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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A CLASS OF NONLINEAR DEGENERATE ELLIPTIC EQUATIONS

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ABSTRACT. In this paper we are interested in the existence of solutions for Dirichlet problem associated with the degenerate nonlinear elliptic equations

$$\begin{cases} -\operatorname{div}\big[\mathcal{A}(x,\nabla u)\,\omega_1+\mathcal{B}(x,u,\nabla u)\,\omega_2\big]+\mathcal{H}(x,u,\nabla u)\,\omega_3=f_0(x)-\displaystyle\sum_{j=1}^{n}D_jf_j(x) \text{ in } \Omega,\\ u(x)=0 \text{ on } \partial\Omega,\end{cases}$$

in the setting of the weighted Sobolev spaces.

1. INTRODUCTION

In this paper we prove the existence of (weak) solutions in the weighted Sobolev space $W_0^{1,p}(\Omega,\omega_1)$ (see Definition 2.3) for the Dirichlet problem

$$(P)\begin{cases} Lu(x)=f_0(x)-\displaystyle\sum_{j=1}^{n}D_jf_j(x) \ \text{ in } \ \Omega,\\ u(x)=0 \ \text{ on } \ \partial\Omega,\end{cases}$$

where L is the partial differential operator

$$Lu(x)=-\operatorname{div}\big[\mathcal{A}(x,\nabla u)\,\omega_1+\mathcal{B}(x,u,\nabla u)\,\omega_2\big]+\mathcal{H}(x,u,\nabla u)\,\omega_3 \tag{1.1}$$

where $D_j=\partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω_1, ω_2 and ω_3 are three weight functions (which represent the degeneration or singularity in the equation (1.1)), $1<q,s<p<\infty$ and the functions $\mathcal{A}_j:\Omega\times\mathbb{R}^n\to\mathbb{R}$, $\mathcal{B}_j:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ $(j=1,...,n)$ and $\mathcal{H}:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ satisfying the following conditions:

(H1) $x\mapsto\mathcal{A}_j(x,\xi)$ is measurable on Ω for all $\xi\in\mathbb{R}^n$,
$\xi\mapsto\mathcal{A}_j(x,\xi)$ is continuous on $\mathbb{R}^n$ for almost all $x\in\Omega$.

Keywords: degenerate nonlinear elliptic equations, weighted Sobolev spaces.

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(H2) There exists a constant $\theta_1 > 0$ such that

$$\langle \mathcal{A}(x,\xi) - \mathcal{A}(x,\xi'), (\xi - \xi') \rangle \geq \theta_1 \, |\xi - \xi'|^p,$$

whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, and $\mathcal{A}(x,\xi) = (\mathcal{A}_1(x,\xi), ..., \mathcal{A}_n(x,\xi))$ (where $\langle .,. \rangle$ denotes here the Euclidian scalar product in $\mathbb{R}^n$).

(H3) $\langle \mathcal{A}(x,\xi), \xi \rangle \geq \lambda_1 |\xi|^p$, where λ_1 is a positive constant.

(H4) $|\mathcal{A}(x,\xi)| \leq K_1(x) + h_1(x)|\xi|^{p/p'}$, where K_1 and h_1 are nonnegative functions, with $h_1 \in L^\infty(\Omega)$ and $K_1 \in L^{p'}(\Omega, \omega_1)$ (with $1/p + 1/p' = 1$).

(H5) $x \mapsto \mathcal{B}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n$,
$(\eta,\xi) \mapsto \mathcal{B}_j(x,\eta,\xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost all $x \in \Omega$.

(H6) $\langle \mathcal{B}(x,\eta,\xi) - \mathcal{B}(x,\eta',\xi'), (\xi - \xi') \rangle > 0$, whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, where $\mathcal{B}(x,\eta,\xi) = (\mathcal{B}_1(x,\eta,\xi), ..., \mathcal{B}_n(x,\eta,\xi))$.

(H7) $\langle \mathcal{B}(x,\eta,\xi), \xi \rangle \geq \lambda_2 |\xi|^q + \Lambda_2 |\eta|^q$, where $\lambda_2 > 0$ and $\Lambda_2 \geq 0$ are constants.

(H8) $|\mathcal{B}(x,\eta,\xi)| \leq K_2(x) + g_1(x)|\eta|^{q/q'} + g_2(x)|\xi|^{q/q'}$, where K_2, g_1 and g_2 are nonnegative functions, with g_1 and $g_2 \in L^\infty(\Omega)$, and $K_2 \in L^{q'}(\Omega, \omega_2)$ (with $1/q + 1/q' = 1$).

(H9) $x \mapsto \mathcal{H}(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n$
$(\eta,\xi) \mapsto \mathcal{H}(x,\eta,\xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost all $x \in \Omega$.

(H10) $[\mathcal{H}(x,\eta,\xi) - \mathcal{H}(x,\eta',\xi')](\eta - \eta') > 0$, whenever $\eta, \eta' \in \mathbb{R}$, $\eta \neq \eta'$.

(H11) $\mathcal{H}(x,\eta,\xi)\eta \geq \lambda_3 |\xi|^s + \Lambda_3 |\eta|^s$, where λ_3 and Λ_3 are nonnegative constants.

(H12) $|\mathcal{H}(x,\eta,\xi)| \leq K_3(x) + h_2(x)|\eta|^{s/s'} + h_3(x)|\xi|^{s/s'}$, where K_3, h_2 and h_3 are nonnegative functions, with $K_3 \in L^{s'}(\Omega, \omega_3)$ (with $1/s + 1/s' = 1$), h_2 and $h_3 \in L^\infty(\Omega)$.

Let Ω be an open set in $\mathbb{R}^n$. By the symbol $\mathcal{W}(\Omega)$ we denote the set of all measurable a.e. in Ω positive and finite functions $\omega = \omega(x)$, $x \in \Omega$. Elements of $\mathcal{W}(\Omega)$ will be called *weight functions*. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \displaystyle\int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [2], [3], [4] and [7]). In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. There are several very concrete problems from practice which lead to such differential equations, e.g. from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction and reaction-diffusion problems (see some examples of applications of degenerate elliptic equations in [1] and [6]).

A class of weights, which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [15]). These classes have found many useful applications in harmonic analysis (see [17]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [13]). There are, in fact, many interesting examples of weights (see [12] for p-admissible weights).

The following theorem will be proved in section 3.

Theorem 1.1. *Let* $1 < q, s < p < \infty$ *and assume (H1)-(H12). If*
(i) $\omega_1 \in A_p$, ω_2 *and* $\omega_3 \in \mathcal{W}(\Omega)$, $\dfrac{\omega_2}{\omega_1} \in L^{r_1}(\Omega, \omega_1)$ *and* $\dfrac{\omega_3}{\omega_1} \in L^{r_2}(\Omega, \omega_1)$, *where* $r_1 = p/(p-q)$ *and* $r_2 = p/(p-s)$*;*
(ii) $f_0/\omega_2 \in L^{q'}(\Omega, \omega_2)$ *and* $f_j/\omega_1 \in L^{p'}(\Omega, \omega_1)$ $(j = 1, ..., n)$*.*
Then the problem (P) has a unique solution $u \in W_0^{1,p}(\Omega, \omega_1)$*. Moreover, we have*

$$\|u\|_{W_0^{1,p}(\Omega,\omega_1)} \leq C \left(C_{p,q} \, \|f_0/\omega_2\|_{L^{q'}(\Omega,\omega_2)} + \sum_{j=1}^{n} \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \right)^{1/(p-1)},$$

where $C = [(C_\Omega^p + 1)/\lambda_1]^{1/(p-1)}$ *(*C_Ω *is the constant in Theroem 2.2 and* $C_{p,q}$ *is the constant in Remark 3.2(i)).*

2. DEFINITIONS AND BASIC RESULTS

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x) dx \right)^{p-1} \leq C,$$

for all balls $B \subset \mathbb{R}^n$, where $|.|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [10], [12] or [17] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that

$$\mu(B(x; 2r)) \leq C \, \mu(B(x; r)),$$

for every ball $B = B(x; r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x) \, dx$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [12]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [17]).

If $\omega \in A_p$, then

$$\left(\frac{|E|}{|B|} \right)^p \leq C \frac{\mu(E)}{\mu(B)},$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [12]). Therefore, if $\mu(E) = 0$ then $|E| = 0$. The measure μ and the Lebesgue measure $|.|$ are mutually absolutely continuous, i.e., they have the same zero sets ($\mu(E) = 0$

if and only if $|E| = 0$); so there is no need to specify the measure when using the ubiquitous expression almost everywhere and almost every, both abbreviated a.e..

