MINIMUM COVERING ENERGY OF SOME THORNY GRAPHS

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Abstract: Thorn graphs are obtained by attaching pendent vertices to each of its vertices. The minimum covering energy of a graph is based on the minimum covering sets of a graph. In this paper minimum covering energies of some thorn graphs are computed.

1. Introduction

According to Huckel Molecular orbital method, the total $\pi$ electron energy is sum of eigen values of the underlying molecular graph [3]. Motivated by HMO total molecular $\pi$ electron energy, I. Gutman [4] conceived the energy of a graph, defined as the sum of absolute values of all the eigenvalues of a graph. There is variety of results available not only on energy, but also on bounds of eigen values etc [4 –7]. Apart from the adjacency matrix other matrices such as Incidence matrix [12], Laplacian Matrix [8], Distance Matrix [9] etc have been defined and corresponding energies are obtained. Recently Adiga, Gutman et al [1] defined the concept of minimum covering energy and obtained results on spectra as well as energy. In this paper we obtain spectra and energy of thorn graphs of a family of graphs.

All the graphs considered in this paper are finite, simple, undirected. Let $G$ be such a graph of order $n$, with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set $E$. A subset $C$ of $V$ is called a covering set of $G$ if every edge of $G$ is incident to at least one vertex of $C$. Any covering set with minimum cardinality is called minimum covering set. Let $C$ be a minimum covering set of a graph $G$. The minimum covering matrix is the $n \times n$ matrix $A_c (G) = (a_{ij})$, where,

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{Otherwise} \end{cases}$$

The characteristic polynomial of $A_c (G)$ is denoted by

$$f_m (G, \lambda) = \det (\lambda I - A_c (G))$$

The minimum covering eigenvalues of the graph $G$ are the eigenvalues of $A_c (G)$. Since $A_c (G)$ is real symmetric, its eigenvalues are real numbers and we label them in non-
increasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n$. The minimum covering energy of $G$ is then defined as $E_c(G) = \sum_{i=1}^{n} |\lambda_i|$. In [1] some basic properties of minimum covering energy are discussed and some upper and lower bounds for $E_c(G)$ are given. Also minimum covering energies of star graphs, complete graphs etc are obtained.

In what follows we consider a class of graphs constructed by attaching $k$ new pendant vertices to each vertex of the underlying graph. These graphs are often referred to as thorny graphs or thorn graphs and have been much studied in the mathematical literature (see, for instance [2,11 ,13]. The thorny graph pertaining to the graph $G$ will be denoted by $G^{*k}$ The spectrum of $G^{*k}$ was determined in [7].

In this paper we compute the minimum covering energy of thorn graphs of all the graphs discussed in [1].

2. Minimum Covering Energy of Thorn Graphs

Let $G$ be any connected graph of order $n$ with vertex set $\{v_1, v_2, \ldots, v_n\}$. The minimum covering set of $G^{*k}$ is simply the vertex set of $G$ if $k > 1$ and there are two minimum covering sets if $k = 1$. So we consider two cases.

Case 1: Let $k > 1$ then minimum covering set of $G^{*k}$ is simply the vertex set of $G$. Thus if, $A_c(G)$ denotes the minimum covering matrix for $G$ then minimum covering matrix for $G^{*k}$ denoted by $A_c^*(G)$, obtained by making all diagonal entries of $A_c(G)$ equal to 1. With pertinent labeling of vertices, minimum covering matrix of $G^{*k}$ has the form,

$$A_c(G^{*k}) = \begin{bmatrix} O & A \\ A^T & A_c^*(G) \end{bmatrix}$$

where $A$ is a matrix of order $n \times n$ with $i$th column having 1’s in ‘$(i-1) + 1$’th to ‘$ik$’th positions and rest 0’s

The minimum covering polynomial of $G^{*k}$ takes the form,

$$f_m(G^{*k}, \lambda) = |\lambda I - A_c(G^{*k})| = |\lambda I^{nk} - A^T A - A_c^*(G)|$$

Using,

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |M| |Q - PM^{-1}N|$$

Since $A^T A = k I_n$ we get,

$$f_m(G^{*k}, \lambda) = \lambda^{nk} \left( \frac{\lambda^2 - k}{\lambda} \right) I - A_c^*(G)$$

(2.1)
Obviously from equation (2.1), the minimum covering polynomial for $G^{+k}$ depends on the characteristic polynomial for $A_c^*(G)$.

From the minimum covering matrix of $K_n$, $K_{m,n}$ $K_{l \times 2}$ (cocktail party graph) and $S^0_n$ (crown graph which is the complete bipartite graph $K_{m,n}$ with the horizontal edges removed) we obtain $A_c^*(K_n)$, $A_c^*(K_{m,n})$, $A_c^*(K_{l \times 2})$ and $A_c^*(S^0_n)$ consequently giving $|\lambda I - A_c^*(K_n)|$, $|\lambda I - A_c^*(K_{m,n})|$, $|\lambda I - A_c^*(K_{l \times 2})|$ and $|\lambda I - A_c^*(S^0_n)|$. Finally replacing $\lambda$ by $\frac{\lambda^2 - k}{\lambda}$ we get the desired minimum covering polynomial as per equation (2.1).

