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INVARIANT FINSLER (α, β)-METRIC $F^2 = 2\alpha\beta$ ON HOMOGENEOUS FINSLER SPACES

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Abstract. In this paper, we consider invariant Finsler (α, β)-metric $F^2 = 2\alpha\beta$ which are induced by Invariant Riemannian metric and invariant vector fields on homogeneous spaces, and we give the flag curvature formula of them. Then some conclusions in the case of naturally reductive homogeneous spaces, and some examples of geodesically complete (α, β)-metric spaces.

1. **Introduction**

Finsler geometry is an interesting field in differential geometry which has found many applications in physics and biology [8]. One of important quantities which can be used for characterizing Finsler spaces is flag curvature. The computation of flag curvature, which is a generalization of the concept of sectional curvature in Riemannian geometry, is very difficult. Therefore finding an explicit formula for computing it can be useful for characterizing Finsler spaces. Also it can help us to finding new examples of spaces with some curvature properties. Working on a general Finsler space for finding an explicit formula for the flag curvature is very computational because of computation in local coordinates. A family of spaces which has many applications in physics is homogeneous spaces (in particular, Lie groups) equipped with invariant metrics. The study of homogeneous spaces (Lie groups) with invariant Riemannian metrics has been a very hot field in last decades (for example see ([1], [11], [16], [23], [24]). During recent years, some of these results extended to Finsler spaces in some special cases ([10], [12], [25],[26]).

Key words and phrases. Finsler metric, Invariant Finsler metric, Flag curvature, Homogeneous space, Lie group, and Riemannian metric.

Invariant Finsler structures on homogeneous manifolds is one of the interesting subjects in Finsler geometry which has been studied by some Finsler geometers, during recent years(for example see [2], [3], [4], [5], [9]). An important family of Finsler metrics is the family of (α, β)-metrics. These metrics are introduced by M. Matsumoto in 1972 (see[7]). The study of Finsler space with (α, β)-metric was studied by many authors([15], [18], [19]) and it is quite old concept, but it is a very important aspects of Finsler geometry and its applications to physics. They seek for some non-Riemannian models for space time. For example, by using (α, β)-metrics, G. S. Asanov introduced Finsleroid Finsler spaces and formulated pseudo-Finsleroid gravitational field equations (see [6] and [7]).

Invariant metrics are of these invariant structures. S. Kobayoshi and K. Nomizu studied many interesting properties of invariant Riemannian metrics and the existence and properties of invariant affine connections on reductive homogeneous spaces (see [1] and [11]).

Also some curvature properties of invariant Riemannian metrics on Lie groups has studied by J. Milnor [14]. So it is important to study invariant Finsler metrics which are a generalization of invariant Riemannian metrics.

Deng and Hou studied invariant Finsler metrics on reductive homogeneous manifolds and gave an algebraic description of these metrics and obtained a necessary and sufficient condition for a homogeneous manifold to have invariant Finsler metrics (see [10],[12]). Also they studied invariant Randers metrics on homogeneous Riemannian manifolds and used this structure to construct Berwald space which is neither Riemannian nor locally Minkowskian. They gave a formula for the flag curvature of invariant Randers metrics on homogeneous manifolds.

The purpose of the present paper is considered invariant (α, β) metric $F^2 = 2\alpha\beta$ which are induced by invariant Riemannian metrics and invariant vector fields on homogeneous spaces, and then we find the explicit formula for the flag curvature of invariant Finsler (α, β) metric $F^2 = 2\alpha\beta$ on naturally reductive homogeneous manifolds $(G/H, g)$, where the metric induced by the invariant Riemannian metric g and an invariant vector field $\tilde{\mathrm{X}}$ which is parallel with g. Also we study the special cases of naturally reductive spaces and bi-invariant metrics. We end the article by giving some examples of geodesically complete (α, β) metric spaces.

