ON THE ALEKSANDROV-RASSIAS PROBLEM IN 2-NORMED SPACE

QIAN ZHANG, YUBO LIU, LIFANG CHANG, MEIMEI SONG

Abstract. Let $X$ and $Y$ be 2-normed linear spaces. If a mapping $f : X \times X \to Y$ preserves two distances with a noninteger ratio, $f$ must be an isometry. In this paper, we provide some results of the Aleksandrov-Rassias problem for mappings which preserves three distances in 2-normed space.

1. INTRODUCTION

The theory of isometry had its beginning in the important paper by Mazur and Ulam in 1932. He proved that every isometry mapping of a normed real linear space onto a normed real linear space is a linear mapping up to translation. When the target space $Y$ is a strictly convex real normed space, for into mapping, Baker[1] proved that every isometry of a normed real linear space into a strictly convex normed real linear space is also a linear isometry up to translation. What happens if we require, instead of one conservative distance for a mapping between normed vector spaces, two conservative distances? Aleksandrov and Rassias give some results about this problem. Aleksandrov-Rassias problem has obtained some results in Hilbert spaces, $X$ and $Y$ are Hilbert spaces with $\dim X \geq 2$, if $T : X \to Y$ preserves two distances with a noninteger ratio, then $T$ is linear isometry up to translation.

In 2001, Xiang Shuhuang[7] introduced that if $f$ preserves two distances and $X,Y$ are real normed vector spaces such that $Y$ is strictly convex and $\dim Y \geq 2$, it is an open problem whether or not $f$ must be an isometry, however, if $f$ preserves three distances, we have the result about isometry.

Aleksandrov-Rassias problem: If $T$ preserves two distances with a noninteger ratio, and $X$ and $Y$ are real normed vector spaces such

\textit{2010 Mathematics Subject Classification.} 46B04.
\textit{Key words and phrases.} Aleksandrov-Rassias problem; 2-normed space; AOPP.
that $Y$ is strictly convex and $\dim X \geq 2$, whether or not $T$ must be an isometry?

Benz and Berens\cite{2} proved the following theorem and pointed out that the condition that $Y$ is strictly convex can not be relaxed. Other authors proved Aleksandrov-Rassias problem in \cite{3,4,5}.

Let $X$ and $Y$ be real normed vector spaces, assume that $\dim X \geq 2$ and $Y$ is strictly convex, suppose $T : X \to Y$ satisfies that $T$ preserves the two distances $\rho$ and $\lambda \rho$ for some integer $\lambda \geq 2$, that is, for all $x, y \in X$ with $\|x - y\| = \rho$, then $\|T(x) - T(y)\| \leq \rho$, and for all $x, y \in X$ with $\|x - y\| = \lambda \rho$, then $\|T(x) - T(y)\| \geq \lambda \rho$, so $T$ is linear isometry up to translation.

It is easy to verify that the condition in above result is equivalent to that $T$ preserves the distances $\rho$ and $\lambda \rho$. Next we need the following definitions we will use in our main result.

**Definition 1.1.** Let $X$ be a real linear space with $\|\cdot, \cdot\| : X^2 \to R$, then $(X, \|\cdot, \cdot\|)$ is called a $2$-normed space if

\begin{enumerate}
\item \(\|x, y\| = 0 \iff x\text{ and } y\text{ are linearly dependent.}\)
\item \(\|x, y\| = \|y, x\|.\)
\item \(\|\alpha x, y\| = |\alpha| \|x, y\|.\)
\item \(\|x, y + z\| \leq \|x, y\| + \|x, z\|.\)
\end{enumerate}

For $\alpha \in R$ and $x, y, z \in X$. The function $\|\cdot, \cdot\|$ is called the $2$-norm on $X$.

**Definition 1.2.**\cite{6} we called $f$ a $2$-isometry if $\|x - y, y - z\| = \|f(x) - f(y), f(y) - f(z)\|$ for all $x, y, z \in X$.

**Definition 1.3.**\cite{6} (AOPP) Let $x, y, z \in X$ with $\|x - y, y - z\| = 1$, then $\|f(x) - f(y), f(y) - f(z)\| = 1$.

