FEKETE-SZEGÖ PROBLEM FOR SOME SUBCLASSES OF COMPLEX ORDER RELATED TO SÁLÁGEAN OPERATOR

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ABSTRACT. In the present investigation, sharp upper bounds of \(|a_3 - \mu a_2^2|\) for function \(f(z)\) belonging to certain subclasses of \(\text{Re} \left[1 + \frac{1}{2} \left\{ (1 - \alpha) \frac{f''(z)}{f'(z)} + \alpha f'(z) - 1 \right\} \right] > 0\) are obtained. Also certain applications of the main results for subclasses of functions defined by convolution with a normalized analytic function are given. In particular, Fekete - Szegö inequalities for certain classes of functions defined through fractional derivatives are obtained.

1. INTRODUCTION

We let \(A\) to denote the class of all analytic functions \(f(z)\) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in D = \{ z \in \mathbb{C} : |z| < 1 \})
\]

and \(\mathbb{S}\) be the subclass of \(A\) consisting of univalent functions. For two analytic functions \(f(z)\) given by (1.1) and \(g(z) = z + \sum_{n=2}^{\infty} b_n z^n\), their convolution (or Hadamard product) is defined to be the function \((f \ast g)(z)\) given by \((f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n\).

For two functions \(f, g \in A\), we say that the function \(f(z)\) is subordinate to \(g(z)\) in \(D\) and write \(f \prec g\) or \(f(z) \prec g(z) \quad (z \in D)\), if there exists an analytic function \(w(z)\) with \(w(0) = 0\) and \(|w(z)| < 1 \quad (z \in D)\), such that

\[
f(z) = g(w(z)) \quad (z \in D).
\]

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In particular, if the function \( g \) is univalent in \( D \), the above subordination is equivalent to \( f (0) = g (0) \) and \( f (D) \subset g (D) \).

Throughout this paper, we assume that \( \phi \) is an analytic univalent function with positive real part in \( D \), \( \phi (D) \) is symmetric with respect to the real axis and starlike with respect to \( \phi (0) = 1 \), and \( \phi ' (0) > 0 \). The Taylor’s series expansion of such function is of the form

\[
(1.2) \quad \phi (z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \text{with} \quad B_1 > 0.
\]

In the present investigation, we obtain Fekete-Szeg\"o inequality for function in a more general class \( \Re (\alpha, \phi) \) which we define below. We also give applications of our results to certain functions defined through Hadamard product and functions defined by fractional derivatives.

**Definition 1.1.** Let \( \alpha \geq 0 \). A function \( f \in A \) given by (1.1) is in the class \( \Re (\alpha, \phi) \), if it satisfies

\[
(1.3) \quad \text{Re} \left[ 1 + \frac{1}{b} \left\{ (1-\alpha) \frac{f(z)}{z} + \alpha f' (z) - 1 \right\} \right] > 0
\]

**Lemma 1.1.** If \( p_1 (z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \) is analytic function with positive real part in \( D \), then

\[
|c_2 - vc_1^2| \leq \begin{cases} -4v + 2 & \text{if } v \leq 0, \\ 2 & \text{if } 0 \leq v \leq 1, \\ 4v - 2 & \text{if } v \geq 1. \end{cases}
\]

When \( v < 0 \) or \( v > 1 \), the equality holds if and only if \( p_1 (z) \) is \( (1+z)/(1-z) \) or one of its rotations. If \( 0 < v < 1 \), then the equality holds if and only if \( p_1 (z) \) is \( (1+z^2)/(1-z^2) \) or one of its rotations. If \( v = 0 \), the equality holds if and only if

\[
p_1 (z) = \left( \frac{1}{2} + \frac{1}{2} \lambda \right) \frac{1+z}{1-z} + \left( \frac{1}{2} - \frac{1}{2} \lambda \right) \frac{1-z}{1+z}, (0 \leq \lambda \leq 1)
\]

or one of its rotations. If \( v = 1 \), the equality holds if and only if \( p_1 (z) \) is the reciprocal of one of the functions such that the equality holds in the case \( v = 0 \). Also the above upper bound is sharp and it can be improved as follows:

When \( 0 < v < 1 \),

\[
|c_2 - vc_1^2| + v |c_1|^2 \leq 2 \quad (0 < v \leq 1/2)
\]
and

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2 \quad (1/2 < v \leq 1).$$

Let a differential operator be defined Sălăgean [10] on a class of analytic functions of the form (1.1) as follows

$$D^0 f (z) = f (z), \quad D^1 f (z) = D f (z) = zf' (z),$$

and in general

$$D^n f (z) = D \left( D^{n-1} f (z) \right), \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

We easily find that

$$(1.4) \quad D^n f (z) = z + \sum_{k=2}^{\infty} k^n a_k z^n \quad (n \in \mathbb{N}_0).$$

2. **Fekete-Szegő Problem for the Function class $\mathcal{R} (\alpha, \phi)$**

By using Lemma 1.1, we prove the following Fekete-Szegő inequalities.

**Theorem 2.1.** Let $b$ be a non-zero complex number. If $f (z)$ given by (1.1) belongs to $\mathcal{N}^b_n (\alpha, \phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{b}{3^n} \left[ \frac{B_2}{1 + 2\alpha} - \frac{\mu B_2^2}{(1 + \alpha)^2} \left( \frac{3}{4} \right)^n \right] & \text{if } \mu \leq \sigma_1, \\ \frac{b B_1}{3^n (1 + 2\alpha)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ - \frac{b}{3^n} \left[ \frac{B_2}{1 + 2\alpha} - \frac{\mu B_2^2}{(1 + \alpha)^2} \left( \frac{3}{4} \right)^n \right] & \text{if } \mu \geq \sigma_2. \end{cases}$$

where

$$\sigma_1 := \frac{(1 + \alpha)^2 (B_2 - B_1)}{b (1 + 2\alpha) B_1^2} \left( \frac{4}{3} \right)^n, \quad \sigma_2 := \frac{(1 + \alpha)^2 (B_2 + B_1)}{b (1 + 2\alpha) B_1^2} \left( \frac{4}{3} \right)^n.$$ 

The result is sharp.

If $f (z) \in \mathcal{R} (\alpha, \phi)$, then there exists a Schwarz function $w (z)$ analytic in $D$ with $w (0) = 0$ and $|w (z)| < 1 \quad (z \in D)$, such that

$$(2.1) \quad 1 + \frac{1}{b} \left\{ (1 - \alpha) \frac{f (z)}{z} + \alpha f' (z) - 1 \right\} = \phi (w (z))$$

where

$$\phi (w (z)) = \int_{\frac{1}{2}}^{1} \left[ \frac{B_2}{1 + 2\alpha} - \frac{\mu B_2^2}{(1 + \alpha)^2} \left( \frac{3}{4} \right)^n \right] \frac{dz}{z^n}.$$
Define the function $p_1$ by

$$p_1 = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

since $w(z)$ is a Schwarz function, we see that $\Re(p_1(z)) > 0 (z \in D)$ and $p_1(0) = 1$. Now, defining the function $p(z)$ by

$$p(z) = 1 + \frac{1}{b} \left\{ (1 - \alpha) \frac{f(z)}{z} + \alpha f'(z) - 1 \right\} = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$$

we find from (2.1) and (2.2) that

$$p(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right).$$

Thus by using (2.2) in (2.4), we obtain

$$b_1 = \frac{1}{2} B_1 c_1, \quad b_2 = \frac{1}{2} B_1 \left( c_2 - \frac{1}{2} c_1^2 \right) + \frac{1}{4} B_2 c_1^2.$$ 

Also, from (2.3) we obtain

$$b_1 = \frac{1}{b} (1 + \alpha) 2^n a_2, \quad b_2 = \frac{1}{b} (1 + 2\alpha) 3^n a_3.$$ 