2. Preliminaries

Definition 2.1. *([1], [11]). A homogeneous space G/H of a connected Lie group G is called reductive if the following conditions are satisfied:*

(1) In the Lie algebra g of G there exists a subspace m such that $g = m + \eta$ (direct sum of vector subspaces).

(2) $ad(h)m \subset m$, for all $h \in H$,
where η is the subalgebra of g corresponding to the identity component H_0 of H and $ad(h)$ denotes the adjoint representation of H in g.

Note that condition (2) implies,

(3) $[\eta, m] \subset m$,

and conversely, if H is connected, then (3) implies (2).

For example, G/H is reductive in either of the following cases:

- *H is compact,*
- *H is connected and semi-simple,*
- *H is a discrete subgroup of $G; \eta = 0$ and $m = g$.*

Let G/H be a reductive homogeneous manifold with invariant Riemannian metric g which the subspace m is the orthogonal complement of η with respect to the inner product on g. Also let

$$V = \{X \in m / ad(h)X = X, < X, X >< 1, \forall h \in H\},$$

where $<,>$ is the inner product induced by g. Then for any $X \in V$ there exist an invariant (α,β)- metrics $F^2 = 2\alpha\beta$ on G/H by the following formula :

$$F^2(xH, Y) \quad = \quad 2\sqrt{g(xH)(Y,Y)}g(xH)(\widetilde{X}, Y), \tag{2.1}$$

where $g(X, X) < 1$, $Y \in T_{xH}G/H$ and $\widetilde{X}$ is the corresponding invariant vector field on G/H to X.

Let M be a smooth n-dimensional manifold and TM be its tangent bundle. A Finsler metric on M is a non-negative function $F : TM \to R$ which has the following properties:
(a) F is smooth on the slit tangent bundle $TM^0 := TM \backslash \{0\}$,
(b) $F(x, \lambda y) = \lambda F(x, y)$ for any $x- \in M, y- \in T_x M$ and $\lambda > 0$,
(c) the $n - \times n$ Hessian matrix

$$[g_{ij}(x,y)] = [\tfrac{1}{2}\tfrac{\partial^2 F^2}{\partial y^i \partial y^j}],$$

is positive definite at every point $(x, y) \in TM^0$.

For a smooth manifold M suppose that g and b are a Riemannian metric and a 1-form respectively as follows:

$$g = g_{ij} dx^i \otimes dx^j, \tag{2.2}$$

$$b = b_i dx^i. \tag{2.3}$$

An important family of Finsler metrics is the family of (α, β)-metrics which is introduced by M. Matsumoto (see [8], [16]). (α, β)-metric $F^2 = 2\alpha\beta$ can written as the following form :

$$F^2 = 2\alpha(x, y)\beta(x, y), \tag{2.4}$$

where

$$\begin{aligned} \alpha(x, y) &= \sqrt{g_{ij}(x) y^i y^j}, \quad and \\ \beta(x, y) &= b_i(x) y^i. \end{aligned}$$

In a natural way, the Riemannian metric g induces an inner product on any cotangent space $T_x^* M$ such that $< dx^i(x), dx^j(x) >= g^{ij}(x)$. The induced inner product on $T_x^* M$ induces a linear isomorphism between $T_x^* M$ and $T_x M$ (for more details see [10]). Then the 1-form b corresponds to a vector field $\tilde{\mathrm{X}}$ on M such that

$$g(y, \widetilde{X}(x)) = \beta(x, y). \tag{2.5}$$

Therefore we can write the (α, β)-metric $F^2 = 2\alpha\beta$ as the following :

$$F^2(x, y) = 2\alpha(x, y) g(\widetilde{X}(x), y). \tag{2.6}$$

The Flag curvature, which is a generalization of the concept of sectional curvature in Riemannian geometry, is one of the fundamental quantities which associates with a Finsler space. Flag curvature is computed by the following formula:

$$K(P, Y) = \frac{g_Y(R(U, Y)Y, U)}{g_Y(Y, Y)g_y(U, U) - g_Y^2(Y, U)}, \tag{2.7}$$

where

$$\begin{aligned} g_Y(U, V) &= (\frac{1}{2})(\frac{\partial^2}{\partial s \partial t})(F^2(Y + sU + tV))|_{s=t=0}, \\ P &= span\{U, Y\}, \\ R(U, Y)Y &= \nabla_U \nabla_Y Y - \nabla_Y \nabla_U Y - \nabla_{[U,Y]}, \end{aligned}$$

and ∇ is the chern connection induced by F (see[15],[17]).