**Definition 1.4.**\cite{6} We call $f$ a $2$-Lipschitz mapping if there is a $k \geq 0$ such that $\|f(z) - f(x), f(y) - f(x)\| \leq k \|z - x, y - x\|$ for all $x, y, z \in X$, the smallest such $k$ is called the $2$-Lipschitz constant.

2. **MAIN RESULTS**

**Lemma 2.1.**\cite{6} For $b, c \in X$, if $b$ and $c$ are linearly dependent with the same direction, that is $c = \alpha b$ for some $\alpha > 0$, then $\|a, b + c\| = \|a, b\| + \|a, c\|$ for all $a \in X$.

**Lemma 2.2.** Let $X, Y$ are $2$-normed space and $f : X \times X \to Y$ is a surjection and satisfied:

\begin{enumerate}
\item $\|x - y, p - q\| \leq 1$, then $\|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\|$.
\item $\|x - y, p - q\| \geq \alpha$, then $\|f(x) - f(y), f(p) - f(q)\| \geq \alpha$.
\end{enumerate}

For all $x, y, p, q \in X$ and then $f$ is an $2$-isometry.
Proof. (a) First, we prove \(\|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\|\) for all \(x, y, p, q \in X\).

Let \(\|x - y, p - q\| \leq \frac{m}{n}\), if \(m = 1\), the result is obvious.

We suppose that \(m \geq 2\). Define \(q_i = q + \frac{i}{m}(p - q), (i = 0, 1, 2, \ldots, m)\), and

\[ q_{i+1} - q_i = \frac{1}{m}(p - q), p - q = \sum_{i=0}^{m-1} (q_{i+1} - q_i) \]

then

\[ \|x - y, q_{i+1} - q_i\| = \|x - y, \frac{1}{m}(p - q)\| = \frac{1}{m} \|x - y, p - q\| \leq \frac{1}{n}, \]

\((i = 0, 1, 2, \ldots, m - 1)\),

\[ \|f(x) - f(y), f(p) - f(q)\| \leq \sum_{i=0}^{m-1} \|f(x) - f(y), f(q_{i+1}) - f(q_i)\| \]

\[ \leq \sum_{i=0}^{m-1} \|x - y, q_{i+1} - q_i\| \]

\[ \leq \frac{m}{n}. \]

By Lemma 2.1

\[ \|x - y, p - q\| = \sum_{i=0}^{m-1} \|x - y, q_{i+1} - q_i\| \]

Thus \(\|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\|\).

(b) We prove \(f\) preserves \(\alpha\). We suppose \(\|x - y, p - q\| = \alpha\). Then there are \(m, n \in N\), and \(\alpha \leq \frac{m}{n}\), by (a), we have

\[ \|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\| \]

by the condition (2) \(\|f(x) - f(y), f(p) - f(q)\| \geq \|x - y, p - q\|\), so

\[ \|f(x) - f(y), f(p) - f(q)\| = \|x - y, p - q\| = \alpha. \]

(c) \(\|f(x) - f(y), f(p) - f(q)\| = \|x - y, p - q\|\) as \(\|x - y, p - q\| < \alpha\).

For \(\alpha > 0\), there must be \(m, n \in N\), so \(\alpha < \frac{m}{n}\), by (a), \(\|f(x) - f(y), f(p) - f(q)\| \leq \|x - y, p - q\|\)

Assume that

\[ \|f(x) - f(y), f(p) - f(q)\| < \|x - y, p - q\|. \]
Let \( z = x + \frac{\alpha}{\|x - y, p - q\|}(y - x) \), so
\[
\|z - x, p - q\| = \alpha, \|z - y, p - q\| = \alpha - \|x - y, p - q\|.
\]

Then by (b) and (a)
\[
\alpha = \|f(z) - f(x), f(p) - f(q)\| \\
\leq \|f(z) - f(y), f(p) - f(q)\| + \|f(x) - f(y), f(p) - f(q)\| \\
< \alpha - \|x - y, p - q\| + \|x - y, p - q\| = \alpha.
\]
This is a contradiction, that implies that \( \|f(x) - f(y), f(p) - f(q)\| = \|x - y, p - q\| \).