Definition 2.1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $1 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f|^p \omega \, dx \right)^{1/p} < \infty.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [18]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.2. Let ω be a A_p-weight ($1 < p < \infty$), and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. We define the weighted Sobolev space $W^{1,p}(\Omega, \omega)$ as the set of functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D_j u \in L^p(\Omega, \omega)$. The norm of u in $W^{1,p}(\Omega, \omega)$ is defined by

$$\|u\|_{W^{1,p}(\Omega,\omega)} = \left(\int_\Omega |u|^p \, \omega \, dx + \sum_{j=1}^n \int_\Omega |D_j u|^p \, \omega \, dx \right)^{1/p}. \tag{2.1}$$

If $\omega \in A_p$, then $W^{1,p}(\Omega, \omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Theorem 2.1.4 in [18]). The spaces $W^{1,p}(\Omega, \omega)$ are Banach spaces.

The space $W^{1,p}_0(\Omega, \omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm (2.1). Equipped with this norm, $W^{1,p}_0(\Omega, \omega)$ is a reflexive Banach space (see [14] for more information about the spaces $W^{1,p}(\Omega, \omega)$). The dual of space $W^{1,p}_0(\Omega, \omega)$ is the space

$$\begin{aligned} &[W^{1,p}_0(\Omega, \omega)]^* \\ &= \{T = f_0 - \text{div}(F),\ F = (f_1, ..., f_n) : \ \frac{f_j}{\omega} \in L^{p'}(\Omega, \omega), j = 0, 1, ..., n\}. \end{aligned}$$

It is evident that a weight function ω which satisfies $0 < c_1 \leq \omega(x) \leq c_2$ for $x \in \Omega$ (where c_1 and c_2 are constants), give nothing new (the space $W^{1,p}_0(\Omega, \omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{1,p}_0(\Omega)$). Consequently, we shall be interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

In this paper we use the following results.

Theorem 2.1. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega, \omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega, \omega)$ such that*
(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$ a.e. on Ω;
(ii) $|u_{m_k}(x)| \leq \Phi(x)$ a.e. on Ω.

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [9]. □

Theorem 2.2. *(The weighted Sobolev inequality) Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ ($1 < p < \infty$). There exist constants C_Ω and δ positive such that for all $u \in W^{1,p}_0(\Omega, \omega)$*

and all k satisfying $1 \le k \le n/(n-1)+\delta$,

$$\|u\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \|\,|\nabla u|\,\|_{L^p(\Omega,\omega)}, \tag{2.2}$$

where C_Ω depends only on n, p, the A_p-constant $C(p,\omega)$ of ω and the diameter of Ω.

Proof. Its suffices to prove the inequality for functions $u \in C_0^\infty(\Omega)$ (see Theorem 1.3 in [8]). To extend the estimates (2.2) to arbitrary $u \in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying the estimates (2.2) to differences $u_{m_1} - u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{kp}(\Omega,\omega)$. Consequently the limit function u will lie in the desired spaces and satisfy (2.2). □

Definition 2.3. We say that an element $u \in W_0^{1,p}(\Omega,\omega_1)$ is a (weak) solution of problem (P) if

$$\int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle\, \omega_1\, dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle\, \omega_2\, dx + \int_\Omega \mathcal{H}(x,u,\nabla u)\, \varphi\, \omega_3\, dx$$
$$= \int_\Omega f_0\, \varphi\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j\varphi\, dx,$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1)$.

Remark 2.3. (i) If $\dfrac{\omega_2}{\omega_1} \in L^{r_1}(\Omega,\omega_1)$ (where $r_1 = p/(p-q)$, $1 < q < p < \infty$) then

$$\|u\|_{L^q(\Omega,\omega_2)} \le C_{p,q}\, \|u\|_{L^p(\Omega,\omega_1)},$$

where $C_{p,q} = \|\omega_2/\omega_1\|^{1/q}_{L^{r_1}(\Omega,\omega_1)}$. In fact, by Hölder's inequality we obtain

$$\begin{aligned} \|u\|^q_{L^q(\Omega,\omega_2)} &= \int_\Omega |u|^q\, \omega_2\, dx = \int_\Omega |u|^q\, \frac{\omega_2}{\omega_1}\, \omega_1\, dx \\ &\le \left(\int_\Omega |u|^{q\,p/q}\, \omega_1\, dx\right)^{q/p} \left(\int_\Omega \big(\omega_2/\omega_1\big)^{p/(p-q)}\, \omega_1\, dx\right)^{(p-q)/p} \\ &= \|u\|^q_{L^p(\Omega,\omega_1)}\, \|\omega_2/\omega_1\|_{L^{r_1}(\Omega,\omega_1)}. \end{aligned}$$

Hence,

$$\|u\|_{L^q(\Omega,\omega_2)} \le C_{p,q} \|u\|_{L^p(\Omega,\omega_1)}.$$

(ii) Analogously, if $\dfrac{\omega_3}{\omega_1} \in L^{r_2}(\Omega,\omega_1)$ (where $r_2 = p/(p-s)$, $1 < s < p < \infty$) then

$$\|u\|_{L^s(\Omega,\omega_3)} \le C_{p,s}\, \|u\|_{L^p(\Omega,\omega_1)},$$

where $C_{p,s} = \|\omega_3/\omega_1\|^{1/s}_{L^{r_2}(\Omega,\omega_1)}$.

(iii) Since $\omega_1 \in A_p$, then by Theorem 2.2 (with $k = 1$), we have

$$\begin{aligned} \|\,|\nabla u|\,\|^p_{L^p(\Omega,\omega_1)} &\le \|u\|^p_{W_0^{1,p}(\Omega,\omega_1)} \\ &= \int_\Omega |u|^p\, \omega_1\, dx + \int_\Omega |\nabla u|^p\, \omega_1\, dx \\ &\le (C_\Omega^p + 1) \|\,|\nabla u|\,\|^p_{L^p(\Omega,\omega_1)}. \end{aligned}$$

Hence,

$$\| \, |\nabla u| \, \|_{L^p(\Omega,\omega_1)} \leq \|u\|_{W_0^{1,p}(\Omega,\omega_1)} \leq (C_\Omega^p + 1)^{1/p} \| \, |\nabla u| \, \|_{L^p(\Omega,\omega_1)}.$$

3. PROOF OF THEOREM 1.1

The basic idea is to reduce the problem (P) to an operator equation $Au = T$ and apply the theorem below.

Theorem 3.1. *Let $A : X \to X^*$ be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space X. Then the following assertions hold:*

(a) For each $T \in X^*$ the equation $Au = T$ has a solution $u \in X$;

(b) If the operator A is strictly monotone, then equation $A\,u = T$ is uniquely solvable in X.

Proof. See Theorem 26.A in [20]. □

To prove Theorem 1.1, we define $\mathbf{B}, \mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3} : W_0^{1,p}(\Omega,\omega_1) \times W_0^{1,p}(\Omega,\omega_1) \to \mathbb{R}$ and $\mathbf{T} : W_0^{1,p}(\Omega,\omega_1) \to \mathbb{R}$ by

$$\mathbf{B}(u,\varphi) = \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B}_3(u,\varphi),$$

$$\begin{aligned}
\mathbf{B_1}(u,\varphi) &= \int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi \rangle \, \omega_1 \, dx, \\
\mathbf{B_2}(u,\varphi) &= \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi \rangle \, \omega_2 \, dx, \\
\mathbf{B}_3(u,\varphi) &= \int_\Omega \mathcal{H}(x,u,\nabla u) \, \varphi \, \omega_3 \, dx \\
\mathbf{T}(\varphi) &= \int_\Omega f_0 \, \varphi \, dx + \sum_{j=1}^n \int_\Omega f_j \, D_j\varphi \, dx.
\end{aligned}$$

Then $u \in W_0^{1,p}(\Omega,\omega_1)$ is a (weak) solution to problem (P) if

$$\mathbf{B}(u,\varphi) = \mathbf{B_1}(u,\varphi) + \mathbf{B_2}(u,\varphi) + \mathbf{B}_3(u,\varphi) = \mathbf{T}(\varphi),$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1)$.

Step 1. For $j = 1, ..., n$ we define the operator $F_j : W_0^{1,p}(\Omega,\omega_1) \to L^{p'}(\Omega,\omega_1)$ as

$$(F_j u)(x) = \mathcal{A}_j(x, \nabla u(x)).$$

We now show that the operator F_j is bounded and continuous.