**Theorem 2.1:** Let $K_n$ be a complete graph of order $n$ then minimum covering energy the thorn graph $K_n^{+k}$ is given by, $2\sqrt{k} (n-1) + \sqrt{n^2 + 4k}$

**Proof:** For $K_n$, $A_c^*(K_n) = J$ (matrix of all 1’s) and hence the characteristic polynomial is, $|\lambda I - A_c^*(K_n)| = \lambda^{n-1}(\lambda - n)$ replacing $\lambda$ by $\frac{\lambda^2 - k}{\lambda}$ we get,

$$|\lambda I - A_c^*(K_n^{+k})| = \lambda^{nk}\left(\frac{\lambda^2 - k}{\lambda}\right)^{n-1}\left(\frac{\lambda^2 - k}{\lambda} - n\right)$$

On simplifying results into,

$$f_m(K_n)^{+k}, \lambda] = |\lambda I - A_c^*(K_n^{+k})| = \lambda^{nk} - n (\lambda^2 - k)^{n-1}(\lambda^2 - n\lambda - k)$$

Therefore minimum covering eigen values are

$$\pm \sqrt{k} (n-1) \text{ times, } \frac{n \pm \sqrt{n^2 + 4k}}{2} \text{ and } 0 \text{ (nk - n) times}$$

Consequently $E_c [(K_n)^{+k}] = 2\sqrt{k} (n-1) + \sqrt{n^2 + 4k}$

**Theorem 2.2:** Let $K_{m,n}$ be a complete bipartite graph of order $m + n$ then minimum covering polynomial of the thorn graph $K_{m,n}^{+k}$ is given by,

$$f_m[(K_{m,n})^{+k}, \lambda] = \lambda^{(m+n)k-m-n}(\lambda^2 - \lambda - k)^{m+n-2}\left[\lambda^4 - 2\lambda^3 - (2k + n^2 - 1)\lambda^2 + 2k\lambda + k^2\right]$$

**Proof:** On similar lines.

**Corollary 2.3:** Putting $m=1$ we get star $K_{1,n}$ with minimum covering polynomial as,

$$f_m[(K_{1,n})^{+k}, \lambda] = \lambda^{(n+1)(k-1)}(\lambda^2 - \lambda - k)^{n-1}\left[\lambda^4 - 2\lambda^3 - (2k + n^2 - 1)\lambda^2 + 2k\lambda + k^2\right]$$

Further when $k = n$

$$f_m[(K_{1,n})^{+n}, \lambda] = \lambda^{(n+1)(n-1)}(\lambda^2 - \lambda - n)^{n-1}\left[\lambda^4 - 2\lambda^3 - (2n + n^2 - 1)\lambda^2 + 2n\lambda + n^2\right]$$

$$= \lambda^{(n^2-1)}(\lambda^2 - \lambda - n)(\lambda + n)(\lambda - 1)[\lambda^2 - (n+1)\lambda - n]$$

So for $n = k$ the minimum covering energy becomes,

$$E_c [(K_{1,n})^{+n}] = (n-1)\sqrt{4n+1} + \sqrt{(n+1)^2 + 4n + n + 1}$$

**Theorem 2.4:** If $K_{l \times 2}$ denotes the cocktail party graph of order $2l$ then,
minimum covering energy of \((K_{l \times 2})^+\) is, \((2l - 1)\sqrt{4k + 1} + \sqrt{(2l - 1)^2 + 4k}\)

**Proof:** On similar lines

**Theorem 2.5:** If \(S_{\sigma^0}\) denotes the crown graph of order \(2n\) then, minimum covering energy of \((S_{\sigma^0})^+\) is given by,

\[
(2n - 1)(\sqrt{k + 1} + \sqrt{k}) + \sqrt{(n - 2)^2 + 4k} + \sqrt{n^2 + 4k}
\]

**Proof:** On similar lines

**Case 2:** Let \(k = 1\) then \(G^+\) has precisely two minimum covering sets namely \(V(G)\) and the new set of pendant vertices. Here we discuss both cases.

**Case 2.1:** Consider \(V(G)\) as a minimum covering set of \(G^+\). The minimum covering matrix of \(G^+\) has the form,

\[
A_c(G) = \begin{bmatrix} I & I \\ I & A(G) \end{bmatrix}
\]

where \(I\) is the identity matrix of order \(n\) and \(A(G)\) is the adjacency matrix of \(G\).

