrange partial equations (12) derived on almost paracontact metric manifolds and closed solutions (25) of equations system were found using symbolic computation program Maple. This approach may be suggested [25] to deal with problems in electrical,magnetic, and gravitational fields force for geodesics on defined space moving objects.

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