3. Flag curvature of invariant Finsler (α, β)-metric $F^2 = 2\alpha\beta$

Curvature properties of (α, β)-metrics have been studied by various authors ([13], [21], [22]). There are several interesting curvatures in Finsler geometry, among them the flag curvature is the most important one, which is the natural generalization of sectional curvature in Riemannian geometry.

In this section, we give an explicit formula for the flag curvature of invariant Finsler (α, β)-metrics $F^2 = 2\alpha\beta$, where α is induced by an invariant Riemannian metric g on the homogeneous space and the Chern connection of F coincides to the Levi-Civita connection of g.

Let G be a compact Lie group, H a closed subgroup, g and η are the Lie algebras of G and H respectively. Suppose that g_0 is a bi-invariant Riemannian metric on G, then the tangent space of the homogeneous space G/H is given by the orthogonal compliment m of η in g with respect to g_0. Each invariant metric g on G/H is determined by its restriction to m. The arising Ad_H-invariant inner product from g on m can extend to an Ad_H -invariant inner product on g by taking g_0 for the components in η. In this way the invariant metric g on G/H determines a unique left invariant metric on G that we also denote by g. The values of g_0 and g at the identity are inner products on g and we determine them by $\langle ., . \rangle_0$

and $\langle .,. \rangle$ respectively. The inner product $\langle .,. \rangle$ determines a positive definite endomorphism ϕ of g such that $\langle X, Y\rangle = \langle \phi X, Y\rangle_0$ for all $X, Y \in g$.

Definition 3.2. *([1]). A homogeneous manifold $M = G/H$ with a G-invariant indefinite Riemannian metric g is said to be naturally reductive if it admits an $ad(H)$-invariant decomposition $g = \eta + m$ satisfying the condition*

$$B(X, [Z, Y]_m) + B([Z, X]_m, Y) = 0 \text{ for } X, Y, Z \in m,$$

where B is the bilinear form on m induced by g and $[,]_m$ is the projection to m with respect to the decomposition $g = \eta + m$.

We use the following formula which has shown by T.puttmann in [24].

Lemma 3.1. *The curvature tensor of the invariant metric $< .,. >$ on the compact homogeneous space G/H is given by*

$$\begin{aligned} < R(X,Y)Z, W > \; = \; & -\{\frac{1}{2}(< B_-(X,Y), [Z,W] >_0 + < [X,Y], B_-(Z,W) >_0) + \\ & +\frac{1}{4}(< [X,W], [Y,Z]_m > - < [X,Z], [Y,W]_m > \\ & -2 < [X,Y], [Z,W]_m >) + (< B_+(X,W), \phi^{-1} B_+(Y,Z) >_0 \\ & - < B_+(X,Z), \phi^{-1} B_+(Y,W) >_0)\}, \end{aligned}$$

where B_+ and B_- are defined by

$$B_+(X,Y) = 1/2([X, \phi Y] + [Y, \phi X]),$$

$$B_-(X,Y) = 1/2([\phi X, Y] + [X, \phi Y]),$$

and $[.,.]_m$ is the projection of $[.,.]$ to m.

