ON COMPLETENESS AND BICOMPLETIONS OF QUASI $b$-METRIC SPACES

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Abstract. The purpose of this paper is to present results on completeness and bicompletions of quasi $b$-metric spaces. We further discuss extension of maps defined on a dense subspace to the whole space.

1. Introduction

In the literature one find a rich theory on completeness of metric spaces and quasi metric spaces for example: Doitchinov’s completion [5], Yoneda’s completion [10], Salbany’s completion [13] and Smyth’s completion [16]. Note that the current completion theory on metric spaces and quasi metric spaces does not cater and address for a completion of some mathematical structures: For example the space $X$, where $X$ is the set of rational numbers equipped with the distance function $d(x,y) = (x - y)^2$. We easily see that $(X,d)$ is neither a metric space nor a quasi metric space. The completion of $X$ is just $\tilde{X}$, the set of real numbers with the distance function $d(x,y) = (x - y)^2$. So certainly, we need a formal and explicit structure to address this motivation.

The notion of $b$-metric spaces was first introduced and studied by Bakhtin [1] in 1989. Due to its importance, Czerwik used the notion to present and study some generalization of contraction mappings in [2] and also in [3]. Subsequently, several researchers motivated by the notion of $b$-metric spaces have obtained interesting extensions of the Banach contraction principle see the papers [4] and [11] for example.

Inspired by the notion of $b$-metric spaces we introduce the concept of quasi $b$-metric spaces and deduce several results in this context. Among other results presented in the paper, we show that every quasi $b$-metric space admits a bicompletion which is unique up to isometry, and we
also show that every quasi $b$-metric space is quasi metrizable. On the other hand, we have provided some examples to show the generality of our results. In particular, we extend the work of [12] and some others.

2. Basic notions and preliminaries

Our basic references for quasi uniform spaces and quasi metric spaces are [6] and [9] respectively. Recall that:

**Definition 2.1.** [9] A **quasi metric space** is a pair $\langle X, d \rangle$ where $d : X \times X \to [0, \infty)$ satisfies the following for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

**Example 2.1.** Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ be defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x \leq y. \end{cases}$$

Then $\langle X, d \rangle$ is a quasi metric space.

**Definition 2.2.** [2] Let $X$ be a set, and $d : X \times X \to [0, \infty)$ be a function satisfies the following for all $x, y, z \in X$, and $s \geq 1$:

(i) $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$. The pair $\langle X, d \rangle$ is called a **$b$-metric space**.

**Example 2.2.** Let $X = \mathbb{R}$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = (x - y)^2$ for all $x, y \in \mathbb{R}$. Then $\langle X, d \rangle$ is a $b$-metric space with constant ($s = 2$). Note that $\langle X, d \rangle$ is neither a quasi metric space nor a metric space.

For more examples of $b$-metric spaces, which shows that $b$-metric space is a generalization of metric space we refer the reader to [7] and [14].

**Remark 2.1.** Let $\langle X, d \rangle$ be a $b$-metric space. Note that:

- If $s = 1$ then, Definition 2.2 reduces to a standard metric space.
- If $s = 1$ and properties (i) and (iii) are satisfied then, Definition 2.2 reduces to a quasi metric space.
- Hence, the class of $b$-metric spaces is larger than the class of metric spaces.

**Definition 2.3.** A **quasi $b$-metric space** is a pair $\langle X, d \rangle$ where $s \geq 1, s \in \mathbb{R}$ and $d : X \times X \to [0, \infty)$ satisfies the following for all $x, y, z \in X$: 

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(i) \(d(x, y) = 0\) if and only if \(x = y\).
(ii) \(d(x, y) \leq s[d(x, z) + d(z, y)]\).

Given a quasi b-metric space \((X, d)\) we define \(d^{-1} : X \times X \to [0, \infty)\) by \(d^{-1}(x, y) = d(y, x)\) for all \(x, y \in X\). Then \((X, d^{-1})\) is also a quasi b-metric space. We call \(d^{-1}\) the conjugate of \(d\) on \(X\). Note that \(d^{-1}\) is a \(b\)-metric on \(X\).