The minimum covering polynomial is then,

\[
f_m(G^+, \lambda) = (\lambda - 1)I - I - I \lambda I - A(G) \lambda = (\lambda - 1)\left(\frac{\lambda^2 - \lambda - 1}{\lambda - 1} I - A(G)\right)
\]

\[
= (\lambda - 1)^n \left(\frac{\lambda^2 - \lambda - 1}{\lambda - 1} I - A(G)\right)
\]

Thus knowing the adjacency polynomial the minimum covering polynomial can be easily obtained from equation (2.3).

**Theorem 2.6:** The minimum covering energy of thorn graph of a complete graph \((K_n)^+\) is, \(2(n - 1)\sqrt{n} + n\)

**Proof:** Using the adjacency polynomial of \(K_n\) we have from equation (2.3)

\[
f_m(K_n^+, \lambda) = (\lambda - 1)^n \left(\frac{\lambda^2 - \lambda - 1}{\lambda - 1} + 1\right) [\frac{\lambda^2 - \lambda - 1}{\lambda - 1} - (n - 1)\] \[
= (\lambda^2 - 2)^n \left[\lambda^2 - n\lambda + (n - 2)\right]
\]

Equating to zero we get eigen values and adding their absolute values the theorem follows.

**Theorem 2.7:** The minimum covering energy of \((K_{m,n})^+\) is given by,

\[
(2n - 2)\sqrt{5} + \sqrt{(1 - \sqrt{mn})^2 + 4(\sqrt{mn} + 1) + 1 + \sqrt{mn}}
\]

**Proof:** On similar lines
Corollary 2.8: When \( m = n \), the minimum covering energy of \((K_{m,n})^{+1}\) will be,

\[
2(n-1)\sqrt{5} + \sqrt{n^2 + 2n + 3 + n + 1}
\]

Theorem 2.9: The minimum covering energy for thorn graph of a cocktail party graph \( K_{l\times2} \) is given by,

\[
l\sqrt{5} + (l-1)\sqrt{13 + (2l-1)}
\]

Proof:

Theorem 2.10: If \( S_n^0 \) denotes the crown graph of order \( 2n \) then, minimum covering energy of \((S_n^0)^{+1}\) is,

\[
2(n-1)(1+\sqrt{2}) + n + \sqrt{n^2 + 4}
\]

Proof: On similar lines

Case 2.2: Consider set of pendent vertices as minimum covering set of \( G^{+1} \). The minimum covering matrix of \( G^{+1} \) has the form,

\[
A_c(G) = \begin{bmatrix}
O & I \\
I & A^*(G)
\end{bmatrix}
\]

where \( I \) is the identity matrix of order \( n \) and \( A^*(G) \) is the adjacency matrix of \( G \) having all diagonal entries 1.


The minimum covering polynomial is then,

\[
f_m(G^{+1}, \lambda) = \begin{vmatrix}
\lambda I & -I \\
-I & \lambda I - A^*(G)
\end{vmatrix}
\]

\[
= \lambda \begin{vmatrix}
\lambda I & -I \\
-I & \lambda I - A^*(G)
\end{vmatrix}
\]

\[
f_m(G^{+1}, \lambda) = \lambda^n \left( \frac{\lambda^2 - 1}{\lambda} \right) I - A^*(G)
\]  

(2.4)

Thus knowing the modified adjacency polynomial the minimum covering polynomial can be easily obtained from equation (2.4).

Theorem 2.11: The minimum covering energy of \( (K_n)^{+1} \) is given by

\[
2(n-1) + \sqrt{n^2 + 4}
\]

Proof: Since \( A^*(K_n) = J \)

\[
f_m(K_n^{+1}, \lambda) = \lambda^n \left( \frac{\lambda^2 - 1}{\lambda} \right) I - J
\]
\[ = \lambda^2 |(\lambda^2 - 1)I - K_n| = \lambda^2 |(\lambda^2 - \lambda - 1)I - K_n| \]

\[ \therefore f_m(K_n^{+1}, \lambda) = (\lambda^2 - 1)^{n-1}[\lambda^2 - n\lambda - 1] \]

Equating to zero we get eigen values and adding their absolute values the theorem follows

**Theorem 2.12**: The minimum covering spectrum of thorn graph \((K_{m,n})^{+1}\) is,

\[ (m + n - 2)\sqrt{5} + \sqrt{mn} + 1 + \sqrt{mn + 2\sqrt{mn} + 5} \]

**Proof**: On similar lines

**Theorem 2.13**: For cocktail party graph \(K_{l \times 2}\) the minimum covering energy of 

\[ (K_{l \times 2})^{+1} \] is, \((2l - 1)\sqrt{5} + \sqrt{4l^2 - 4l + 5}\)

**Proof**: On similar lines

**Theorem 2.14**: If \(S_n^0\) denotes the crown graph of order \(2n\) then,

minimum covering energy of \((S_n^0)^{+1}\) is, \(2(n - 1)(1 + \sqrt{2}) + \sqrt{n^2} + 4 + \sqrt{n^2 - 4n + 8}\)

**Proof**: On similar lines

**REFERENCES**


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