Theorem 3.1. *Let G/H be a homogeneous manifold with invariant Riemannian metric g and F be an invariant(α, β)-metrics of type $2\alpha\beta$ defined by the $ad(H)$-invariant vector X,*

$$F^2(xH, Y) = 2\sqrt{g(xH)(Y,Y)}\, g(xH)(\widetilde{X}, Y), \tag{3.1}$$

where $g(X,X) < 1$, $Y \in T_{xH}G/H$ and $\widetilde{X}$ is the corresponding invariant vector field on G/H to X. Also suppose that the vector field $\widetilde{X}$ is parallel with respect to g and $(G/H, g)$ is naturally reductive.

Then the flag curvature of the flag (P, Y) in $T_H(G/H)$ is given by

$$K(P,Y) = \frac{A}{B-C}, \tag{3.2}$$

where

$$\begin{aligned}
A &= \frac{1}{\sqrt{g(Y,Y)}}(g(\alpha,U)g(Y,X)+g(\alpha,Y)g(U,X)-\frac{g(\alpha,Y)g(Y,X)g(U,Y)}{g(Y,Y)}+g(U,Y)g(\alpha,X)),\\
B &= 2g(Y,X)(g(U,U)g(Y,X)+2g(U,Y)g(U,X)-\frac{g^2(U,Y)g(Y,X)}{g(Y,Y)}),
\end{aligned}$$

and

$$C = \frac{1}{g(Y,Y)}(g^2(Y,Y)g^2(U,X)-2g(Y,Y)g(U,X)g(U,Y)g(Y,X)+g^2(U,Y)g^2(Y,X)),$$

where U is any vector in P such that span $\{Y,U\} = P$ and α in A is defined by

$$\alpha = \frac{1}{4}[Y,[U,Y]_m]_m + [Y,[U,Y]_\eta]. \tag{3.3}$$

Note that $[,]_m$ and $[,]_\eta$ are the projections of $[,]$ to m and η, respectively, and we assume that U is orthogonal to Y with respect to g.

Proof. Let $\widetilde{X}$ is parallel with respect to g, therefore F is of Berwald type. Also the Chern connection of F and the Riemannian connection of g coincide (See [15] page 305), therefore we have

$$R^F(U,V)W = R^g(U,V)W, \tag{3.4}$$

where R^F and R^g are the curvature tensors of F and g respectively.(Now let $R := R^F = R^g$). Notice g is naturally reductive, thus by using proposition 3.4 proved by S. Kobayashi and K. Nomizu of [1](page 202)we have

$$\begin{aligned}
R((U,V)W)_0 &= \frac{1}{4}[U,[V,W]_m]_m - \frac{1}{4}[V,[U,W]_m]_m\\
&\quad -\frac{1}{2}[[U,V]_m,W]_m - [[U,V]_\eta,W] \quad for \quad U,V,W \in m.
\end{aligned}$$

Then

$$
\begin{aligned}
R((U,Y)Y)_0 &= \frac{1}{4}[U,[Y,Y]_m]_m - \frac{1}{4}[Y,[U,Y]_m]_m \\
&\quad -\frac{1}{2}[[U,Y]_m, Y]_m - [[U,Y]_\eta, Y], \\
&= \frac{1}{4}[Y,[U,Y]_m]_m + [Y,[U,Y]_\eta].
\end{aligned}
$$

On other hand,

$$
K(P,Y) = \frac{g_Y(R(U,Y)Y,U)}{g_(Y,Y)g_Y(U,U) - g_Y^2(Y,U)}, \tag{3.5}
$$

where

$$
\begin{aligned}
g_Y(U,V) &= \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\{F^2(Y+sU+tV)\}|_{s=t=0.} \\
&= \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\{\sqrt{g(Y+sU+tV, Y+sU+tV)}g(X, Y+sU+tV)\}_{s=t=0}.
\end{aligned}
$$