(i) Using (H4) we obtain

$$\begin{aligned}
\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_j u(x)|^{p'}\omega_1\,dx \\
&= \int_\Omega |\mathcal{A}_j(x,\nabla u)|^{p'}\omega_1\,dx \\
&\le \int_\Omega \left(K_1 + h_1|\nabla u|^{p/p'}\right)^{p'}\omega_1\,dx \\
&\le C_p\int_\Omega \left[(K_1^{p'} + h_1^{p'}|\nabla u|^p)\omega_1\right]dx \\
&= C_p\left[\int_\Omega K_1^{p'}\,\omega_1\,dx + \int_\Omega h_1^{p'}|\nabla u|^p\omega_1\,dx\right] \\
&\le C_p\left(\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^p_{L^p(\Omega,\omega_1)}\right) \\
&\le C_p\left(\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}\right), \qquad (3.1)
\end{aligned}$$

where the constant C_p depends only on p. Therefore, in (3.1) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega_1)}\le C_p^{1/p'}\left(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\,\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega_1)}\right).$$

(ii) Let $u_m\to u$ in $W_0^{1,p}(\Omega,\omega_1)$ as $m\to\infty$. We need to show that $F_j u_m\to F_j u$ in $L^{p'}(\Omega,\omega_1)$. We will apply the Lebesgue Dominated Convergence Theorem. If $u_m\to u$ in $W_0^{1,p}(\Omega,\omega_1)$, then $|\nabla u_m|\to|\nabla u|$ in $L^p(\Omega,\omega_1)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_1\in L^p(\Omega,\omega_1)$ such that

$$\begin{aligned}
&u_{m_k}(x)\to u(x) \text{ a.e. in } \Omega,\\
&D_j u_{m_k}(x)\to D_j u(x) \text{ a.e. in } \Omega,\\
&|\nabla u_{m_k}(x)|\le\Phi_1(x) \text{ a.e. in } \Omega.
\end{aligned}$$

Next, applying (H4) we obtain

$$\begin{aligned}
\|F_j u_{m_k} - F_j u\|^{p'}_{L^{p'}(\Omega,\omega_1)} &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'}\omega_1\,dx \\
&= \int_\Omega |\mathcal{A}_j(x,\nabla u_{m_k}) - \mathcal{A}_j(x,\nabla u)|^{p'}\,\omega_1\,dx \\
&\le C_p\int_\Omega \left(|\mathcal{A}_j(x,\nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x,\nabla u)|^{p'}\right)\omega_1\,dx \\
&\le C_p\left[\int_\Omega \left(K_1 + h_1|\nabla u_{m_k}|^{p/p'}\right)^{p'}\omega_1\,dx\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \int_\Omega \left(K_1 + h_1|\nabla u|^{p/p'}\right)^{p'} \omega_1\,dx\Big] \\
&\leq 2\,C_p \int_\Omega \left(K_1 + h_1\Phi_1^{p/p'}\right)^{p'} \omega_1\,dx \\
&\leq 2\,C_p \left[\int_\Omega K_1^{p'}\,\omega_1\,dx + \int_\Omega h_1^{p'}\Phi_1^p\,\omega_1\,dx\right] \\
&\leq 2\,C_p \left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \int_\Omega \Phi_1^p\,\omega_1\,dx\right] \\
&= 2\,C_p \left[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega_1)} + \|h_1\|^{p'}_{L^\infty(\Omega)} \|\Phi_1\|^p_{L^p(\Omega,\omega_1)}\right].
\end{aligned}
$$

By condition (H1), we have

$$F_j u_{m_k}(x) = \mathcal{A}_j(x, \nabla u_{m_k}(x)) \to \mathcal{A}_j(x, \nabla u(x)) = F_j u(x),$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega_1)} \to 0,$$

that is,

$$F_j u_{m_k} \to F_j u \quad \text{in} \quad L^{p'}(\Omega,\omega_1).$$

We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]) that

$$F_j u_m \to F_j u \quad \text{in} \quad L^{p'}(\Omega,\omega_1). \tag{3.2}$$

Step 2. We define the operator $G_j : W_0^{1,p}(\Omega,\omega_1) \to L^{q'}(\Omega,\omega_2)$ by

$$(G_j u)(x) = \mathcal{B}_j(x, u(x), \nabla u(x)).$$

This operator is continuous and bounded. In fact,

(i) Using (H8) and Remark 2.3(i) we obtain

$$
\begin{aligned}
&\|G_j u\|^{q'}_{L^{q'}(\Omega,\omega_2)}\\
&= \int_\Omega |G_j u(x)|^{q'}\omega_2\,dx\\
&= \int_\Omega |\mathcal{B}_j(x,u,\nabla u)|^{q'}\omega_2\,dx\\
&\le \int_\Omega \left(K_2 + g_1|u|^{q/q'} + g_2|\nabla u|^{q/q'}\right)^{q'}\omega_2\,dx\\
&\le C_q \int_\Omega \left[(K_2^{q'} + g_1^{q'}|u|^q + g_2^{q'}|\nabla u|^q)\omega_2\right]dx\\
&= C_q\left[\int_\Omega K_2^{q'}\,\omega_2\,dx + \int_\Omega g_1^{q'}|u|^q\,\omega_2\,dx + \int_\Omega g_2^{q'}|\nabla u|^q\omega_2\,dx\right]\\
&\le C_q\left(\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|g_1\|^{q'}_{L^\infty(\Omega)}\|u\|^q_{L^q(\Omega,\omega_2)} + \|g_2\|^{q'}_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^q_{L^q(\Omega,\omega_2)}\right)\\
&\le C_q\left(\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|g_1\|^{q'}_{L^\infty(\Omega)}C^q_{p,q}\|u\|^q_{L^p(\Omega,\omega_1)}\right.\\
&\left.+ C^q_{p,q}\|g_2\|^{q'}_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^q_{L^p(\Omega,\omega_1)}\right)\\
&\le C_q\left(\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q}\big(\|g_1\|^{q'}_{L^\infty(\Omega)} + \|g_2\|^{q'}_{L^\infty(\Omega)}\big)\|u\|^q_{W^{1,p}_0(\Omega,\omega_1)}\right),
\end{aligned}
\tag{3.3}
$$

where the C_q depends only on q. Therefore, in (3.3), we obtain

$$
\begin{aligned}
&\|G_j u\|_{L^{q'}(\Omega,\omega_2)}\\
&\le C_q^{1/q'}\left(\|K_2\|_{L^{q'}(\Omega,\omega_2)} + C^{q-1}_{p,q}\big(\|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}\big)\,\|u\|^{q-1}_{W^{1,p}_0(\Omega,\omega_1)}\right).
\end{aligned}
\tag{3.4}
$$

(ii) Let $u_m \to u$ in $W^{1,p}_0(\Omega,\omega_1)$ as $m\to\infty$. We need to show that $G_j u_m \to G_j u$ in $L^{q'}(\Omega,\omega_2)$. We will apply the Lebesgue Dominated Theorem. If $u_m\to u$ in $W^{1,p}_0(\Omega,\omega_1)$, then $u_m\to u$ in $L^p(\Omega,\omega_1)$ and $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega,\omega_1)$. Using Theorem 2.1, there exist a subsequence $\{u_{m_k}\}$ and functions Φ_1, $\Phi_2 \in L^p(\Omega,\omega_1)$ such that

$$
\begin{aligned}
&u_{m_k}(x) \to u(x) \text{ a.e. in } \Omega,\\
&|u_{m_k}(x)| \le \Phi_2(x) \text{ a.e. in } \Omega,\\
&D_j u_{m_k}(x) \to D_j u(x) \text{ a.e. in } \Omega,\\
&|\nabla u_{m_k}(x)| \le \Phi_1(x) \text{ a.e. in } \Omega.
\end{aligned}
$$