(d) \( f \) preserve that the distance \( \frac{\alpha}{2} \).

Let \( \|x - z, p - q\| = \frac{\alpha}{2} \), by (a), then \( \|f(x) - f(z), f(p) - f(q)\| \leq \frac{\alpha}{2} \),
let
\[
u = f(x) + \frac{\alpha}{2} \frac{f(z) - f(x)}{\|f(z) - f(x), f(p) - f(q)\|}
\]
there exsits a \( v \in X \), such that \( f(v) = u \) by \( f \) is a surjection. then
\[
\|u - f(x), f(p) - f(q)\| = \frac{\alpha}{2} < \alpha,
\]
by (2), \( \|v - x, p - q\| < \alpha \). by (c), \( \|v - x, p - q\| = \|u - f(x), f(p) - f(q)\| = \frac{\alpha}{2} \).

Then
\[
\|u - f(z), f(p) - f(q)\| \geq \frac{\alpha}{2}(n - 1).
\]
Otherwise if \( \|u - f(z), f(p) - f(q)\| < \frac{\alpha}{2}(n - 1) \), we can find a sequence
\( v_i \in X, (i = 1, 2, \cdots, n - 1) \), such that \( v_0 = v, v_{n-1} = z \), which implies
\( \|f(v) - f(z), f(p) - f(q)\| < \frac{\alpha}{2}(n - 1) \), by \( f \) is surjection, there exsits
\( v_i \in X \), then
\[
f(v_i) = f(v) + \frac{i}{n - 1}(f(z) - f(v)), (i = 0, 1, 2, \cdots, n - 1),
\]
so \( f(v_i) - f(v_{i+1}) = \frac{1}{n-1}(f(z) - f(v)) \) and \( f(v_i) - f(v_{i+1}) \) collinear, hence
\[
f(v) - f(z) = \sum_{i=0}^{n-2} (f(v_i) - f(v_{i+1}))
\]
by Lemma 2.1,
\[
\|f(v) - f(z), f(p) - f(q)\| = \sum_{i=0}^{n-2} \|f(v_i) - f(v_{i+1}), f(p) - f(q)\| < \frac{(n - 1)\alpha}{2}
\]
So
\[ \| f(v_i) - f(v_{i+1}), f(p) - f(q) \| \leq \frac{\alpha}{2} < \alpha, \]
by the condition (2), thus \( \| v_i - v_{i+1}, p - q \| < \alpha (i = 0, 1, 2, \cdots, n-1) \)
and by (c)
\[ \| v_i - v_{i+1}, p - q \| = \| f(v_i) - f(v_{i+1}), f(p) - f(q) \| < \frac{\alpha}{2} (i = 0, 1, 2, \cdots, n-1), \]
that implies
\[ \| v - z, p - q \| = \| \sum_{i=0}^{n-1} (v_i - v_{i+1}), p - q \| \leq \sum_{i=1}^{n-1} \| v_i - v_{i+1}, p - q \| \leq \frac{n-1}{2} \frac{n}{\alpha} \]
by (c),
\[ \| v - x, p - q \| = \| f(v) - f(x), f(p) - f(q) \| = \| u - f(x), f(p) - f(q) \| = \frac{\alpha}{2} \]
Moreover
\[ \| x - z, p - q \| \leq \| x - v, p - q \| + \| v - z, p - q \| < \frac{n-1}{2} + \frac{n}{2} = \frac{n}{2} \]
This is a contradiction with \( \| x - z, p - q \| = \frac{n}{2} \alpha \). and
\[ u - f(z) = f(x) - f(z) + \frac{\alpha}{2} \frac{f(z) - f(x)}{\| f(z) - f(x), f(p) - f(q) \|} \]
then
\[ \| u - f(z), f(p) - f(q) \| = \| f(x) - f(z), f(p) - f(q) \| (1 - \frac{\alpha}{2\| f(z) - f(x), f(p) - f(q) \|}) = \| f(x) - f(z), f(p) - f(q) \| - \frac{\alpha}{2} \]
So
\[ \frac{\alpha}{2} (n - 1) \leq \| u - f(z), f(p) - f(q) \| = \| f(z) - f(x), f(p) - f(q) \| - \frac{\alpha}{2} \]
that implies \( \| f(z) - f(x), f(p) - f(q) \| = \frac{n}{2} \alpha \).
(e) \( f \) is isometry from \( X \) to \( Y \). That is \( \| f(x) - f(y), f(p) - f(q) \| = \| x - y, p - q \| \).
For any \( x, y, p, q \in X \) and \( \alpha > 0 \), there exists \( n \) such that \( \| x - y, p - q \| <\)
Hence, by the condition, \[ \| f(x) - f(y), f(p) - f(q) \| \leq \| x - y, p - q \|. \]
Assume that
\[ \| f(x) - f(y), f(p) - f(q) \| < \| x - y, p - q \|. \]
Let
\[ z = x + \frac{\alpha}{2} n \]
So \[ \| z - x, p - q \| = \frac{\alpha}{2} n, \| z - y, p - q \| = \frac{\alpha}{2} n - \| x - y, p - q \|. \] So by (d),
\[ \| f(z) - f(x), f(p) - f(q) \| = \| z - x, p - q \| = \frac{\alpha}{2} n. \]
By (c),(d) and the assumption
\[ \frac{\alpha}{2} n = \| f(z) - f(x), f(p) - f(q) \| \]
\[ \leq \| f(z) - f(y), f(p) - f(q) \| + \| f(x) - f(y), f(p) - f(q) \| \]
\[ < \frac{\alpha}{2} n - \| x - y, p - q \| + \| x - y, p - q \| = \frac{\alpha}{2} n \]
that is a contradiction, that implies \[ \| f(x) - f(y), f(p) - f(q) \| = \| x - y, p - q \|. \]
we can say that \( f \) preserves the two distance 1 and \( \alpha \) in above Lemma.