By a direct computation we get

$$
\begin{aligned}
g_Y(U,V) &= \frac{1}{\sqrt{g(Y,Y)}}\{g(U,V)g(Y,X) + g(U,Y)g(V,X) - \frac{g(U,Y)g(Y,X)g(V,Y)}{g(Y,Y)} + \\
&\quad + g(V,Y)g(U,X)\}, && (3.6) \\
g_Y(Y,Y) &= 2\sqrt{g(Y,Y)}g(Y,X), && (3.7) \\
g_Y(Y,Y) &= 2\sqrt{g(Y,Y)}g(Y,X), && (3.8) \\
g_Y(U,U) &= \frac{1}{\sqrt{g(Y,Y)}}\{g(U,U)g(Y,X) + 2g(U,Y)g(U,X) \\
&\quad - \frac{g(U,Y)g(Y,X)g(U,Y)}{g(Y,Y)}\}, && (3.9) \\
g_Y(Y,U) &= \frac{1}{\sqrt{g(Y,Y)}}\{g(Y,Y)g(U,X) - g(U,Y)g(Y,X)\}, && (3.10)
\end{aligned}
$$

and

$$
g_Y(R(Y,U)Y,U) = g_Y(\alpha, U), \tag{3.11}
$$

where

$$
\alpha = \tfrac{1}{4}[Y,[U,Y]_m]_m + [Y,[U,Y]_\eta].
$$

Combing the above formulas with the equation (3.5), then completes the proof.

Corollary 3.1. *In a special case we can assume a Lie group G as a reductive homogeneous space with $H = e, \eta = 0$ and $m = g$, then our formula for the flag curvature is simpler because in this case we have* $\alpha = \frac{1}{4}[Y,[U,Y]]$.

Theorem 3.2. *Let G, H, g, η, g, g_0 and ϕ be as above. Assume that $\widetilde{X}$ is an invariant vector field on G/H which is parallel with respect to g and $\sqrt{g(\widetilde{X},\widetilde{X})} < 1$ and $X := \widetilde{X}_H$. Suppose that $F^2 = 2\alpha\beta$ is the (α,β)- metric induced by g and $\widetilde{X}$. Assume that (P,Y) is a flag in $T_H(G/H)$ such that $\{Y,U\}$ is an orthonormal basis of P with respect to $<.,.>$. Then the flag curvature of the flag (P,Y) in $T_H(G/H)$ is given by*

$$K(P,Y) = \frac{\gamma < Y,X > +\alpha < U,Y >}{2 < Y,X >^2 +\alpha^2}, \tag{3.12}$$

where

$$\begin{aligned}
\alpha &= < R(U,Y)Y,X >,\\
&= -\frac{1}{4}(< [\phi U,Y] + [U,\phi Y],[Y,X] >_0 + < [U,Y],[\phi Y,X] + [Y,\phi X] >_0)\\
&\quad -\frac{3}{4} < [Y,U],[Y,X]_m > -\frac{1}{2} < [U,\phi X] + [X,\phi U],\phi^{-1}([Y,\phi Y]) >_0\\
&\quad +\frac{1}{4} < [U,\phi Y] + [Y,\phi U],\phi^{-1}([Y,\phi X] + [X,\phi Y]) >_0,
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
\gamma &= < R(U,Y)Y,U >,\\
&= -\frac{1}{2} < [\phi U,Y] + [U,\phi Y],[Y,U] >_0\\
&\quad -\frac{3}{4} < [Y,U],[Y,U]_m > - < [U,\phi U],\phi^{-1}([Y,\phi Y]) >_0 +\\
&\quad +\frac{1}{4} < [U,\phi Y] + [Y,\phi U],\phi^{-1}([Y,\phi U] + [U,\phi Y]) >_0 .
\end{aligned} \tag{3.14}$$

Proof. Let $\widetilde{X}$ is parallel with respect to g, and therefore the Chern connection of F coincides on the Levi-Civita connection of g (see[8]). So the Finsler metric F and the Riemannian metric g have the same curvature tensor.