Remark 2.2. Note that a quasi metric space is a quasi b-metric space with a constant \(s = 1\). Therefore the class of quasi b-metric spaces is larger than the class of quasi metric spaces.

Definition 2.4. Let \((X, d)\) be a quasi b-metric space. Then:
(i) A sequence \(\{x_n\}\) in \((X, d)\) \(d\)-converges to \(x \in X\) if
\[
\lim_{n} d(x_n, x) = 0.
\]
(ii) A sequence \(\{x_n\}\) in \((X, d)\) \(d^{-1}\)-converges to a point \(x \in X\) if
\[
\lim_{n} d^{-1}(x_n, x) = 0.
\]
(iii) A sequence \(\{x_n\}\) in \((X, d)\) \(d^*\)-converges to a point \(x \in X\) if
\[
\lim_{n} d^*(x_n, x) = 0.
\]

Remark 2.3. Let \((X, d)\) be a quasi b-metric space. For a sequence \(\{x_n\}\) in \(X\) and \(x \in X\), \(\lim_n d(x_n, x) = 0\) implies \(\lim_n d^*(x_n, x) = 0\) and \(\lim_n d^{-1}(x_n, x) = 0\).

Example 2.3. Let \(X = \mathbb{Q}\) be equipped with \(d : X \times X \to [0, \infty)\) defined by
\[
d(x, y) = \begin{cases} 
1 & \text{if } x > y, \\
(y - x)^3 & \text{if } x \leq y.
\end{cases}
\]
Then \((X, d)\) is a quasi b-metric space but not a quasi metric space.

3. Bicompletions of quasi b-metric spaces

Definition 3.1. Let \((X, d)\) be a quasi b-metric space. Then:
(i) A sequence \(\{x_n\}\) in \((X, d)\) is called a \(d^*\)-Cauchy if
\[
\lim_{n,m} d^*(x_n, x_m) = 0.
\]
(ii) A quasi b-metric space \((X, d)\) is said to be bicomplete if every \(d^*\)-Cauchy sequence \(d^*-\)converges to a point \(x \in X\).

Remark 3.1. A quasi b-metric space \((X, d)\) is said to be bicomplete if \((X, d^*)\) is a complete b-metric space.

Definition 3.2. Let \((X, d)\) be a quasi b-metric space and \(A\) be a subset of \(X\). Then \(A\) is \(d^*\)-dense whenever it is dense in \(X\) with respect to \(d^*\).
Theorem 3.1. Let \((X, d)\) be a quasi \(b\)-metric space. Then every quasi \(b\)-metric space has a bicompletion.

Proof: Let \((X, d)\) be a quasi \(b\)-metric space and \(C\) be the set of all \(d^*\)-Cauchy sequences in \((X, d)\). Define the relation \(\sim\) on \(C\) as follows: \(\{x_n\} \sim \{y_n\}\) if \(\lim_n d(x_n, y_n) = 0\). It is easy to verify that \(\sim\) is an equivalence relation in \(C\). Let \(\tilde{X}\) be the set of all equivalence classes for \(\sim\) and \(\tilde{X} = \{\{x_n\} : x_n \in C\}\). Define \(\tilde{d} : \tilde{X} \times \tilde{X} \rightarrow [0, \infty)\) by \(\tilde{d}([\{x_n\}], [\{y_n\}]) = \lim_n d(x_n, y_n)\).