**Theorem 2.3.** Let \( X \) and \( Y \) be real 2-normed space. Assume that \( Y \) is strictly convex, suppose \( f : X \to Y \) satisfied AOPP and \( f \) is a 2-Lipschitz mapping with \( k = 1 \), that is \[ \| f(x) - f(y), f(p) - f(q) \| \leq \| x - y, p - q \| \] for all \( x, y, p, q \in X \). Then \( f \) is a 2-isometry.

**Proof.** Let \( x, y, p, q \in X \), and \( \| x - y, p - q \| = \frac{1}{2} \), set \( z = x + 2(y - x) \)
Then \( \| x - z, p - q \| = 1, \| z - y, p - q \| = \frac{1}{2} \)
And by the condition 2-Lipschitz and AOPP,
\[ \| x - y, p - q \| \]
\[ \geq \| f(x) - f(y), f(p) - f(q) \| \]
\[ \geq \| f(z) - f(x), f(p) - f(q) \| - \| f(z) - f(y), f(p) - f(q) \| \]
\[ \geq 1 - \| z - y, p - q \| \geq \frac{1}{2} \]
By the condition, \[ \| f(x) - f(y), f(p) - f(q) \| \leq \| x - y, p - q \| = \frac{1}{2} \]
Hence
\[ \| f(x) - f(y), f(p) - f(q) \| = \frac{1}{2} \]
Similarly
\[ \|f(z) - f(y), f(p) - f(q)\| = \frac{1}{2} \]
And
\[ \|f(z) - f(x), f(p) - f(q)\| = \|f(z) - f(y), f(p) - f(q)\| + \|f(y) - f(x), f(p) - f(q)\| = 1 \]
Because Y is strictly convex, then \( f(y) = \frac{f(x) + f(z)}{2} \) and \( \|f(y) - f(x), f(p) - f(q)\| = \frac{1}{2} \), so f preserves distances 1 and \( \frac{1}{2} \), so f is an isometry due to lemma2.2.

**Theorem 2.4.** Let X and Y be real 2-normed spaces. Assume that \( \dim X \geq 2 \) and Y is strictly convex, suppose \( f : X \times X \to Y \) satisfies the property that f preserves the three distances 1, a and 1 + a, where a is any positive constant. Then f is a 2-isometry.