By using formula 3.6 and the fact that $\{Y, U\}$ is an orthonormal basis for P with respect to g, we have

$$(3.15) g_Y(R(U,Y)Y,U) =< R(U,Y)Y,U >< Y,X > + < R(U,Y)Y,X >< U,Y >,$$

and

$$(3.16) \qquad g_Y(Y,Y)g_Y(U,U) - g_Y^2(U,Y) = 2 < Y,X >^2 + < R(U,Y)Y,X >^2 .$$

We can obtain the relation (3.13) and (3.14) by using Puttmann's formula ([24]). Substituting the relations (3.15)and (3.16)in the equation (3.5), we complete the proof.

The above formula for the flag curvature reduces to a simpler equation, stated as following:

Theorem 3.3. *Let G/H be a naturally reductive homogeneous space. Then the flag curvature of the flag (P, Y) in $T_H(G/H)$ is given*

$$(3.17) \qquad K(P,Y) \quad = \quad \frac{< R(U,Y)Y,U >< Y,X > + < R(U,Y)Y,X >< U,Y >}{2 < Y,X >^2 + < R(U,Y)Y,X >^2},$$

where

$$R(U,Y)Y = \frac{1}{4}[Y,[U,Y]_m]_m + [Y,[U,Y]_\eta]$$

If the invariant (α, β)-Metric $F^2 = 2\alpha\beta$ is defined by a bi-invariant Riemannian metric on a Lie group then there is a simpler formula for the flag curvature, we give this formula in the following theorem.

Theorem 3.4. *Let G be a Lie group, g be a bi-invariant Riemannian metric on G, and $\widetilde{X}$ be a left invariant vector field on G. Suppose that $F^2 = 2\alpha\beta$ is the (α, β)-Metric defined by g and $\widetilde{X}$ on G such that the Chern connection of F coincides on the Levi-Civita connection of g. Then for the flag curvature of the flag $P = span\{Y, U\}$, where $\{Y, U\}$ is an orthonormal basis for P with respect to g, we have:*

$$(3.18) \qquad K(P,Y) = \frac{< [U,Y],Y,U >< Y,X > + < [U,Y],Y,X >< U,Y >}{< [U,Y],Y,X >^2 -8 < Y,X >^2}.$$

Proof. Let g is bi-invariant. Therefore we have $R(U,Y)Y = -\frac{1}{4}[[U,Y],Y]$. Now by using Theorem (3.2) the proof is completed.

4. **Examples of geodesically complete (α,β)- metric**

We begin this section with definition, and we give some examples of geodesically complete (α,β)-metric $F^2 = 2\alpha\beta$.

Definition 4.3. [28]. *The Riemannian manifold (M, g) is said to be homogeneous if the group of isometries of M acts transitively on M.*

Theorem 4.5. *Let (M, g) is a homogeneous Riemannian manifold. Suppose that F is (α,β)-metric $F^2 = 2\alpha\beta$ of Berwald type defined by g and a 1-form b. Then (M, F) is geodesically complete.*

Proof. F is of Berwald type; therefore the Chern connection of F and the Levi-Civita connection of g coincide and hence their geodesics coincide. On the other hand (M, g) is a homogeneous Riemannian manifold; hence (M, g) is geodesically complete (see [28]page 185). Therefore (M, F) is geodesically complete.

Corollary 4.2. *Let M is connected; then by using the Hofp-Rinow theorem for Finsler manifolds, (M, F) is complete.*

Corollary 4.3. *Let G be a Lie group and g be a left invariant Riemannian metric on G. And suppose that X is a parallel vector field with respect to the Levi-Civita connection of g such that$\sqrt{g(X,X)} < 1$. Then the (α,β)-metric $F^2 = 2\alpha\beta$ defined by g, X and the relation* (2.6) *is geodesically complete.*

Now we consider an abelian Lie group equipped with a left invariant Riemannian metric. We know that this space is flat. In this case we have the following theorem,

Theorem 4.6. *Let G be an abelian Lie group equipped with a left invariant Riemannian metric g, and let g be the Lie algebra of G. Suppose that $X \in g$ is a left invariant vector field with $\sqrt{g(X,X)} < 1$. Then the (α,β)-metric $F^2 = 2\alpha\beta$ defined by the formula* (2.6) *is a flat geodesically complete locally Minkowskian metric on G.*