We now show that \(\tilde{d}\) is well defined. Let \(\{x_n\}\) and \(\{y_n\}\) be two \(d^*\)-Cauchy sequences in \(C\) then \(\lim_{n,m} d^*(x_n, x_m) = 0\) and \(\lim_{n,m} d^*(y_n, y_m) = 0\). Since, 
\[
d(x_n, y_n) \leq s d(x_n, x_m) + s^2 d(x_m, y_m) + s^2 d(y_m, y_n)
\]
for \(s \geq 1\). Then 
\[
d(x_n, y_n) - s^2 d(x_m, y_m) \leq s d(x_n, x_m) + s^2 d(y_m, y_n).
\]
Note that 
\[
d(x_n, y_n) - d(x_m, y_m) \leq |d(x_n, y_n) - s^2 d(x_m, y_m)| \leq s d(x_n, x_m) + s^2 d(y_m, y_n).
\]
Similarly, we have 
\[
d(x_m, y_m) - d(x_n, y_n) \leq |d(x_m, y_m) - s^2 d(x_n, y_n)| \leq s d(x_n, x_m) + s^2 d(y_n, y_m).
\]
Hence, 
\[
|d(x_n, y_n) - d(x_m, y_m)| \leq s d(x_n, x_m) + s^2 d(y_m, y_n) \rightarrow 0.
\]
Since \(\lim |d(x_n, y_n) - d(x_m, y_m)| \rightarrow 0\), this implies that \(\{d(x_n, y_n)\}\) is a \(d^*\)-Cauchy sequences in \([0, \infty)\). Since \([0, \infty)\) is complete when equipped with the usual metric then, the limit of \(\{d(x_n, y_n)\}\) exists. Thus \(\tilde{d}\) is well defined.

We show that \(\tilde{d}\) is a quasi \(b\)-metric on \(\tilde{X}\). Let \(\{x_n\}\), \(\{y_n\}\) and \(\{z_n\}\) \(\in \tilde{X}\).

Suppose that \(\tilde{d}(\{x_n\}, \{y_n\}) = 0\). Then \(\{x_n\} = \{y_n\}\).

Conversely, suppose that \(\{x_n\} = \{y_n\}\). Then \(\tilde{d}(\{x_n\}, \{y_n\}) = \tilde{d}(\{x_n\}, \{x_n\}) = 0\).

Therefore \(\tilde{d}(\{x_n\}, \{y_n\}) = 0\).

We know that for all \(x_n, y_n, z_n \in X\), \(d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)\).

That is, 
\[
\tilde{d}(\{x_n\}, \{z_n\}) = \lim_n d(x_n, z_n) \leq \lim_n s[d(x_n, y_n) + d(y_n, z_n)].
\]

\[
= s[\tilde{d}(\{x_n\}, \{y_n\}) + \tilde{d}(\{y_n\}, \{z_n\})].
\]
Therefore \( \tilde{d} \) is a quasi b-metric on \( \tilde{X} \), with the same parameter \( s \).

Let \( f : X \to \tilde{X} \) be a map and be defined by \( f(x) = \{ \{x_n\} : x_n = x \) for all \( n \} \). Since \( \tilde{d}(f(x), f(y)) = \lim_n d(\{x_n\}, \{y_n\}) = \lim_n d(x_n, y_n) = d(x, y) \). Thus \( f \) is an isometry. \( f \) is injective since \( \{ \{x_n\} : x_n = x \} \neq \{ \{y_n\} : y_n = y \} \) if \( x \neq y \). Thus \( X \) can be regarded as a subspace of \( \tilde{X} \).

Let \( x = \{x_n\} \in X \). Given \( \epsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that for any \( n, m > N \), \( d(x_n, x_m) < \epsilon \). In particular, \( d(x_n, x_{N+1}) < \epsilon \) and \( d(x_{N+1}, x_n) < \epsilon \) for \( n > N \). Hence, \( \tilde{d}(x, x_{N+1}) \leq \epsilon \) and \( \tilde{d}(x_{N+1}, x) \leq \epsilon \). Thus \( X \) is \( d^* \)-dense in \( \tilde{X} \).

Finally, we show that \((\tilde{X}, \tilde{d})\) is bicomplete. Let \( x = \{x_n\} \) be a \( d^* \)-Cauchy sequence in \( X \) and \( \epsilon > 0 \). Then \( \tilde{d}(x_n, x_m) < \epsilon \) for \( m, n > N \). Fix \( n > N \) and let \( m \to \infty \) we get \( \tilde{d}(x_n, x) < \epsilon, \tilde{d}(x, x_n) < \epsilon \). Hence \( \tilde{d}(x_n, x) \to 0 \) and \( \tilde{d}(x, x_n) \to 0 \) as \( n \to \infty \).