Next, applying (H8) and Remark 2.3(i) we obtain

$$
\begin{aligned}
&\|G_j u_{m_k} - G_j u\|^{q'}_{L^{q'}(\Omega,\omega_2)}\\
&= \int_\Omega |G_j u_{m_k}(x) - G_j u(x)|^{q'} \omega_2\, dx\\
&= \int_\Omega |\mathcal{B}_j(x, u_{m_k}, \nabla u_{m_k}) - \mathcal{B}_j(x, u, \nabla u)|^{q'}\, \omega_2\, dx\\
&\leq C_q \int_\Omega \left(|\mathcal{B}_j(x, u_{m_k}, \nabla u_{m_k})|^{q'} + |\mathcal{B}_j(x, u, \nabla u)|^{q'} \right) \omega_2\, dx\\
&\leq C_q \left[\int_\Omega \left(K_2 + g_1 |u_{m_k}|^{q/q'} + g_2 |\nabla u_{m_k}|^{q/q'} \right)^{q'} \omega_2\, dx \right.\\
&\left. + \int_\Omega \left(K_2 + g_1 |u|^{q/q'} + g_2 |\nabla u|^{q/q'} \right)^{q'} \omega_2\, dx \right]\\
&\leq 2\, C_q \int_\Omega \left(K_2 + g_1 \Phi_2^{q/q'} + g_2 \Phi_1^{q/q'} \right)^{q'} \omega_2\, dx\\
&\leq 2\, C_q \left[\int_\Omega K_2^{q'}\, \omega_2\, dx + \int_\Omega g_1^{q'} \Phi_2^q\, \omega_2\, dx + \int_\Omega g_2^{q'} \Phi_1^q\, \omega_2\, dx \right]\\
&\leq 2\, C_q \left[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|g_1\|^{q'}_{L^\infty(\Omega)} \int_\Omega \Phi_2^q\, \omega_2\, dx \right.\\
&\left. + \|g_2\|^{q'}_{L^\infty(\Omega)} \int_\Omega \Phi_1^q\, \omega_2\, dx \right]\\
&= 2\, C_q \left[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + \|g_1\|^{q'}_{L^\infty(\Omega)}\ \|\Phi_2\|^q_{L^q(\Omega,\omega_2)} \right.\\
&\left. + \|g_2\|^{q'}_{L^\infty(\Omega)} \|\Phi_1\|^q_{L^q(\Omega,\omega_2)} \right]\\
&\leq 2\, C_q \left[\|K_2\|^{q'}_{L^{q'}(\Omega,\omega_2)} + C^q_{p,q} \|g_1\|^{q'}_{L^\infty(\Omega)}\ \|\Phi_2\|^q_{L^p(\Omega,\omega_1)} \right.\\
&\left. + C^q_{p,q}\, \|g_2\|^{q'}_{L^\infty(\Omega)} \|\Phi_1\|^q_{L^p(\Omega,\omega_1)} \right].
\end{aligned}
$$

By condition (H5), we have

$$
G_j u_{m_k}(x) = \mathcal{B}_j(x, u_{m_k}(x), \nabla u_{m_k}(x)) \to \mathcal{B}_j(x, u(x), \nabla u(x)) = G_j u(x),
$$

as $m_k \to +\infty$. Therefore, by the Lebesgue Dominated Convergence Theorem, we obtain

$$
\|G_j u_{m_k} - G_j u\|_{L^{q'}(\Omega,\omega_2)} \to 0,
$$

that is,

$$
G_j u_{m_k} \to G_j u \ \text{ in } \ L^{q'}(\Omega, \omega_2).
$$

We conclude from the Convergence Principle in Banach spaces (see Proposition 10.13 in [19]) that

$$G_j u_m \to G_j u \ \text{ in } \ L^{q'}(\Omega,\omega_2). \tag{3.5}$$

Step 3. We define the operator $H : W_0^{1,p}(\Omega,\omega_1) \to L^{s'}(\Omega,\omega_3)$ by

$$(Hu)(x) = \mathcal{H}(x,u(x),\nabla u(x)).$$

We also have that the operator H is continuous and bounded. In fact,
(i) Using $(H12)$ and Remark 2.3(ii) we obtain

$$\begin{aligned}
&\|Hu\|^{s'}_{L^{s'}(\Omega,\omega_3)} = \int_\Omega |Hu|^{s'}\,\omega_3\,dx\\
&= \int_\Omega |\mathcal{H}(x,u,\nabla u)|^{s'}\,\omega_3\,dx\\
&\le \int_\Omega \left(K_3 + h_2|u|^{s/s'} + h_3|\nabla u|^{s/s'}\right)^{s'}\omega_3\,dx\\
&\le C_s \int_\Omega (K_3^{s'} + h_2^{s'}|u|^s + h_3^{s'}|\nabla u|^s)\omega_3\,dx\\
&\le C_s \int_\Omega K_3^{s'}\,\omega_3\,dx + \|h_2\|^{s'}_{L^\infty(\Omega)}\int_\Omega |u|^s\,\omega_3\,dx + \|h_2\|^{s'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^s\omega_3\,dx\Bigg]\\
&\le C_s\Bigg(\|K_3\|^{s'}_{L^{s'}(\Omega,\omega_3)} + \|h_2\|^{s'}_{L^\infty(\Omega)}C^s_{p,s}\|u\|^s_{L^p(\Omega,\omega_1)} + \|h_3\|^{s'}_{L^\infty(\Omega)}C^s_{p,s}\|\,|\nabla u|\,\|^s_{L^p(\Omega,\omega_1)}\Bigg)\\
&\le C_s\Bigg(\|K_3\|^{s'}_{L^{s'}(\Omega,\omega_3)} + C^s_{p,s}(\|h_2\|^{s'}_{L^\infty(\Omega)} + \|h_3\|^{s'}_{L^\infty(\Omega)})\,\|u\|^s_{W_0^{1,p}(\Omega,\omega_1)}\Bigg),
\end{aligned} \tag{3.6}$$

where the constant C_s depends only on s. Hence, in (3.6), we obtain

$$\|Hu\|_{L^{s'}(\Omega,\omega_3)} \le C_s\left[\|K_3\|_{L^{s'}(\Omega,\omega_3)} + C^{s-1}_{p,s}(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)})\|u\|^{s-1}_{W_0^{1,p}(\Omega,\omega_1)}\right].$$

(ii) Applying (H12) and Remark 2.3(ii), by the same argument used in Step 1(ii), we obtain analogously, if $u_m \to u$ in $W_0^{1,p}(\Omega,\omega_1)$ then

$$Hu_m \to Hu, \ \text{ in } \ L^{s'}(\Omega,\omega_3). \tag{3.7}$$

Step 4. Since $\dfrac{f_0}{\omega_2} \in L^{q'}(\Omega, \omega_2)$ and $\dfrac{f_j}{\omega_1} \in L^{p'}(\Omega, \omega_1)$ $(j = 1, ..., n)$ then $\mathbf{T} \in [W_0^{1,p}(\Omega, \omega_1)]^*$. Moreover, we have

$$\begin{aligned}
|\mathbf{T}(\varphi)| &\leq \int_\Omega |f_0||\varphi|\,dx + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,dx \\
&= \int_\Omega \frac{|f_0|}{\omega_2}|\varphi|\omega_2\,dx + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega_1}|D_j\varphi|\,\omega_1\,dx \\
&\leq \|f_0/\omega_2\|_{L^{q'}(\Omega,\omega_2)}\|\varphi\|_{L^q(\Omega,\omega_2)} + \left(\sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right)\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\leq \left(C_{p,q}\,\|f_0/\omega_2\|_{L^{q'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}.
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
|\mathbf{B}(u,\varphi)| &\leq |\mathbf{B_1}(u,\varphi)| + |\mathbf{B_2}(u,\varphi)| + |\mathbf{B}_3(u,\varphi)| \\
&\leq \int_\Omega |\mathcal{A}(x,\nabla u)||\nabla\varphi|\,\omega_1\,dx + \int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega_2\,dx \\
&+ \int_\Omega |\mathcal{H}(x,u,\nabla u)|\,\omega_3. \qquad (3.8)
\end{aligned}$$