Proof. Let $U, V, W \in g$, now by using the Koszul-s formula and the fact that G is abelian we have $\nabla_Y X = 0$, for any $Y \in g$. Hence X is parallel with respect to ∇ and F is of

Berwald type. Also the curvature tensor $R=0$ of g coincides on the curvature tensor of F and therefore the flag curvature of F is zero. F is a flat Berwald metric therefore by Proposition 10.5.1 (p. 275) of [15], F is locally Minkowskian.

Example 1. *($E(2)$ group of rigid motions of Euclidean 2-space). We consider the Lie group $E(2)$ as follows:*

$$E(2) \quad = \quad \left[\begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \setminus a, b, \theta \in R \right]. \tag{4.1}$$

The Lie algebra of $E(2)$ is of the form

$$e(2) = span\{x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\}, \tag{4.2}$$

where

$$[x,y]=0, \quad [y,z]=x, \quad [z,x]=y. \tag{4.3}$$

Let g be the left invariant Riemannian metric induced by the following inner product,

$$< x,x >=< y,y >=< z,z >= \lambda^2, \quad < x,y >=< y,z >=< z,x >= 0, \quad \lambda > 0. \tag{4.4}$$

In [27] they showed that the left invariant vector fields which are parallel with respect to the Levi-Civita connection of this space are of the form $U = uz$. Also they proved that $R = 0$. Let $\sqrt{< U,U >} < 1$, in other words let $0 < |u| < 1/(2\lambda)$. Hence, the left invariant (α, β)-metric $F^2 = 2\alpha\beta$ defined by g and U with formula (2.6) is of Berwald type. Also since F is of Berwald type therefore the curvature tensor of F and g coincide and F is of zero constant flag curvature. Hence F is locally Minkowskian.

Example 2. *Another example of flat geodesically complete locally Minkowskian (α, β)-metric $F^2 = 2\alpha\beta$ is described as follows. Let $g = span\{x, y, z\}$ be a Lie algebra such that*

$$[x,y] = \alpha y + \alpha z, \quad [y,z] = 2\alpha x, \quad [z,x] = \alpha y + \alpha z, \quad \alpha \in R. \tag{4.5}$$

Also consider the inner product described by (4.4) on g. Let G is a Lie group with Lie algebra g, and g is the left invariant Riemannian metric induced by the above inner product $< ., . >$ *on* G.
A direct computation shows that $R = 0$, *therefore* (G, g) *is a flat Riemannian manifold. Also in* [28] *they proved that vector fields which are parallel with respect to the Levi-Civita connection of* (G, g) *are of the form* $U = uy - uz$. *Now suppose that* $\sqrt{2}|u|\lambda = \sqrt{< U, U >} < 1$ *or equivalently let* $0 < |u| < 1/\sqrt{2}\lambda$.

Therefore the invariant (α, β)*-metric* $F^2 = 2\alpha\beta$ *defined by* g *and* U *is a flat geodesically complete locally Minkowskian metric on* G. *Also if* G *is connected,* (G, F) *is complete.*

5. **Conclusion**

It is an important problem to compute the geometric quantities such as curvature properties of homogeneous spaces. In particular, J.Milnor used the formula of the sectional curvature of left invariant Riemannian metric on a Lie group to study the curvature properties of such spaces.

In this paper, we considered the explicit formula for the flag curvature of invariant Finsler (α, β)- metric $F^2 = 2\alpha\beta$ on naturally reductive homogeneous manifolds $(G/H, g)$, where the metric induced by the invariant Riemannian metric g and an invariant vector field $\tilde{\mathrm{X}}$ which is parallel with g.

Also we study the special cases of naturally reductive spaces and bi-invariant metrics. We end the article by giving some examples of geodesically complete (α, β) metric spaces.

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