**Proof.** (1) Let \( x, y \in X \), \( \|x - y, p - q\| = 2 + a \), set \( x_1 = x + \frac{a}{2 + a}(y - x) \), \( x_2 = x + \frac{1 + a}{2 + a}(y - x) \)
Then
\[ \|x_1 - x, p - q\| = 1, \|x_1 - x_2, p - q\| = a, \]
\[ \|y - x_1, p - q\| = 1 + a, \|x_2 - x, p - q\| = 1 + a, \|y - x_2, p - q\| = 1 \]
It follows that
\[ \|f(x_1) - f(x), f(p) - f(q)\| = 1, \|f(x_1) - f(x_2), f(p) - f(q)\| = a \]
\[ \|f(y) - f(x_1), f(p) - f(q)\| = 1 + a, \|f(x_2) - f(x), f(p) - f(q)\| = 1 + a, \]
\[ \|f(y) - f(x_2), f(p) - f(q)\| = 1. \] Since Y is strictly convex, let
\[ f(x_1) - f(x) = \alpha(f(x_2) - f(x))(\alpha > 0) \]
Then
\[ 1 = \|f(x_1) - f(x), f(p) - f(q)\| = \alpha\|f(x_2) - f(x), f(p) - f(q)\| = \alpha(a + 1) \]
So \( \alpha = \frac{1}{a+1} \) and \( f(x_1) - f(x) = \frac{1}{a+1}(f(x_2) - f(x)) \)
We have
\[ f(x_1) = f(x) + \frac{1}{1+a}(f(x_2) - f(x)) \]
And
\[ f(x) = \frac{1+a}{a} f(x_1) - \frac{1}{a} f(x_2). \]
Since Y is strictly convex, let
\[ f(y) - f(x_2) = \alpha'(f(x_2) - f(x_1)) \]
Then
\[ 1 = \| f(y) - f(x_2), f(p) - f(q) \| = \alpha' \| f(x_2) - f(x_1), f(p) - f(q) \| = \alpha' a \]

So \( \alpha' = \frac{1}{a} \) and \( f(y) - f(x_2) = \frac{1}{a} (f(x_2) - f(x_1)) \)

We have \( f(x_2) = f(x_1) + \frac{a}{1+a} (f(y) - f(x_1)) \)

And \( f(y) = \frac{1+a}{a} f(x_2) - \frac{1}{a} f(x_1) \).

Thus \( \| f(x) - f(y), f(p) - f(q) \| = 2 + a \) for all \( x, y \in X \) with \( \| x - y, p - q \| = 2 + a \), so \( f \) preserves the distance \( 2+a \).

(2) Let \( \| x-y, p-q \| = 2a \), set \( x_1 = x + \frac{a-1}{2+2a} (y-x), x_2 = x + \frac{a}{2+2a} (y-x) \)

Then
\[ \| x_1 - x, p - q \| = 1 + a, \| x_1 - x_2, p - q \| = 1, \]
\[ \| y - x_1, p - q \| = 1 + a, \| x_2 - x, p - q \| = 2 + a, \| y - x_2, p - q \| = a, \]

since \( f \) preserves distances \( 1, a, 1+a \) and \( 2+a \), in a similar way, we obtain that
\[ \| f(y) - f(x), f(p) - f(q) \| = 2a. \]

By (1),(2) and Lemma 2.2, we have \( f \) preserves 1 and \( 2a \), then \( f \) is a 2-isometry.

**Corollary 2.5.** Let \( X \) and \( Y \) be real 2-normed spaces. Assume that \( \dim X \geq 2 \) and \( Y \) is strictly convex, suppose \( f : X \times X \to Y \) satisfies the property that \( f \) preserves the three distances \( a, b \) and \( a+b \), where \( a, b \) is any positive constant. Then \( f \) is a 2-isometry.

**References**


COLLEGE OF SCIENCE, TIANJIN UNIVERSITY OF TECHNOLOGY, TIANJIN 300384, CHINA