Therefore we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{3^n 2 (1 + 2\alpha)} \left[ c_2 - \nu c_1 \right].$$

where

$$v := \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} \frac{b \mu B_1 (1 + 2\alpha)}{(1 + \alpha)^2} \left( \frac{3}{4} \right)^n \right].$$

Our result now follows by an application of lemma 1.1. To show that the bounds are sharp, we define the functions $K_{\phi_n}^\alpha (n = 2, 3, 4, \ldots)$ by

$$1 + \frac{1}{b} \left\{ (1 - \alpha) \frac{[K_{\phi_n}^\alpha(z)]}{z} + \alpha [K_{\phi_n}^\alpha]'(z) - 1 \right\} = \phi(z^{n-1}),$$

$$K_{\phi_n}^\alpha(0) = 0 = [K_{\phi_n}^\alpha]'(0) - 1$$

and the function $F_{\lambda}^\alpha$ and $G_{\lambda}^\alpha (0 \leq \lambda \leq 1)$ by

$$1 + \frac{1}{b} \left\{ (1 - \alpha) \frac{[F_{\lambda}^\alpha(z)]}{z} + \alpha [F_{\lambda}^\alpha]'(z) - 1 \right\} = \phi(z^{n-1}),$$

$$F_{\lambda}^\alpha(0) = 0 = [F_{\lambda}^\alpha]'(0) - 1$$
1 + \frac{1}{b} \left\{ (1 - \alpha) \frac{[G_\alpha^n (z)]}{z} + \alpha [G_\alpha^n]' (z) - 1 \right\} = \phi (z^{n-1}),

G_\alpha^n (0) = 0 = [G_\alpha^n]' (0) - 1

Clearly the functions $K_\alpha^n, F_\lambda^n$ and $G_\lambda^n \in \mathcal{R} (\alpha, \phi)$. Also we write $K_\alpha^n := K_{\phi z}^\alpha$.

If $\mu \leq \sigma_1$ or $\mu \geq \sigma_2$, then the equality holds if and only if $f$ is $K_\alpha^n$ or one of its rotations. When $\sigma_1 \leq \mu \leq \sigma_2$, the equality holds if and only if $f$ is $K_\alpha^n$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f$ is $F_\lambda^n$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f$ is $G_\lambda^n$ or one of its rotations.

Remark 2.1. If $\sigma_1 \leq \mu \leq \sigma_2$, then, in view of Lemma 1.1, Theorem 2.1 can be improved. Let $\sigma_3$ be given by

$\sigma_3 := \frac{(1 + \alpha)^2 B_2}{b (1 + 2\alpha) B_1^2} \left( \frac{4}{3} \right)^n$.

Let $f \in \mathcal{R} (\alpha, \phi)$. If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{1}{b (1 + 2\alpha) B_1^2} \left( \frac{4}{3} \right)^n [(1 + \alpha)^2 (B_2 - B_1) \left( \frac{3}{4} \right)^n + \mu b (1 + 2\alpha) B_1^2 \left( \frac{4}{3} \right)^n |a_2|^2 \leq \frac{b B_1}{3^n (1 + 2\alpha)}.$$ 

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{1}{b (1 + 2\alpha) B_1^2} \left( \frac{4}{3} \right)^n [(1 + \alpha)^2 (B_2 + B_1) \left( \frac{3}{4} \right)^n - \mu b (1 + 2\alpha) B_1^2 \left( \frac{4}{3} \right)^n |a_2|^2 \leq \frac{b B_1}{3^n (1 + 2\alpha)}.$$ 

For $\phi (z) = (1 + Cz) / (1 + Dz), -1 \leq D < C \leq 1$. Theorem 2.1 leads to the following results:

Corollary 2.2. Let $-1 \leq D < C \leq 1$. If $f \in \mathcal{R} (\alpha, (1 + Cz) / (1 + Dz))$, then
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{b(D-C)}{3^n(1+2\alpha)} \left[ D - \frac{b\mu(D-C)(1+2\alpha)}{2(1+\alpha)^\alpha} \left( \frac{3}{4} \right)^n \right] & \text{if } \mu \leq -\frac{2}{b} \left[ \frac{(1+D)(1+\alpha)^2}{(1+2\alpha)(C-D)} \left( \frac{4}{3} \right)^n \right], \\ \frac{b(C-D)}{3^n(1+2\alpha)} & \text{if } \mu = \frac{2}{b} \left[ \frac{(1+D)(1+\alpha)^2}{(1+2\alpha)(C-D)} \left( \frac{4}{3} \right)^n \right], \\ -\frac{b(D-C)}{3^n(1+2\alpha)} \left[ D - \frac{b\mu(D-C)(1+2\alpha)}{2(1+\alpha)^\alpha} \left( \frac{3}{4} \right)^n \right] & \text{if } \mu \leq \frac{2}{b} \left[ \frac{(1-D)(1+\alpha)^2}{(1+2\alpha)(C-D)} \left( \frac{4}{3} \right)^n \right]. \end{cases} \]

For \( \phi(z) = (1 + z) / (1 - z) \), Theorem 2.1 leads to the following results:

**Corollary 2.3.** Let \(-1 \leq D < C \leq 1\) If \( f \in \mathbb{R}(\alpha, (1 + z) / (1 - z)) \), then

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{2b}{3^n(1+2\alpha)} \left[ 1 - \frac{b\mu(1+2\alpha)}{(1-\alpha)^\alpha} \left( \frac{3}{4} \right)^n \right] & \text{if } \mu \leq 0, \\ -\frac{2b}{3^n(1+2\alpha)} \left[ 1 - \frac{b\mu(1+2\alpha)}{(1-\alpha)^\alpha} \left( \frac{3}{4} \right)^n \right] & \text{if } 0 \leq \mu \leq \frac{2(1+\alpha)^2}{b(1+2\alpha)} \left( \frac{4}{3} \right)^n, \\ \frac{2b}{3^n(1+2\alpha)} \left[ 1 - \frac{b\mu(1+2\alpha)}{(1-\alpha)^\alpha} \left( \frac{3}{4} \right)^n \right] & \text{if } \mu \geq \frac{2(1+\alpha)^2}{b(1+2\alpha)} \left( \frac{4}{3} \right)^n. \end{cases} \]

For \( C = 1 - 2\beta \) with \( 0 \leq \beta < 1 \) and \( D = -1 \), Corollary 2.2 reduces to the following result:

**Corollary 2.4.**

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{2b(1-\beta)}{3^n(1+2\alpha)} \left[ 1 - \frac{b\mu(1-\beta)(1+2\alpha)}{(1-\alpha)^\alpha} \left( \frac{3}{4} \right)^n \right] & \text{if } \mu \leq 0, \\ \frac{2b(1-\beta)}{3^n(1+2\alpha)} & \text{if } 0 \leq \mu \leq \frac{2(1+\alpha)^2}{b(1-\beta)(1+2\alpha)} \left( \frac{4}{3} \right)^n, \\ \frac{2b(1-\beta)}{3^n(1+2\alpha)} \left[ 1 - \frac{b\mu(1-\beta)(1+2\alpha)}{(1-\alpha)^\alpha} \left( \frac{3}{4} \right)^n \right] & \text{if } \mu \geq \frac{2(1+\alpha)^2}{b(1-\beta)(1+2\alpha)} \left( \frac{4}{3} \right)^n. \end{cases} \]

3. **Application to Functions Defined by Fractional Derivatives**

In order to introduce the class \( \mathbb{R}^\lambda(\alpha, \phi) \), we need the following:

**Definition 3.1.** see ([2, 3], see also [7, 8]). Let \( f(z) \) be analytic in a simply connected region of the \( z \)-plane containing the origin. The fractional derivative of \( f \) of order \( \lambda \) is defined by
\[
D_\lambda^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1),
\]
where the multiplicity of \((z-\zeta)^\lambda\) is removed by requiring that \(\log(z-\zeta)\) is real for \(z-\zeta > 0\). Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [3] introduced the operator \(\Omega^\lambda : \mathcal{A} \rightarrow \mathcal{A}\) defined by

\[
(\Omega^\lambda f)(z) = \Gamma(2-\lambda) z^\lambda D_\lambda^\lambda f(z) \quad (\lambda \neq 2, 3, 4, \ldots).
\]
The class \(\mathcal{R}^\lambda(\alpha, \phi)\) consists of functions \(f \in \mathcal{A}\) for which \(\Omega^\lambda f \in \mathcal{R}(\alpha, \phi)\). Note that \(\mathcal{R}^\lambda(\alpha, \phi)\) is the special case of the class \(\mathcal{R}^g(\alpha, \phi)\) when

\[
g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1) \Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n.
\]

Let \(g(z) = z + \sum_{n=2}^{\infty} g_n z^n\) \((g_n > 0)\). Since \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n\) \(\in \mathcal{R}^g(\alpha, \phi)\) if and only if \((f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in \mathcal{R}(\alpha, \phi)\), we obtain the coefficient estimate for functions in the class \(\mathcal{R}^\lambda(\lambda, \phi)\) from the corresponding estimate for functions in the class \(\mathcal{R}(\lambda, \phi)\). Applying Theorem 2.1 for the function \((f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \cdots\), we get the following theorem after an obvious change of the parameter \(\mu\):

**Theorem 3.1.** Let the function \(\phi(z)\) be given by \(\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots\). If \(f(z)\) given by (1.1) belongs to \(\mathcal{R}^g(\alpha, \phi)\), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{b}{g_3} \left[ \frac{B_2}{1+2\alpha} - \frac{\mu b g_3 B_1^2}{(1+\alpha)^2 g_3^2} \right] & \text{if } \mu \leq \sigma_1, \\
\frac{b B_1}{g_3(1+2\alpha)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\
-\frac{b}{g_3} \left[ \frac{B_2}{1+2\alpha} - \frac{\mu b g_3 B_1^2}{(1+\alpha)^2 g_3^2} \right] & \text{if } \mu \geq \sigma_2.
\end{cases}
\]

where

\[
\sigma_1 := \frac{g_3^2 (1+\alpha)^2 (B_2 - B_1)}{bg_3 (1+2\alpha) B_1^2}, \quad \sigma_1 := \frac{g_3^2 (1+\alpha)^2 (B_2 + B_1)}{bg_3 (1+2\alpha) B_1^2}
\]

The result is sharp.
Since
\[ (\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1) \Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n, \]
we have
\[ g_2 := \frac{\Gamma(3) \Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \]
and
\[ g_3 := \frac{\Gamma(4) \Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \]
For \( g_2 \) and \( g_3 \) given by (3.2) and (3.3), Theorem 3.1 reduces to the following:

**Theorem 3.2.** Let the function \( \phi(z) \) be given by \( \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \). If \( f(z) \) given by (1.1) belongs to \( R_\lambda^\alpha(\alpha, \phi) \), then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{h(2-\lambda)(3-\lambda)}{6} \left[ \frac{B_2}{1+2\alpha} - \frac{\mu h(2-\lambda)B_1^2}{2(1+\alpha)^2(3-\lambda)} \right] & \text{if } \mu \leq \sigma_1, \\
\frac{h(2-\lambda)(3-\lambda)B_1}{6(1+2\alpha)} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\
\frac{h(2-\lambda)(3-\lambda)}{6} \left[ \frac{B_2}{1+2\alpha} - \frac{\mu h(2-\lambda)B_1^2}{2(1+\alpha)^2(3-\lambda)} \right] & \text{if } \mu \geq \sigma_2.
\end{cases}
\]

where
\[ \sigma_1 := \frac{2(3-\lambda)(1+\alpha)^2(B_2 - B_1)}{3b(2-\lambda)(1+2\alpha)B_1^2}, \quad \sigma_2 := \frac{2(3-\lambda)(1+\alpha)^2(B_2 + B_1)}{3b(2-\lambda)(1+2\alpha)B_1^2}. \]
The result is sharp.

**References**


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