In (3.8) we have, by (H4),

$$\begin{aligned}
&\int_\Omega |\mathcal{A}(x,\nabla u)|\,|\nabla\varphi|\,\omega_1\,dx \leq \int_\Omega \left(K_1 + h_1|\nabla u|^{p/p'}\right)|\nabla\varphi|\,\omega_1\,dx \\
&\leq \|K_1\|_{L^{p'}(\Omega,\omega_1)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{p/p'}_{L^p(\Omega,\omega_1)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\leq \left(\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{W_0^{1,p}(\Omega,\omega_1)}\right)\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

and by (H8) and Remark 2.3(i)

$$\begin{aligned}
&\int_\Omega |\mathcal{B}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega_2\,dx \leq \int_\Omega \left(K_2 + g_1|u|^{q/q'} + g_2|\nabla u|^{q/q'}\right)|\nabla\varphi|\,\omega_2\,dx \\
&\leq \|K_2\|_{L^{q'}(\Omega,\omega_2)}\|\,|\nabla\varphi|\,\|_{L^q(\Omega,\omega_2)} + \|g_1\|_{L^\infty(\Omega)}\|u\|^{q/q'}_{L^q(\Omega,\omega_2)}\|\,|\nabla\varphi|\,\|_{L^q(\Omega,\omega_2)} \\
&+ \|g_2\|_{L^\infty(\Omega)}\|\,|\nabla u|\,\|^{q/q'}_{L^q(\Omega,\omega_2)}\|\,|\nabla\varphi|\,\|_{L^q(\Omega,\omega_2)} \\
&\leq C_{p,q}\|K_2\|_{L^{q'}(\Omega,\omega_2)}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} + C_{p,q}^{q-1}\|g_1\|_{L^\infty(\Omega)}\,\|u\|^{q-1}_{L^p(\Omega,\omega_1)}\,C_{p,q}\,\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&+ \|g_2\|_{L^\infty(\Omega)}C_{p,q}^{q-1}\|\,|\nabla u|\,\|^{q-1}_{L^p(\Omega,\omega_1)}C_{p,q}\|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\leq \left[C_{p,q}\|K_2\|_{L^{q'}(\Omega,\omega_2)} + \left(C_{p,q}^q\|g_1\|_{L^\infty(\Omega)} + C_{p,q}^q\|g_2\|_{L^\infty(\Omega)}\right)\|u\|^{q-1}_{W_0^{1,p}(\Omega,\omega_1)}\right]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

and by (H12) and Remark 2.3(ii)

$$\begin{aligned}
&\int_\Omega |\mathcal{H}(x,u,\nabla u)|\,|\varphi|\,\omega_3\,dx \le \int_\Omega \left(K_3 + h_2|u|^{s/s'} + h_3|\nabla u|^{s/s'}\right)|\varphi|\,\omega_3\,dx\\
&\le \int_\Omega K_3\,|\varphi|\,\omega_3\,dx + \|h_2\|_{L^\infty(\Omega)}\int_\Omega |u|^{s/s'}|\varphi|\,\omega_3\,dx + \|h_3\|_{L^\infty(\Omega)}\int_\Omega |\nabla u|^{s/s'}|\varphi|\,\omega_3\,dx\\
&\le \|K_3\|_{L^{s'}(\Omega,\omega_3)}\|\varphi\|_{L^s(\Omega,\omega_3)} + \|h_2\|_{L^\infty(\Omega)}\|u\|^{s/s'}_{L^s(\Omega,\omega_3)}\|\varphi\|_{L^s(\Omega,\omega_3)}\\
&\le C_{p,s}\|K_3\|_{L^{s'}(\Omega)}\|\varphi\|_{L^p(\Omega,\omega_1)} + \|h_2\|_{L^\infty(\Omega)}C^{s-1}_{p,s}\|u\|^{s-1}_{L^p(\Omega,\omega_1)}C_{p,s}\|\varphi\|_{L^p(\Omega,\omega_1)}\\
&+\|h_3\|_{L^\infty(\Omega)}C^{s-1}_{p,s}\|\,|\nabla u|\,\|^{s-1}_{L^p(\Omega,\omega_1)}C_{p,s}\|\varphi\|_{L^p(\Omega,\omega_1)}\\
&\le \left[C_{p,s}\,\|K_3\|_{L^{s'}(\Omega,\omega_3)} + C^s_{p,s}(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)})\|u\|^{s-1}_{W^{1,p}_0(\Omega,\omega_1)}\right]\|\varphi\|_{W^{1,p}_0(\Omega,\omega_1)}.
\end{aligned}$$

Hence, in (3.8) we obtain, for all $u,\varphi\in W^{1,p}_0(\Omega,\omega_1)$

$$\begin{aligned}
&|\mathbf{B}(u,\varphi)|\\
&\le\Big[\|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega_1)} + C_{p,q}\|K_2\|_{L^{q'}(\Omega,\omega_2)}\\
&+C^q_{p,q}\big(\|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}\big)\|u\|^{q-1}_{W^{1,p}_0(\Omega,\omega_1)} + C_{p,s}\|K_3\|_{L^{s'}(\Omega,\omega_3)}\\
&+C^s_{p,s}(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)})\|u\|^{s-1}_{W^{1,p}_0(\Omega,\omega_1)}\Big]\|\varphi\|_{W^{1,p}_0(\Omega,\omega_1)}.
\end{aligned}$$

Since $\mathbf{B}(u,.)$ is linear, for each $u\in W^{1,p}_0(\Omega,\omega_1)$, there exists a linear and continuous functional on $W^{1,p}_0(\Omega,\omega_1)$ denoted by Au such that $(Au|\varphi)=\mathbf{B}(u,\varphi)$ for all $u,\varphi\in W^{1,p}_0(\Omega,\omega_1)$ (here $(f|x)$ denotes the value of the linear functional f at the point x). Moreover

$$\begin{aligned}
\|Au\|_* \le\ & \|K_1\|_{L^{p'}(\Omega,\omega_1)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p-1}_{W^{1,p}_0(\Omega,\omega_1)} + C_{p,q}\|K_2\|_{L^{q'}(\Omega,\omega_2)}\\
&+C^q_{p,q}\big(\|g_1\|_{L^\infty(\Omega)} + \|g_2\|_{L^\infty(\Omega)}\big)\|u\|^{q-1}_{W^{1,p}_0(\Omega,\omega_1)}\\
&+C_{p,s}\|K_3\|_{L^{s'}(\Omega,\omega_3)} + C^s_{p,s}(\|h_2\|_{L^\infty(\Omega)} + \|h_3\|_{L^\infty(\Omega)})\|u\|^{s-1}_{W^{1,p}_0(\Omega,\omega_1)}.
\end{aligned}$$

where $\|Au\|_* = \sup\{|(Au|\varphi)| = |B(u,\varphi)| : \varphi\in W^{1,p}_0(\Omega,\omega_1), \|\varphi\|_{W^{1,p}_0(\Omega,\omega_1)} = 1\}$ is the norm of the operator Au. Hence, we obtain the operator

$$\begin{aligned}
A:\quad & W^{1,p}_0(\Omega,\omega_1)\to[W^{1,p}_0(\Omega,\omega_1)]^*\\
& u\mapsto Au.
\end{aligned}$$

Consequently, problem (P) is equivalent to the operator equation

$$Au=\mathbf{T},\ u\in W^{1,p}_0(\Omega,\omega_1).$$

Step 5. Using (H2), (H6) and (H10), we obtain (for $u_1, u_2 \in W_0^{1,p}(\Omega,\omega_1)$, $u_1 \neq u_2$)

$$\begin{aligned}
&(Au_1 - Au_2|u_1 - u_2) = \mathbf{B}(u_1, u_1 - u_2) - \mathbf{B}(u_2, u_1 - u_2)\\
&= \int_\Omega \langle \mathcal{A}(x,\nabla u_1), \nabla(u_1 - u_2)\rangle\, \omega_1\, dx + \int_\Omega \langle \mathcal{B}(x,u_1,\nabla u_1), \nabla(u_1 - u_2)\rangle\, \omega_2\, dx\\
&+ \int_\Omega \mathcal{H}(x,u_1,\nabla u_1)(u_1 - u_2)\, \omega_3\, dx\\
&- \int_\Omega \langle \mathcal{A}(x,\nabla u_2), \nabla(u_1 - u_2)\rangle\, \omega_1\, dx - \int_\Omega \langle \mathcal{B}(x,u_2,\nabla u_2, \nabla(u_1 - u_2)\rangle\, \omega_2\, dx\\
&- \int_\Omega \mathcal{H}(x,u_2,\nabla u_2)(u_1 - u_2)\, \omega_3\, dx\\
&= \int_\Omega \langle \mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2), \nabla(u_1 - u_2)\rangle\, \omega_1\, dx\\
&+ \int_\Omega \langle \mathcal{B}(x,u_1,\nabla u_1) - \mathcal{B}(x,u_2,\nabla u_2), \nabla(u_1 - u_2)\rangle\, \omega_2\, dx\\
&+ \int_\Omega \Big(\mathcal{H}(x,u_1,\nabla u_1) - \mathcal{H}(x,u_2,\nabla u_2)\Big)(u_1 - u_2)\, \omega_3\, dx\\
&\geq \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^p\, \omega_1\, dx\\
&> 0.
\end{aligned}$$