**Remark 3.2.** Let \( f \) be a map from a quasi b-metric space \((X, d)\) into \((\tilde{X}, \tilde{d})\) such that for each \( x \in X, f(x) = [\{\tilde{x}\}] \) where \([\{\tilde{x}\}]\) is the equivalence class of constant sequences. Then for each \( x, y \in X, \tilde{d}(f(x), f(y)) = d(x, y) \), and hence, \( f \) is an isometry.

**Corollary 3.1.** Every b-metric space admits a completion.

We now show by means of an example that Theorem 3.1 holds.

**Example 3.1.** Let \((X, d)\) be defined as in Example 2.3. Observe that \((X, d)\) is not bicomplete. Its bicompletion is \( \tilde{X} = \mathbb{R} \) with \( \tilde{d} : \tilde{X} \times \tilde{X} \to [0, \infty) \) defined by

\[
\tilde{d}(x, y) = \begin{cases} 
1 & \text{if } x > y, \\
(y - x)^3 & \text{if } x \leq y.
\end{cases}
\]

Note that both \((X, d)\) and \((\tilde{X}, \tilde{d})\) have the same parameter.

4. **Quasi b-metric spaces and quasi uniformities**

Let \((X, d)\) be a quasi b-metric space, then \( d \) on \( X \) induces a topology \( \tau_d \) on \( X \) which has as a base the family of open balls \{\(B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}\}, where \( B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\} \) for all \( x \in X \) and \( \varepsilon > 0 \). In fact, we obtain a bitopological space \((X, \tau_d, \tau_{d-1})\).

A topological space \((X, \tau)\) is called quasi metrizable if there is a quasi metric \( d \) on \( X \) such that \( \tau = \tau_d \). In this case we say that \( d \) is compatible with \( \tau \), and that \( \tau \) is a quasi metrizable topology.

**Definition 4.1.** [6] A quasi uniformity on a set \( X \) is a filter \( \mathcal{U} \) on \( X \times X \) such that:

(i) For each \( U \in \mathcal{U}, \triangle \subset U \).
(ii) If \( U \in \mathcal{U} \), then \( V \circ V \subset U \) for some \( V \in \mathcal{U} \). The pair \((X, \mathcal{U})\) is called **quasi uniform space** and the members of \( \mathcal{U} \) are called **entourages**.

It worth noting that if \( \mathcal{U} \) is a quasi uniformity on \( X \), then \( \mathcal{U}^{-1} = \{U^{-1} | U \in \mathcal{U}\} \) is also a quasi uniformity on \( X \) and is called the **conjugate** of \( \mathcal{U} \) where \( U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\} \). A quasi uniformity \( \mathcal{U} \) is a uniformity provided \( \mathcal{U} = \mathcal{U}^{-1} \).

For a quasi uniformity \((X, \mathcal{U})\), let \( \mathcal{U}^* = \mathcal{U} \vee \mathcal{U}^{-1} \) then \( \mathcal{U}^* \) is a uniformity on \( X \), define \( U[x] = \{y \in X : (x, y) \in U\} \). Then each quasi uniformity \( \mathcal{U} \) on \( X \) induces a topology \( \tau_\mathcal{U} \) on \( X \), defined as follows: \( \tau_\mathcal{U} = \{A \subset X : \text{for each } x \in A \text{ there is } U \in \mathcal{U} \text{ such that } U[x] \subset A\} \).

**Definition 4.2.** [6] A quasi uniform space \((X, \mathcal{U})\) is called **bicomplete** if \((X, \mathcal{U}^*)\) is a complete uniform space. In this case we say that \( \mathcal{U} \) is a **bicomplete quasi uniformity**.

It was proved in [6] that every quasi uniform space \((X, \mathcal{U})\) admits a unique bicompletion \((\tilde{X}, \tilde{\mathcal{U}})\) up to quasi isomorphism.