Therefore, the operator A is strictly monotone. Moreover, from (H3),(H7), (H11) and Remark 2.3(iii), we obtain

$$\begin{aligned}
(Au|u) &= \mathbf{B}(u,u)\\
&= \mathbf{B_1}(u,u) + \mathbf{B_2}(u,u) + \mathbf{B}_3(u,u)\\
&= \int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla u\rangle\, \omega_1\, dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla u\rangle\, \omega_2\, dx\\
&+ \int_\Omega \mathcal{H}(x,u,\nabla u)\, u\, \omega_3\, dx\\
&\geq \lambda_1 \int_\Omega |\nabla u|^p\, \omega_1\, dx + \lambda_2 \int_\Omega |\nabla u|^q\, \omega_2\, dx + \Lambda_2 \int_\Omega |u|^q\, \omega_2\, dx\\
&+ \lambda_3 \int_\Omega |\nabla u|^s\, \omega_3\, dx + \Lambda_3 \int_\Omega |u|^s\, \omega_3\, dx\\
&\geq \lambda_1 \int_\Omega |\nabla u|^p\, \omega_1\, dx\\
&\geq \frac{\lambda_1}{(C_\Omega^p + 1)}\, \|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}.
\end{aligned}$$

Hence, since $1 < q, s < p < \infty$, we have

$$\frac{(Au|u)}{\|u\|_{W_0^{1,p}(\Omega,\omega_1)}} \to +\infty, \text{ as } \|u\|_{W_0^{1,p}(\Omega,\omega_1)} \to +\infty,$$

that is, A is coercive.

Step 6. We need to show that the operator A is continuous.

Let $u_m \to u$ in X as $m \to \infty$. We have,

$$\begin{aligned}
&|\mathbf{B_1}(u_m,\varphi) - \mathbf{B_1}(u,\varphi)| \\
&\leq \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,\nabla u_m) - \mathcal{A}_j(x,\nabla u)||D_j\varphi|\,\omega_1\,dx \\
&= \sum_{j=1}^n \int_\Omega |F_j u_m - F_j u||D_j\varphi|\,\omega_1\,dx \\
&\leq \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)}\right) \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\leq \left(\sum_{j=1}^n \|F_j u_m - F_j u\|_{L^{p'}(\Omega,\omega_1)}\right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

and, by Remark 2.3(i),

$$\begin{aligned}
|\mathbf{B_2}(u_m,\varphi) - \mathbf{B_2}(u,\varphi)| &\leq \sum_{j=1}^n \int_\Omega |\mathcal{B}_j(x,u_m,\nabla u_m) - \mathcal{B}_j(x,u,\nabla u)||D_j\varphi|\,\omega_2\,dx \\
&= \sum_{j=1}^n \int_\Omega |G_j u_m - G_j u||D_j\varphi|\,\omega_2\,dx \\
&\leq \left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\right) \|\,|\nabla\varphi|\,\|_{L^q(\Omega,\omega_2)} \\
&\leq C_{p,q}\left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\right) \|\,|\nabla\varphi|\,\|_{L^p(\Omega,\omega_1)} \\
&\leq C_{p,q}\left(\sum_{j=1}^n \|G_j u_m - G_j u\|_{L^{q'}(\Omega,\omega_2)}\right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)},
\end{aligned}$$

and by Remark 2.3(ii)

$$\begin{aligned}
|\mathbf{B}_3(u_m,\varphi) - \mathbf{B}_3(u,\varphi)| &\leq \int_\Omega |\mathcal{H}(x,u_m,\nabla u_m) - \mathcal{H}(x,u,\nabla u)|\,|\varphi|\,\omega_3\,dx \\
&= \int_\Omega |Hu_m - Hu|\,|\varphi|\,\omega_3\,dx \\
&\leq \|Hu_m - Hu\|_{L^{s'}(\Omega,\omega_3)}\,\|\varphi\|_{L^s(\Omega,\omega_3)} \\
&\leq C_{p,s}\,\|Hu_m - Hu\|_{L^{s'}(\Omega,\omega_3)}\,\|\varphi\|_{L^p(\Omega,\omega_1)} \\
&\leq C_{p,s}\,\|Hu_m - Hu\|_{L^{s'}(\Omega,\omega_3)}\,\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}
\end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1)$. Hence,

$$
\begin{aligned}
&|\mathbf{B}(u_m,\varphi)-\mathbf{B}(u,\varphi)| \\
&\leq |\mathbf{B_1}(u_m,\varphi)-\mathbf{B_1}(u,\varphi)|+|\mathbf{B_2}(u_m,\varphi)-\mathbf{B_2}(u,\varphi)|+|\mathbf{B_3}(u_m,\varphi)-\mathbf{B_3}(u,\varphi)| \\
&\leq \Bigg[\sum_{j=1}^{n}\left(\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}+C_{p,q}\,\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\omega_2)}\right) \\
&+C_{p,s}\|Hu_m-Hu\|_{L^{s'}(\Omega,\omega_3)}\Bigg]\|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\|Au_m-Au\|_* \;\leq\; & \sum_{j=1}^{n}\left(\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega_1)}+C_{p,q}\|G_ju_m-G_ju\|_{L^{q'}(\Omega,\omega_2)}\right) \\
+\; & C_{p,s}\|Hu_m-Hu\|_{L^{s'}(\Omega,\omega_3)}.
\end{aligned}
$$

Hence, using (3.2), (3.5) and (3.7) we have $\|Au_m-Au\|_*\to 0$ as $m\to+\infty$, that is, A is continuous and this implies that A is hemicontinuous.

Therefore, by Theorem 3.1, the operator equation $Au=\mathbf{T}$ has a unique solution $u\in W_0^{1,p}(\Omega,\omega_1)$ and it is the unique solution for problem (P).

Step 7. Estimates for $\|u\|_{W_0^{1,p}(\Omega,\omega_1)}$. In particular, by setting $\varphi=u$ in Definition 2.3, we have

$$
\mathbf{B}(u,u)=\mathbf{B_1}(u,u)+\mathbf{B_2}(u,u)+\mathbf{B}_3(u,u)=\mathbf{T}(u). \tag{3.9}
$$

Hence, using (H3),(H7), (H11) and Remark 2.3(ii) we obtain

$$
\begin{aligned}
&\mathbf{B_1}(u,u)+\mathbf{B_2}(u,u)+\mathbf{B}_3(u,u) \\
&=\int_\Omega \langle \mathcal{A}(x,\nabla u),\nabla u\rangle\,\omega_1\,dx+\int_\Omega \langle \mathbf{B}(x,u,\nabla u),\nabla u\rangle\,\omega_2\,dx+\int_\Omega H(x,u,\nabla u)\,u\,\omega_3\,dx \\
&\geq \lambda_1\int_\Omega |\nabla u|^p\,\omega_1\,dx+\lambda_2\int_\Omega |\nabla u|^q\,\omega_2\,dx+\Lambda_2\int_\Omega |u|^q\,\omega_2\,dx \\
&+\lambda_3\int_\Omega |\nabla u|^s\,\omega_3\,dx+\Lambda_3\int_\Omega |u|^s\,\omega_3\,dx \\
&\geq \lambda_1\int_\Omega |\nabla u|^p\,\omega_1\,dx \\
&\geq \frac{\lambda_1}{(C_\Omega^p+1)}\|u\|^p_{W_0^{1,p}(\Omega,\omega_1)}, \qquad (3.10)
\end{aligned}
$$

and

$$\begin{aligned}
\mathbf{T}(u) &= \int_\Omega f_0\, u\, dx + \sum_{j=1}^n \int_\Omega f_j\, D_j u\, dx \\
&\leq \|f_0/\omega_2\|_{L^{q'}(\Omega,\omega_2)} \|u\|_{L^q(\Omega,\omega_2)} + \left(\sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \right) \| \,|\nabla u|\, \|_{L^p(\Omega,\omega_1)} \\
&\leq \left(C_{p,q}\, \|f_0/\omega_2\|_{L^{q'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \right) \|u\|_{W_0^{1,p}(\Omega,\omega_1)} \\
&= M\, \|u\|_{W_0^{1,p}(\Omega,\omega_1)}, \qquad (3.11)
\end{aligned}$$

where $M = C_{p,q}\, \|f_0/\omega_2\|_{L^{q'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)}$. Hence in (3.9), using (3.10) and (3.11), we obtain

$$\frac{\lambda_1}{(C_\Omega^p + 1)}\, \|u\|^p_{W_0^{1,p}(\Omega,\omega_1)} \leq M\, \|u\|_{W_0^{1,p}(\Omega,\omega_1)}.$$

Therefore

$$\begin{aligned}
\|u\|_{W_0^{1,p}(\Omega,\omega_1)} &\leq \left(\frac{C_\Omega^p + 1}{\lambda_1} \right)^{1/(p-1)} M^{1/(p-1)} \\
&= C \left(C_{p,q}\, \|f_0/\omega_2\|_{L^{q'}(\Omega,\omega_2)} + \sum_{j=1}^n \|f_j/\omega_1\|_{L^{p'}(\Omega,\omega_1)} \right)^{1/(p-1)},
\end{aligned}$$

where $C = \big((C_\Omega^p + 1)/\lambda_1\big)^{1/(p-1)}$.

Example. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, the weight functions $\omega_1(x,y) = (x^2+y^2)^{-1/2}$, $\omega_2(x,y) = (x^2+y^2)^{-1/3}$, $\omega_3(x,y) = (x^2+y^2)^{-1}$ ($\omega_1 \in A_4$, $p = 4$, $q = 3$ and $s = 2$), and the function

$$\mathcal{A} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$$
$$\mathcal{A}((x,y),\xi) = h_1(x,y)\, |\xi|^2\, \xi,$$

where $h_1(x,y) = 2\, \mathrm{e}^{(x^2+y^2)}$, and

$$\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$$
$$\mathcal{B}((x,y),\eta,\xi) = g_2(x,y)\, |\xi|\, \xi,$$

where $g_2(x,y) = 2 + \cos(x^2+y^2)$, and

$$\mathcal{H} : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$$
$$\mathcal{H}((x,y),\eta,\xi) = \eta\, h_2(x,y),$$

where $h_2(x,y) = 1 + \cos^2(xy)$. Let us consider the partial differential operator

$$Lu(x,y) = -\mathrm{div}\big(\mathcal{A}((x,y),\nabla u)\, \omega_1(x,y) + \mathcal{B}((x,y),u,\nabla u)\, \omega_2(x,y)\big) + \mathcal{H}((x,y),u,\nabla u)\, \omega_3(x,y).$$

Therefore, by Theorem 1.1, the problem

$$(P)\begin{cases} Lu(x) &= \dfrac{\cos(xy)}{(x^2+y^2)} - \dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{(x^2+y^2)}\right) \text{ in } \Omega, \\ u(x) &= 0 \text{ on } \partial\Omega, \end{cases}$$

has a unique solution $u \in W_0^{1,4}(\Omega,\omega_1)$.

Corollary 3.2. *Assume $1 < q < p < \infty$. Let the assumptions of Theorem 1.1 be fulfilled and let $\{f_{0m}\}$ and $\{f_{jm}\}$ ($j = 1, ..., n$) be sequences of functions satisfying*
(i) $\dfrac{f_{0m}}{\omega_2} \to \dfrac{f_0}{\omega_2}$ *in* $L^{q'}(\Omega,\omega_2)$;
(ii) $\dfrac{f_{jm}}{\omega_1} \to \dfrac{f_j}{\omega_1}$ *in* $L^{p'}(\Omega,\omega_1)$ *as* $m \to \infty$.
If $u_m \in W_0^{1,p}(\Omega,\omega_1)$ is a solutions of the problem

$$(P_m)\begin{cases} Lu_m(x) = f_{0m}(x) - \displaystyle\sum_{j=1}^{n} D_j f_{jm}(x) \quad \text{in} \quad \Omega, \\ u_m(x) = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

then $u_n \to u$ in $W_0^{1,p}(\Omega,\omega_1)$ and u is a solution of problem (P).

Proof. We will split the demonstration in two steps.
Step 1. If $u_1, u_2 \in W_0^{1,p}(\Omega,\omega_1)$ are solutions of

$$(P1)\begin{cases} Lu_1(x) = f_0(x) - \displaystyle\sum_{j=1}^{n} D_j f_j(x) \quad \text{in} \quad \Omega, \\ u_1(x) = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

$$(P2)\begin{cases} Lu_2(x) = \tilde{f}_0(x) - \displaystyle\sum_{j=1}^{n} D_j \tilde{f}_j(x) \quad \text{in} \quad \Omega, \\ u_2(x) = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

then for all $\varphi \in W_0^{1,p}(\Omega,\omega_1)$ we have

$$\int_\Omega \langle \mathcal{A}(x,\nabla u_1),\nabla\varphi\rangle\,\omega_1\,dx + \int_\Omega \langle \mathcal{B}(x,u_1,\nabla u_1),\nabla\varphi\rangle\,\omega_2\,dx + \int_\Omega \mathcal{H}(x,u_1,\nabla u_1)\,\varphi\,\omega_3\,dx$$
$$= \int_\Omega f_0\,\varphi\,dx + \sum_{j=1}^{n}\int_\Omega f_j D_j\varphi\,dx,$$

and

$$\int_\Omega \langle \mathcal{A}(x,\nabla u_2),\nabla\varphi\rangle\,\omega_1\,dx + \int_\Omega \langle \mathcal{B}(x,u_2,\nabla u_2),\nabla\varphi\rangle\,\omega_2\,dx + \int_\Omega \mathcal{H}(x,u_2,\nabla u_2)\,\varphi\,\omega_3\,dx$$
$$= \int_\Omega \tilde{f}_0\,\varphi\,dx + \sum_{j=1}^{n}\int_\Omega \tilde{f}_j D_j\varphi\,dx.$$

In particular for $\varphi = u_1 - u_2$, we obtain

$$\begin{aligned}
&\int_\Omega \langle \mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2), \nabla(u_1-u_2)\rangle\, \omega_1\, dx \\
&+ \int_\Omega \langle \mathcal{B}(x,u_1,\nabla u_1) - \mathcal{B}(x,u_2,\nabla u_2), \nabla(u_1-u_2)\rangle\, \omega_2\, dx \\
&\int_\Omega (\mathcal{H}(x,u_1,\nabla u_1) - \mathcal{H}(x,u_2,\nabla u_2))(u_1-u_2)\, \omega_3\, dx \\
&= \int_\Omega (f_0-\tilde{f}_0)(u_1-u_2)\, dx + \sum_{j=1}^n \int_\Omega (f_j-\tilde{f}_j) D_j(u_1-u_2)\, dx. \qquad (3.12)
\end{aligned}$$

In (3.12) we have

(i) By (H2) and Remark 2.3(iii)

$$\begin{aligned}
\int_\Omega \langle \mathcal{A}(x,\nabla u_1) - \mathcal{A}(x,\nabla u_2), \nabla(u_1-u_2)\rangle\, \omega_1\, dx &\geq \theta_1 \int_\Omega |\nabla(u_1-u_2)|^p\, \omega_1\, dx \\
&\geq \frac{\theta_1}{C_\Omega^p+1}\, \|u_1-u_2\|^p_{W_0^{1,p}(\Omega,\omega_1)}.
\end{aligned}$$

(ii) By (H6) we have

$$\int_\Omega \mathcal{B}(x,u_1,\nabla u_1) - \mathcal{B}(x,u_2,\nabla u_2), \nabla(u_1-u_2)\rangle\, \omega_2\, dx \geq 0.$$

(iii) By (H10) we obtain

$$\int_\Omega (\mathcal{H}(x,u_1,\nabla u_1) - \mathcal{H}(x,u_2,\nabla u_2))(u_1-u_2)\, \omega_3\, dx \geq 0.$$

(iv) By Remark 2.3(i)

$$\begin{aligned}
&\left| \int_\Omega (f_0-\tilde{f}_0)\,(u_1-u_2)\, dx + \sum_{j=1}^n \int_\Omega (f_j-\tilde{f}_j)\, D_j(u_1-u_2)\, dx \right| \\
&\leq \left(C_{p,q} \left\| \frac{f_0-\tilde{f}_0}{\omega_2} \right\|_{L^{q'}(\Omega,\omega_2)} + \sum_{j=1}^n \left\| \frac{f_j-\tilde{f}_j}{\omega_1} \right\|_{L^{p'}(\Omega,\omega_1)} \right) \|u_1-u_2\|_{W_0^{1,p}(\Omega,\omega_1)}.
\end{aligned}$$

Therefore in (3.12) we obtain

$$\begin{aligned}
&\|u_1-u_2\|_{W_0^{1,p}(\Omega,\omega_1)} \\
&\leq C \left[C_{p,q} \left\| \frac{f_0-\tilde{f}_0}{\omega_2} \right\|_{L^{q'}(\Omega,\omega_2)} + \sum_{j=1}^n \left\| \frac{f_j-\tilde{f}_j}{\omega_1} \right\|_{L^{p'}(\Omega,\omega_1)} \right]^{1/(p-1)}, \qquad (3.13)
\end{aligned}$$

where $C = ((C_\Omega^p+1)/\theta_1)^{1/(p-1)}$.