**Definition 4.3.** A quasi b-metric space \((X, d)\) is **quasi uniformizable** if there exists a quasi uniform space \((X, \mathcal{U})\) such that \( \tau_d = \tau_\mathcal{U} \).

**Theorem 4.1.** Every quasi b-metric space is quasi metrizable.

**Proof:** For each \( n \in \mathbb{N} \), define

\[
U_n = \{(x, y) \in X \times X : d(x, y) < \frac{1}{n}\}.
\]

We shall prove that \( \{U_n\} \) is a base for the quasi uniformity on \( X \) whose induced topology coincides with \( \tau_d \). Note that for each \( n \in \mathbb{N} \)

\[
\{(x, x) : x \in X\} \subseteq U_n, U_{n+1} \subset U_n.
\]

For each \( n \in \mathbb{N} \), choose \( m \in \mathbb{N} \) such that \( m > (s + 2s^2)n \). Next we show that \( U_m \circ U_m \circ U_m \subseteq U_n \). So let \((x, z) \in U_m \circ U_m \circ U_m \subseteq U_n \). We find \( y, w \in X \) such that \((x, y) \in U_m, (y, w) \in U_m \) and \((w, z) \in U_m \). Then \( d(x, y) < \frac{1}{m}, d(y, w) < \frac{1}{m} \) and \( d(w, z) < \frac{1}{m} \). Hence,

\[
d(x, z) \leq sd(x, y) + s^2d(y, w) + s^2d(w, z) < \frac{2s^2 + s}{m} < \frac{1}{n}.
\]

Therefore \((x, z) \in U_n \). Thus \( \{U_n : n \in \mathbb{N}\} \) is a base for a quasi uniformity \( \mathcal{U} \) on \( X \). Therefore a quasi b-metric space is quasi uniformizable,
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we denote by $U_d$ the quasi uniformity induced by $d$. Since for each $x \in X$ and each $n \in \mathbb{N}$,

$$U_n(x) = \{ y \in X : d(x,y) < \frac{1}{n} \} = B(x, \frac{1}{n}).$$

Clearly, $U_d$ has a countable base, hence $(X, \tau_d)$ is quasi metrizable [6] and [8]. In fact, $	au_d = \tau_{U_d}$. □

**Corollary 4.1.** Every b-metric space is metrizable.

Every quasi b-metric space $(X,d)$ induces a quasi uniformity $(X,U_d)$ as shown in Theorem 4.1. In addition, every quasi uniformity $(X,U_d)$ has a bicompletion $(\tilde{X},\tilde{U}_d)$ [6].

**Remark 4.1.** Given a quasi b-metric space $(X,d)$, let $(\tilde{X},\tilde{d})$ be the bicompletion of $(X,d)$ as in Theorem 3.1 and $(\tilde{X},\tilde{U}_d)$ be the quasi uniformity of $(\tilde{X},\tilde{d})$ as in Theorem 4.1. Further, let $(\tilde{X},\tilde{U}_d)$ be the bicompletion of $(X,U_d)$, see, for instance, [6].

Then:

**Theorem 4.2.** Let $(X,d)$ be a quasi b-metric space. Then, $(\tilde{X},\tilde{U}_d)$ is quasi isomorphic to $(\tilde{X},\tilde{U}_d)$. In particular, $\tau_{U_d} = \tau_{\tilde{U}_d}$.

5. EXTENSIONS OF MAPS ON QUASI b-METRIC SPACES

**Definition 5.1.** Let $f : (X,d) \to (Y,\rho)$ be a map between two quasi b-metric spaces. Then:

(i) A mapping $f : (X,d) \to (Y,\rho)$ is $(d,\rho)$-continuous if $\{x_n\}$ d-converges to $x$ implies that $f(x_n)$ $\rho$-converges to $f(x)$.

(ii) A mapping $f : (X,d) \to (Y,\rho)$ is $(d^{-1},\rho^{-1})$-continuous if $\{x_n\}$ $d^{-1}$-converges to $x$ implies that $f(x_n)$ $\rho^{-1}$-converges to $f(x)$.