Step 2. If $u_m, u_s \in W_0^{1,p}(\Omega,\omega_1)$ are solutions of (P_m) and (P_s) respectively, then by (3.13) we have

$$\begin{aligned}
&\|u_m - u_s\|_{W_0^{1,p}(\Omega,\omega_1)} \\
&\leq C\left[C_{p,q}\left\|\frac{f_{0m}-f_{0s}}{\omega_2}\right\|_{L^{q'}(\Omega,\omega_2)} + \sum_{j=1}^{n}\left\|\frac{f_{jm}-f_{js}}{\omega_1}\right\|_{L^{p'}(\Omega,\omega_1)}\right]^{1/(p-1)}.
\end{aligned}$$

Therefore, $\{u_m\}$ is a Cauchy sequence in $W_0^{1,p}(\Omega,\omega_1)$. Hence, there exists $u \in W_0^{1,p}(\Omega,\omega_1)$ such that $u_m \to u$ in $W_0^{1,p}(\Omega,\omega_1)$. We have that u is a solution of problem (P). In fact, since u_m is a solution of problem (P_m), we obtain for all $\varphi \in W_0^{1,p}(\Omega,\omega_1)$

$$\begin{aligned}
&\int_\Omega \langle \mathcal{A}(x,\nabla u),\nabla\varphi\rangle\,\omega_1\,dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u),\nabla\varphi\rangle\,\omega_2\,dx + \int_\Omega \mathcal{H}(x,u,\nabla u)\,\varphi\,\omega_3\,dx \\
&= \int_\Omega \langle \mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla u_m),\nabla\varphi\rangle\,\omega_1\,dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u)-\mathcal{B}(x,u_m,\nabla u_m),\nabla\varphi\rangle\,\omega_2\,dx \\
&+ \int_\Omega (\mathcal{H}(x,u,\nabla u)-\mathcal{H}(x,u_m,\nabla u_m))\,\varphi\,\omega_3\,dx \\
&+ \int_\Omega \langle \mathcal{A}(x,\nabla u_m),\nabla\varphi\rangle\,\omega_1\,dx + \int_\Omega \langle \mathcal{B}(x,u_m,\nabla u_m),\nabla\varphi\rangle\,\omega_2\,dx \\
&+ \int_\Omega \mathcal{H}(x,u_m,\nabla u_m)\,\varphi\,\omega_3\,dx \\
&= \int_\Omega \langle \mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla u_m),\nabla\varphi\rangle\,\omega_1\,dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u)-\mathcal{B}(x,u_m,\nabla u_m),\nabla\varphi\rangle\,\omega_2\,dx \\
&+ \int_\Omega (\mathcal{H}(x,u,\nabla u)-\mathcal{H}(x,u_m,\nabla u_m))\,\varphi\,\omega_3\,dx \\
&+ \int_\Omega (f_{0m}-f_0)\,\varphi\,dx + \sum_{j=1}^{n}(f_{jm}-f_j)\,D_j\varphi\,dx \\
&+ \int_\Omega f_0\,\varphi\,dx + \sum_{j=1}^{n}\int_\Omega f_j\,D_j\varphi\,dx. \qquad (3.14)
\end{aligned}$$

In (3.14) we have
(i) By Step 1 and (3.2)

$$\left|\int_\Omega \langle \mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla u_m),\nabla\varphi\rangle\,\omega_1\,dx\right| \leq \sum_{j=1}^{n}\int_\Omega |\mathcal{A}_j(x,\nabla u)-\mathcal{A}_j(x,\nabla u_m)|\,|D_j\varphi|\,\omega_1\,dx$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \int_\Omega |F_j u - F_j u_m|\, |D_j \varphi|\, \omega_1\, dx \\
&\le \left(\sum_{j=1}^{n} \|F_j u - F_j u_m\|_{L^{p'}(\Omega,\omega_1)} \right) \|\ |\nabla \varphi|\ \|_{L^p(\Omega,\omega_1)} \\
&\le \left(\sum_{j=1}^{n} \|F_j u - F_j u_m\|_{L^{p'}(\Omega,\omega_1)} \right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)} \\
&\to 0 \text{ as m} \to \infty.
\end{aligned}
$$

(ii) By Remark 2.3 (i), Step 2 and (3.5)

$$
\begin{aligned}
&\left| \int_\Omega \langle \mathcal{B}(x,u,\nabla u) - \mathcal{B}(x,u_m,\nabla u_m), \nabla \varphi \rangle\, \omega_2\, dx \right| \\
&\le \sum_{j=1}^{n} \int_\Omega |\mathcal{B}_j(x,u,\nabla u) - \mathcal{B}_j(x,u_m,\nabla u_m)|\, |D_j \varphi|\, \omega_2\, dx \\
&= \sum_{j=1}^{n} \int_\Omega |G_j u - G_j u_m|\, |D_j \varphi|\, \omega_2\, dx \\
&\le \left(\sum_{j=1}^{n} \|G_j u - G_j u_m\|_{L^{q'}(\Omega,\omega_2)} \right) \|\, |\nabla \varphi|\, \|_{L^q(\Omega,\omega_2)} \\
&\le C_{p,q} \left(\sum_{j=1}^{n} \|G_j u - G_j u_m\|_{L^{q'}(\Omega,\omega_2)} \right) \|\, |\nabla \varphi|\, \|_{L^p(\Omega,\omega_1)} \\
&\le C_{p,q} \left(\sum_{j=1}^{n} \|G_j u - G_j u_m\|_{L^{q'}(\Omega,\omega_2)} \right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)} \\
&\to 0 \text{ as m} \to \infty.
\end{aligned}
$$

(iii) By Remark 2.3(ii), Step 3 and (3.7)

$$
\begin{aligned}
&\left| \int_\Omega (\mathcal{H}(x,u,\nabla u) - \mathcal{H}(x,u_m,\nabla u_m))\, \varphi\, \omega_3\, dx \right| \\
&\le \int_\Omega |\mathcal{H}(x,u,\nabla u) - \mathcal{H}(x,u_m,\nabla u_m)|\, |\varphi|\, \omega_3\, dx \\
&= \int_\Omega |Hu - Hu_m|\, |\varphi|\, \omega_3\, dx \\
&\le \|Hu - Hu_m\|_{L^{s'}(\Omega,\omega_3)}\, \|\varphi\|_{L^s(\Omega,\omega_3)} \\
&\le C_{p,s}\, \|Hu - Hu_m\|_{L^{s'}(\Omega,\omega_3)}\, \|\varphi\|_{L^p(\Omega,\omega_1)} \\
&\le C_{p,s}\, \|Hu - Hu_m\|_{L^{s'}(\Omega,\omega_3)}\, \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)} \\
&\to 0 \text{ as m} \to \infty.
\end{aligned}
$$

(iv) And

$$\begin{aligned}
&\left| \int_\Omega (f_{0m} - f_0)\,\varphi\,dx + \sum_{j=1}^{n} (f_{jm} - f_m)\,D_j\varphi\,dx \right| \\
&\leq \left(C_{p,q} \left\| \frac{f_{0m} - f_0}{\omega_2} \right\|_{L^{q'}(\Omega,\omega_2)} + \sum_{j=1}^{n} \left\| \frac{f_{jm} - f_m}{\omega_1} \right\|_{L^{p'}(\Omega,\omega_1)} \right) \|\varphi\|_{W_0^{1,p}(\Omega,\omega_1)} \\
&\to 0 \text{ as m} \to \infty.
\end{aligned}$$

Therefore in (3.14) we obtain when $m \to \infty$

$$\begin{aligned}
&\int_\Omega \langle \mathcal{A}(x,\nabla u), \nabla\varphi\rangle\,\omega_1\,dx + \int_\Omega \langle \mathcal{B}(x,u,\nabla u), \nabla\varphi\rangle\,\omega_2\,dx + \int_\Omega \mathcal{H}(x,u,\nabla u)\,\varphi\,\omega_3\,dx \\
&= \int_\Omega f_0\,\varphi\,dx + \sum_{j=1}^{n} \int_\Omega f_j\,D_j\varphi\,dx,
\end{aligned}$$

for all $\varphi \in W_0^{1,p}(\Omega,\omega_1)$, i.e., u is a solution of problem (P). □

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