(iii) A mapping $f : (X,d) \to (Y,\rho)$ is $(d^*,\rho^*)$-continuous if $\{x_n\}$ $d^*$-converges to $x$ implies that $f(x_n)$ $\rho^*$-converges to $f(x)$.

**Definition 5.2.** Let $(X,d)$ and $(Y,\rho)$ be two quasi b-metric spaces. A mapping $f : (X,d) \to (Y,\rho)$ is said to be quasi uniformly continuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that $x,y \in X$ and $d(x,y) < \delta$ implies that $\rho(f(x), f(y)) < \epsilon$.

**Lemma 5.1.** Let $(X,d)$ and $(Y,\rho)$ be two quasi b-metric spaces. A mapping $f : (X,d) \to (Y,\rho)$ is quasi uniformly continuous if and only if $f : (X,U_d) \to (Y,U_\rho)$ is quasi uniformly continuous between the quasi uniform spaces.
**Proposition 5.1.** Let $a, d \in \mathbb{R}$ and $k \geq 1$. If $a \leq kd$ and $d \leq ka$ hold, then $a = d$.

**Theorem 5.1.** Let $(X, d)$ be a quasi $b$-metric space and $(Y, \rho)$ be a bicomplete quasi $b$-metric space. If $f : (A, d) \to (Y, \rho)$ is quasi uniformly continuous, from a $d^*$-dense subspace $(A, d)$ of $(X, d)$ to $(Y, \rho)$, then there exists a unique extension $f^* : (X, d) \to (Y, \rho)$ such that $f^|_A = f$ and $f^*$ is quasi uniformly continuous. In particular, $f^*$ is an isometry whenever $f$ is so.

**Proof:** By Lemma 5.1, $f$ is quasi uniformly continuous from the quasi uniform space $(X, \mathcal{U}_d|_{A \times A})$ to the quasi uniform space $(Y, \mathcal{U}_\rho)$. Since $A$ is $d^*$-dense in $X$, it follows that $f$ has a unique quasi uniformly continuous extension $f^* : (X, \mathcal{U}_d) \to (Y, \mathcal{U}_\rho)$, see Theorem 3.29 in [6]. Thus $f^*$ is quasi uniformly continuous from the quasi $b$-metric space $(X, d)$ to the bicomplete quasi $b$-metric space $(Y, \rho)$ by Lemma 5.1.

Let $f$ be an isometry from $(A, d)$ to $(Y, \rho)$ and $x, y \in X$. Suppose that $f$ is $(d^* - \rho^*)$-continuous. Then there exists $\{x_n\} \cup \{y_n\}$ in $A$ such that $\lim_n d^*(x_n, x) = 0$ and $\lim_n d^*(y_n, y) = 0$. By $(d^* - \rho^*)$-continuity of $f^*$ it follows that $\lim_n \rho^*(f^*(x_n), f^*(x)) = 0$ and $\lim_n \rho^*(f^*(y_n), f^*(y)) = 0$. Then there exists $N \in \mathbb{N}$ such that $d(x, x_n) < \epsilon, d(y, y_n) < \epsilon, \rho(f^*(x), f^*(x_n)) < \delta$ for all $n \geq N$. Thus we have,

\[
d(x, y) \leq sd(x, x_n) + s^2d(x_n, y_n) + s^2d(y_n, y) \\
\leq s\epsilon + s^2\rho(f^*(x_n), f^*(y_n)) + s^2\epsilon \\
\leq s^2\rho(f^*(x_n), f^*(y_n))
\]

and

\[
\rho(f^*(x_n), f^*(y_n)) \leq \alpha\rho(f^*(x_n), f^*(x)) \\
+ \alpha^2\rho(f^*(x), f^*(y)) + \alpha^2\rho(f^*(y), f^*(y_n)) \\
\leq \alpha\delta + \alpha^2\rho(f^*(x), f^*(y)) + \alpha^2\delta \\
\leq \alpha^2\rho(f^*(x), f^*(y))
\]

for all $n \in N$ and $\alpha \geq 1$. Therefore

\[
d(x, y) \leq \alpha s^2\rho(f^*(x), f^*(y)) \tag{1}
\]

Similarly,

\[
\rho(f^*(x), f^*(y)) \leq \alpha\rho(f^*(x), f^*(x_n)) + \alpha^2\rho(f^*(x_n), f^*(y_n)) \\
+ \alpha^2\rho(f^*(y_n), f^*(y)) \\
\leq \alpha\delta + \alpha^2d(x_n, y_n) + \alpha^2\delta \\
\leq \alpha^2d(x_n, y_n),
\]
and
\[
\begin{align*}
  d(x_n, y_n) & \leq s d(x_n, x) + s^2 d(x, y) + s^2 d(y, y_n) \\
  & \leq s \varepsilon + s^2 d(x, y) + s^2 \varepsilon \\
  & \leq \alpha^2 s^2 d(x, y)
\end{align*}
\]
for all \( n \in \mathbb{N} \) and \( s \geq 1 \). Therefore
\[
\rho(f^*(x), f^*(y)) \leq \alpha^2 s^2 d(x, y) \tag{2}
\]
By (1), (2) and Proposition 5.1, we have \( \rho(f^*(x), f^*(y)) = d(x, y) \).
Hence \( f^* \) is an isometry from \((X, d)\) to \((Y, \rho)\).

We see from Theorem 3.1 and Theorem 5.1 that: Every quasi \( b \)-metric space admits a bicompletion which is unique up to isometry. We now present the following results as Corollaries. Note that Corollary 5.1 and Corollary 5.3 are well known.

**Corollary 5.1.** Let \((X, d)\) be a quasi metric space and \((Y, \rho)\) be a bicomplete quasi metric space. If \( f : (A, d) \rightarrow (Y, \rho) \) is quasi uniformly continuous, from a \( d^* \)-dense subspace \((A, d)\) of \((X, d)\) to \((Y, \rho)\), then there exists a unique extension \( f^* : (X, d) \rightarrow (Y, \rho) \) such that \( f^*|_A = f \) and \( f^* \) is quasi uniformly continuous. In particular, \( f^* \) is an isometry whenever \( f \) is so.

**Corollary 5.2.** Let \((X, d)\) be a \( b \)-metric space and \((Y, \rho)\) be a complete \( b \)-metric space. If \( f : (A, d) \rightarrow (Y, \rho) \) is uniformly continuous, from a dense subspace \((A, d)\) of \((X, d)\) to \((Y, \rho)\), then there exists a unique extension \( f^* : (X, d) \rightarrow (Y, \rho) \) such that \( f^*|_A = f \) and \( f^* \) is uniformly continuous. In particular, \( f^* \) is an isometry whenever \( f \) is so.

**Corollary 5.3.** Let \((X, d)\) be a metric space and \((Y, \rho)\) be a complete metric space. If \( f : (A, d) \rightarrow (Y, \rho) \) is uniformly continuous, from a dense subspace \((A, d)\) of \((X, d)\) to \((Y, \rho)\), then there exists a unique extension \( f^* : (X, d) \rightarrow (Y, \rho) \) such that \( f^*|_A = f \) and \( f^* \) is uniformly continuous. In particular, \( f^* \) is an isometry whenever \( f \) is so.

We conclude the paper with the following example:

**Example 5.1.** Let \((X, d)\) be defined as in Examples 2.3 and 3.1. Consider \( Y = \mathbb{R} \) be equipped with the usual metric. Define a mapping \( f : (A, d) \rightarrow (Y, \rho) \) by \( f(x) = x + 1 \) for all \( x \in A \). Then \( f \) is quasi uniformly continuous. Let \( F : (X, d) \rightarrow (Y, \rho) \) be defined by \( F(x) = x + 1 \) for all \( x \in X \), where \( X = \mathbb{R} \). Then \( F \) extend \( f \) uniquely to \( X \) and it is quasi uniformly continuous. In particular, Theorem 5.1 holds.
References


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